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The center cyclicity of the Lorenz, Chen and Lü systems

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The Hopf points in $\mathbb{R}^3$

We consider real polynomial families of differential equations

$$\begin{aligned} \dot{x} &= -y + \mathcal{F}_1(x, y, z; \mu), \\ \dot{y} &= \ \ x + \mathcal{F}_2(x, y, z; \mu), \\ \dot{z} &= \lambda z + \mathcal{F}_3(x, y, z; \mu), \end{aligned} \tag{1}$$

- Parameter space $E \subseteq \{(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^p\}$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$;
- $\mathcal{F}_j$ $(j = 1, 2, 3)$ contain only nonlinear terms in (x, y, z).

Hopf singular point

The origin is a *Hopf singularity* of all the family (1): it possesses the eigenvalues $\pm i \in \mathbb{C}$ and $\lambda \in \mathbb{R}^*$.

Local dynamics of a member on a center manifold $\mathcal{W}^c$

Let W^c be a local center manifold at the origin of system $(1)_{(\lambda,\mu)=(\lambda^\dagger,\mu^\dagger)}$:

- The origin is a center of $(1)_{(\lambda,\mu)=(\lambda^\dagger,\mu^\dagger)}$ if all the orbits on W^c are periodic;
- Otherwise, the origin is a saddle-focus: a focus on each W^c.

The center problem for a Hopf singularity in $\mathbb{R}^3$

To decide for which parameters $(\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^p$ the origin of (1) is a center or not.

The Lyapunov solution to the center problem

Lyapunov Center Theorem

The origin is a center for system $(1)_{(\lambda,\mu)=(\lambda^\dagger,\mu^\dagger)}$ if and only if it admits a real analytic local first integral of the form

$$H(x,y,z) = x^2 + y^2 + \cdots$$

in a neighborhood of the origin in $\mathbb{R}^3$.

Remark

When there is a center, the W^c is unique and analytic.

Formal Lyapunov function

There is a formal series $H(x, y, z) = x^2 + y^2 + \cdots \in \mathbb{R}[[x, y, z]]$ such that

$$\mathcal{X}(H) = \sum_{j \geqslant 2} \widetilde{\eta}_j(\lambda, \mu)(x^2 + y^2)^j, \tag{2}$$

where $\mathcal{X}$ is the vector field associated to family (1).

Focus quantities: $\widetilde{\eta}_j(\lambda, \mu)$

$$\widetilde{\eta}_j(\lambda, \mu) = \frac{\eta_j(\lambda, \mu)}{d_j(\lambda)} \in \mathbb{Q}(\lambda)[\mu], \qquad \Big(d_j(\lambda) = 0 \Rightarrow \lambda \in i\mathbb{Q}\Big).$$

Unfolding the Hopf points

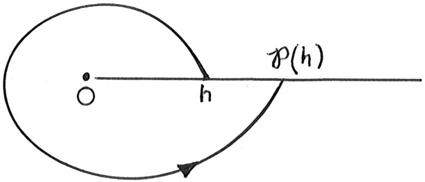
In order to capture the full range of perturbations of Hopf points starting with a member of family (1) we need to consider the larger family:

$$
\begin{aligned}
\dot{x} &= \alpha x - y + \mathcal{F}_1(x, y, z; \mu), \\
\dot{y} &= x + \alpha y + \mathcal{F}_2(x, y, z; \mu), \\
\dot{z} &= \lambda z + \mathcal{F}_3(x, y, z; \mu),
\end{aligned}
\tag{3}
$$

- Parameter set: $E' = E \times \mathbb{R}$.
- $\mathcal{X}(H) = \sum_{j \geqslant 1} \widetilde{\eta}_j(\lambda, \mu, \alpha)(x^2 + y^2)^j$ where $\mathcal{X}$ is the vector field associated to family (3).
- $\tilde{\eta}_1(\lambda, \mu, \alpha) = 2\alpha$ and $\widetilde{\eta}_j(\lambda, \mu, \alpha)$ are analytic functions.

