A two-parameter family of vector fields with inclination-flip and orbit-flip homoclinic connections.

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We consider the following three-dimensional two-parameter family:

$$\begin{cases} \dot{x} = a + yz, \\ \dot{y} = -y + x^2, \\ \dot{z} = b - 4x, \end{cases} \tag{1}$$

with $a \neq 0$ and $b \neq 0$.

⇒ See **A. Algaba *et al.* Commun Nonlinear Sci Numer Simulat (2019) 77,pp. 324-337.**([1])

One equilibrium: $E = \left(\frac{b}{4}, \frac{b^2}{16}, \frac{-16a}{b^2} \right)$.

- If $a = 0$ and $b = 1$: Sprott's system.
- If $a \neq 0$ and $b = 1$: Wang and Chen's system.
- If $a \neq 0$ and $b = 0$: there are no equilibria.
- If $a = 0$ and $b = 0$: there is a continuum of equilibria (axis z).

We translate the equilibrium E to the origin by means of the change,

$$X = x - \frac{b}{4}, \quad Y = y - \frac{b^2}{16}, \quad Z = z + \frac{16}{b^2}$$

that transforms the system (1) into

$$\begin{cases} \dot{X} = \dfrac{-16a}{b^2} Y + \dfrac{b^2}{16} Z + YZ, \\ \dot{Y} = \dfrac{b}{2} X - Y + X^2, \\ \dot{Z} = -4X, \end{cases} \tag{2}$$

The linearization matrix of (2) is given by

$$\begin{pmatrix} 0 & \frac{-16a}{b^2} & \frac{b^2}{16} \\ \frac{b}{2} & -1 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

Its characteristic polynomial is:

$$p = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3,$$

where

$$p_1 = 1, \quad p_2 = \frac{8a}{b} + \frac{b^2}{4}, \quad p_3 = \frac{b^2}{4}.$$

- **HOPF BIFURCATION**: $\boxed{a = 0, b \neq 0.}$

  Eigenvalues: $\lambda_1 = -1, \lambda_{2,3} = \pm\omega_0 i$, with $\omega_0 = \dfrac{b}{2}$.

Considering the system (2) in these critical values and by means of a linear change, the system (2) transforms into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 & 0 \\ -\omega_0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \\ F_3(u,v) \end{pmatrix}. \tag{3}$$

By means of a third-order approximation of the center manifold,

$$w = A_1 u^2 + A_2 uv + A_3 v^2 + ...,$$

we get the second-order system on the computer center manifold,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} G_1(u,v) \\ G_2(u,v) \end{pmatrix}. \tag{4}$$

By means of the reparameterization

$$u \to -\bar{u}, \quad v \to \bar{v}, \quad t \to \frac{2}{b}\tau,$$

and the recursive algorithm developed by Gamero et al.([3]), we obtain a third-order normal form for the reduced system (4),

$$\begin{cases} \dot{r} = \alpha_1 r^3 + \cdots , \\ \dot{\theta} = 1 + \beta_1 r^2 + \cdots , \end{cases} \tag{5}$$

where

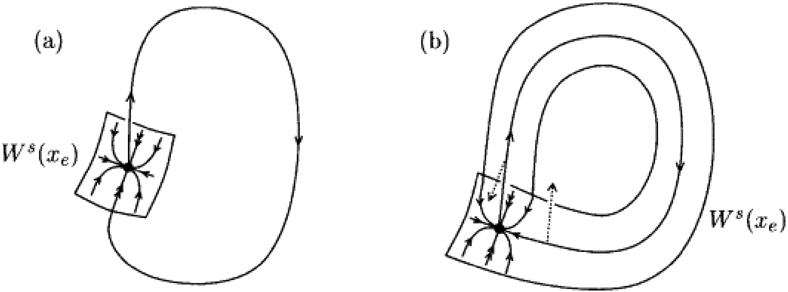
$$\alpha_1 \quad = \quad -\frac{2(b^4 - 40\,b^2 + 16)}{b^3\,(b^2+1)\,(b^2+4)}, \qquad \beta_1 = \frac{2(6b^6 - 13b^4 + 136b^2 - 16)}{3b^4\,(b^2+1)\,(b^2+4)}. \tag{6}$$

Theorem

The equilibrium E of system (1) exhibits a Hopf bifurcation when $a = 0$ and $b \neq 0$. This bifurcation is subcritical if $b \in (b_4, b_3) \cup (b_1, b_2)$ and supercritical when $b \in (-\infty, b_4) \cup (b_3, 0) \cup (0, b_1) \cup (b_2, +\infty)$, where

$$b_1 = 2(\sqrt{3} - \sqrt{2}), \quad b_2 = 2(\sqrt{3} + \sqrt{2}), \quad b_3 = -2(\sqrt{3} - \sqrt{2}), \quad b_4 = -2(\sqrt{3} + \sqrt{2}).$$

A DEGENERATE HOMOCLINIC BIFURCATION



THREE CASES OF AN HOMOCLINIC FLIP BIFURCATION.

$$\boxed{\lambda^s < 0 < \lambda^u < \lambda^{uu}} \qquad \boxed{\alpha = \frac{-\lambda_{uu}}{\lambda_s} \qquad \beta = \frac{-\lambda_u}{\lambda_s}}$$

Inclination-flip	Orbit-flip
A: $\beta > 1$	**A**: $\beta > 1$
B: $\alpha > 1,\ \frac{1}{2} < \beta < 1$	**B**: $\beta < 1,\ \alpha > 1$
C: $\alpha < 1$, ó $\beta < \frac{1}{2}$	**C**: $\alpha < 1$

- Case A: only a periodic orbit is created.
- Case B: three new curves emerge: SNOP,PD bifurcation and a double-period homoclinic bifurcation.
- Case C: infinitely many homoclinic bifurcations, saddle-node and period-doubling cascades emerge.

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- Giraldo A, Krauskopf B, Osinga HM. Saddle invariant objects and their global manifolds in a neighborhood of a homoclinic flip bifurcation of case B. SIAM J Appl Dyn Syst 2017;16:640–86.
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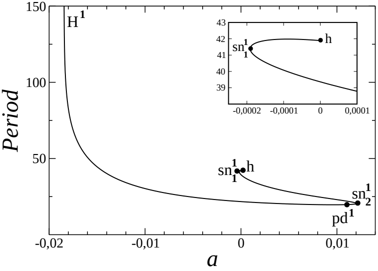
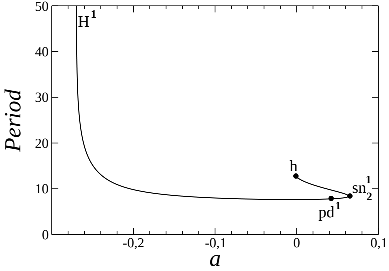
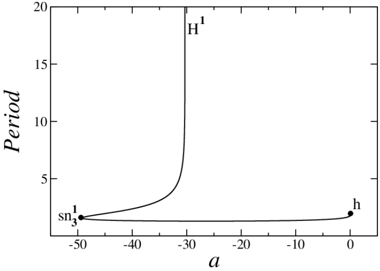