The cyclicity of the Hopf points

Given a family of vector fields $(1)_{(\lambda,\mu)}$ we have a local Poincaré return map at the Hopf point



The cyclicity of O is the maximum number of limit cycles that can bifurcate from it under small perturbations within family $(3)_{(\lambda,\mu,\alpha)}$.

Main goal: to get the cyclicity (or a bound) in the center case

Difficulties

- Lack of analyticity of W^c when $\alpha = 0$ (in general).
- There is no W^c when $\alpha \neq 0$ (the singularity becomes hyperbolic).
- λ is a trouble parameter when (for further convenience) we allow $\lambda \in i\mathbb{Q} \subset \mathbb{C}$.

Target

The goal is to overcome these difficulties, presenting a method for bounding the cyclicity in the center case without any kind of reduction to W^c.

The displacement map around the Hopf singularity

Polar-directional blow-up $\Phi : \mathbb{R}^3 \to \mathbb{S}^1 \times \mathbb{R} \times \mathbb{R}$

$(x, y, z) \mapsto (\theta, r, w) \qquad x = r\cos\theta, \quad y = r\sin\theta, \quad z = r\,w.$

- Blows up $(x, y, z) = (0, 0, 0)$ to the set $\{(\theta, r, w) : r = 0\}$.
- Φ is a diffeomorphism outside the solid cone $C_\tau = \{(x, y, z) : z^2 \geqslant \tau(x^2 + y^2)\}$ for any $\tau > 0$.

Family $(3)_{(\lambda,\mu,\alpha)}$ is written as the analytic system

$$\frac{dr}{d\theta} = R(\theta, r, w; \mu), \quad \frac{dw}{d\theta} = (\lambda - \alpha)w + W(\theta, r, w; \lambda, \mu) \qquad (4)$$

on some cylinder $\{(\theta, r, w) : |r| \ll 1, w \in \mathcal{K}\}$ where $\mathcal{K} \subset \mathbb{R}$ is an arbitrary compact neighborhood of 0 in $\mathbb{R}$ ($\mathcal{K} = \{|w| < \sqrt{\tau}\}$).

The displacement map around the Hopf singularity

Small amplitude periodic solutions of $\mathcal{X}$ around the origin

There is a one-to-one correspondence between 2π-periodic solutions of (4) and small amplitude periodic orbits of $(3)_{(\lambda,\mu,\alpha)}$.

- When $\alpha = 0$:
  - Any small periodic orbit is contained in W^c.
  - $W^c \cap C_\tau = \{(0,0,0)\}$.
- When $\alpha \neq 0$: we prove that under perturbation ($|\alpha| \ll 1$) the normally hyperbolic W^c is replaced by a normally hyperbolic invariant two-manifold $\mathcal{M}$ through $(0,0,0)$ such that:
  - Any small periodic orbit is contained in $\mathcal{M}$.
  - $\mathcal{M} \cap C_\tau = \{(0,0,0)\}$.

The reduced displacement map $\delta(r_0; \lambda, \mu, \alpha)$

- Let $\Psi(\theta; r_0, w_0; \lambda, \mu, \alpha)$ be the unique solution of (4) with initial condition (r_0, w_0).
- *Poincaré map*: $\Pi(r_0, w_0; \lambda, \mu, \alpha) = \Psi(2\pi; r_0, w_0; \lambda, \mu, \alpha)$.
- *Displacement map*: $d(r_0, w_0; \lambda, \mu, \alpha) = \Pi(r_0, w_0; \lambda, \mu, \alpha) - (r_0, w_0)$.

$$d(r_0, w_0; \lambda, \mu, \alpha) = (d_1(r_0, w_0; \lambda, \mu, \alpha), d_2(r_0, w_0; \lambda, \mu, \alpha)).$$

Applying a Lyapunov-Schmidt reduction to the displacement map:

- The *reduced displacement map*

$$\delta(r_0; \lambda, \mu, \alpha) := d_1(r_0, \bar{w}(r_0, \lambda, \mu, \alpha); \lambda, \mu, \alpha) = \sum_{j \geqslant 1} v_j(\lambda, \mu, \alpha) r_0^j$$

- $v_1(\lambda, \mu, \alpha) = e^{2\pi\alpha} - 1$.
- $\mathcal{X}_{(\lambda^*, \mu^*, 0)}$ has a center at the origin $\Leftrightarrow \delta(r_0; \lambda^*, \mu^*, 0) \equiv 0$.

Some ideals in the Noetherian ring $\mathbb{R}(\lambda)[\mu]$

Poincaré-Lyapunov quantities (when $\alpha = 0$)

$v_j(\lambda, \mu, 0) = \frac{V_j(\lambda,\mu)}{D_j(\lambda)} \in \mathbb{R}(\lambda)[\mu], \quad \Big(D_j(\lambda) = 0 \Rightarrow \lambda \in i\mathbb{Q}\Big).$

Bautin ideal (when $\alpha = 0$)

$\mathscr{B} = \langle v_j : j \in \mathbb{N} \rangle$

- $\mathscr{B}_k = \langle v_3, \ldots, v_k \rangle$.
- $\mathscr{I} = \langle \widetilde{\eta}_j : j \in \mathbb{N} \rangle$
- $\mathscr{I}_k = \langle \widetilde{\eta}_2, \ldots, \widetilde{\eta}_k \rangle$

Relation between the ideals $\mathscr{B}$ and $\mathscr{I}$

Theorem

The following holds:

- $\mathscr{B} = \langle v_{2k+1} : k \geqslant 1 \rangle = \mathscr{I}$;
- The minimal bases of $\mathscr{B}$ and $\mathscr{I}$ have the same finite cardinality;

The Bautin depth: the cardinality of the minimal basis of $\mathscr{B}$

$$\#_{\min}\mathscr{B}$$

A uniform bound on the cyclicity of centers

Using the classical strategy:

$$\delta(r_0;\lambda,\mu) = \sum_{k=1}^{\#_{\min}\mathscr{B}} v_{j_k}(\lambda,\mu)\left[1+\psi_k(r_0;\lambda,\mu)\right] r_0^{j_k} \tag{5}$$

where $\psi_k(r_0;\lambda,\mu)$ are analytic functions at $r_0 = 0$, $\psi_k(0;\lambda,\mu) = 0$ and

$$\left\{ v_{j_1},\ldots, v_{j_{\#_{\min}\mathscr{B}}} \right\}$$

is a minimal basis of $\mathscr{B}$.

Theorem

The cyclicity of any center at the origin perturbing in E' is at most $\#_{\min}\mathscr{B}$.

But, how to compute $\#_{\min}\mathscr{B}$?

Main tool (inspired by the work of Colin Christopher)

Center variety $V_{\mathscr{C}} \stackrel{\text{def}}{=} \mathbf{V}(\langle \eta_j : j \in \mathbb{N} \rangle) \subset \mathbb{R}^{p+1}$

An element of the family (1) corresponding to (λ, μ) has a center at the origin if and only if $(\lambda, \mu) \in V_{\mathscr{C}} \cap E$.

The mapping $F_\kappa : \mathbb{R}^* \times \mathbb{R}^p \to \mathbb{R}^\kappa$

For any $\kappa \leq \#_{\min} \mathscr{B}$, define

$$F_\kappa(\lambda, \mu) = (\widetilde{\eta}_{j_1}(\lambda, \mu), \ldots, \widetilde{\eta}_{j_\kappa}(\lambda, \mu)), \tag{6}$$

where $\{\widetilde{\eta}_{j_1}(\lambda, \mu), \ldots, \widetilde{\eta}_{j_\kappa}(\lambda, \mu)\}$ is the minimal basis of the ideal $\mathscr{I}_{j_\kappa}$ in $\mathbb{R}(\lambda)[\mu]$.

Main tool (inspired by the work of Colin Christopher)

Let $\mathrm{Cyc}(\mathcal{X}_{(\lambda^*,\mu^*)}, 0)$ be the cyclicity of a point $(\lambda^*, \mu^*) \in V_{\mathscr{C}} \cap E$ under perturbations in E'.

Theorem

Let κ and F_κ be as before and C be an irreducible component of $V_{\mathscr{C}}$. Suppose:

- $\kappa \leqslant p + 1$;
- $\mathrm{rank}(d_P F_\kappa) = \kappa$ at some point $P = (\lambda^*, \mu^*) \in C \cap E$.

Then the following holds:

(i) $\mathrm{Cyc}(\mathcal{X}_{(\lambda^*,\mu^*)}, 0) \geq \kappa$;

(ii) If moreover $\mathrm{codim}\{(\lambda, \mu, 0) \in \mathbb{R}^{p+2} : (\lambda, \mu) \in C\} = \kappa + 1$ then $\mathrm{Cyc}(\mathcal{X}_{(\lambda^*,\mu^*)}, 0) = \kappa$.

The Lorenz family

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = -bz + xy, \qquad (7)$$

with parameters $(\rho, \sigma, b) \in \mathbb{R}^3$ such that $b\,\sigma \neq 0$ else no singularity is isolated.

- The origin is always a singularity;
- When $b\,(\rho - 1) > 0$ there also exists the symmetric singularities

$$E_{\pm} := (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1).$$

NOTE: System (7) is invariant under the involution $(x, y, z) \mapsto (-x, -y, z)$.

Hopf points in the Lorenz family

Let $E \subset \mathbb{R}^3$ be set of parameters (σ, ρ, b) for which the singularity is a Hopf point.

- For the origin: $E = \mathbf{V}(\sigma + 1) \bigcap \{(\sigma, \rho, b) : \rho - 1 > 0\}$;
- For the points $E_{\pm}$:

$$E = \mathbf{V}((\sigma+\rho)b+\sigma^2+(3-\rho)\sigma+\rho) \bigcap \{(\sigma, \rho, b) \in \mathbb{R}^3 : (\sigma, \rho) \in H_E\}$$

being

$$\begin{aligned} H_E \overset{\text{def}}{=} & \{(\sigma, \rho) : -\sigma^2 - (3-\rho)\sigma - \rho > 0, \sigma \neq 0\} \\ & \cap [\{(\sigma, \rho) : \sigma + \rho < 0\} \cup \{(\sigma, \rho) : \rho - 1 > 0\}], \end{aligned}$$

DEFINITION: $b = \xi(\sigma, \rho) = \frac{-\sigma^2-(3-\rho)\sigma-\rho}{\sigma+\rho}$.

Bibliography

The next two theorems characterize the centers and the cyclicity of saddle-foci and have been proved independently by several authors:

- A. Algaba, M. C. Dominguez-Moreno, M. Merino, and A. J. Rodriguez-Luis, *Study of the Hopf Bifurcation in the Lorenz, Chen and Lü systems*, Nonlinear Dyn. **79** (2015) 885–902.
- Q. Wang, W. Huang, and J. Feng, *Multiple limit cycles and centers on center manifolds for Lorenz system*, Appl. Math. Comput. **238** (2014) 281–288.

Centers in the Lorenz family

Theorem

The singularities of the Lorenz family are centers if and only if $(\sigma, \rho, b) \in V_{\mathscr{C}} \cap E$ with center varieties

- The origin: $V_{\mathscr{C}} = \mathbf{V}(\sigma + 1, b + 2)$;
- For the points $E_{\pm}$:
  $V_{\mathscr{C}} = \mathbf{V}(\sigma + 1, (\sigma + \rho)b + \sigma^2 + (3 - \rho)\sigma + \rho)$.

Sketch of the proof (for the points $E_\pm$)

To solve the center problem we need only work with the numerators $\eta_j := \eta_j(\sigma, \rho, \xi(\sigma, \rho))$ of the focus quantities $\tilde{\eta}_j$:

1. Let $I_k = \langle \eta_2, \ldots, \eta_k \rangle$ in the ring $\mathbb{R}[\sigma, \rho]$.
2. Certainly $V_{\mathscr{C}} \cap E \subset \mathbf{V}(I_4) = \mathbf{V}(\sqrt{I_4})$.
3. Use SINGULAR to compute the prime decomposition $\sqrt{I_4} = \cap_{j=1}^{7} J_j$ with $J_7 = \langle \sigma + 1 \rangle$.
4. We check that $\mathbf{V}(J_j) \cap E = \varnothing$ for all $j \neq 7$, hence $V_{\mathscr{C}} \cap E \subset \mathbf{V}(J_7)$.
5. The reverse inclusion $\mathbf{V}(J_7) \subset V_{\mathscr{C}} \cap E$ follows because $V(x, y, z) = z - \frac{1}{2}x^2$ is an inverse Jacobi multiplier when $\sigma = -1$

Cyclicity of the saddle-foci in the Lorenz family

Theorem

For the Lorenz family (7) the sharp upper bound on the cyclicity of any saddle-focus (at either the origin or $E_\pm$), under perturbation within full family (7), is three.

Two saddle-focus of maximum order: There are exactly two points

$$(\sigma_j^*, \rho_j^*, b_j^*) \in \mathbf{V}(I_3) \cap E, \quad \eta_4(\sigma_j^*, \rho_j^*, b_j^*) \neq 0, \quad j = 1, 2.$$

These two points corresponds with saddle-foci at $E_\pm$ having (simultaneously) cyclicity three.

Cyclicity of the centers in the Lorenz family

Theorem

For the Lorenz family (7), with the exception of the system that corresponds to the parameter choice $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$ (for which the origin is not a center), the cyclicity of any center (at either the origin or $E_{\pm}$), under perturbation within full family (7), is one.

Open problem

To know the cyclicity (≥ 2) of the center at $E_{\pm}$ for the Lorenz system with $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$.

Sketch of the proof (for the points $E_\pm$)

1. $\mathrm{codim}(V_{\mathscr{C}} \cap E) = 2$ (intersect transversely).
2. Let the map $F_1 : \mathbb{R}^2 \to \mathbb{R}$ with $F_1(\sigma, \rho) = \tilde{\eta}_2(\sigma, \rho, \xi(\sigma, \rho))$ and an arbitrary point $P = (\sigma, \rho) = (-1, \rho)$ with $(\sigma, \rho, \xi(\sigma, \rho)) \in V_{\mathscr{C}} \cap E$.
3. Then we directly compute the rank of the 1×2 matrix

$$\mathrm{rank}(d_P F_1) = \mathrm{rank} \begin{pmatrix} \dfrac{(\rho^2 + 6\rho - 9)\sqrt{1-\rho}}{4\sqrt{2}(\rho - 3)(\rho - 1)^2(2\rho - 3)} & 0 \end{pmatrix} = 1$$

except for $\rho = -3 - 3\sqrt{2}$.

NOTE: $(\sigma, \rho) \notin H_E$ for $\rho \in \{-3 + 3\sqrt{2}, 3, 1, 3/2\}$.

Byproduct: Cyclicity of the centers in the Lü and Chen family

The Lü family is given by

$$\dot{x} = A(y - x), \quad \dot{y} = Cy - xz, \quad \dot{z} = -Bz + xy, \tag{8}$$

and the Chen family by

$$\dot{x} = A(y - x), \quad \dot{y} = (C - A)x + Cy - xz, \quad \dot{z} = -Bz + xy, \tag{9}$$

with parameters $(A, B, C) \in \mathbb{R}^3$.

Reduction when $C \neq 0$: $(x, y, z, t) \mapsto (-x/C, -y/C, -z/C, -Ct)$

After this linear scaling the Chen and Lü families reduce to special cases of the Lorenz family.

Theorem

For both the Lü and the Chen families, the cyclicity of any center (either the origin or not), under perturbation within full family, is one.

Publications

- I.A. García, S. Maza & D.S. Shafer, *Cyclicity of polynomial nondegenerate centers on center manifolds*, J. Differential Equations **265** (2018), 5767–5808.
- I.A. García, S. Maza & D.S. Shafer, *Center cyclicity of Lorenz, Chen and Lü systems*. To appear in Nonlinear Anal.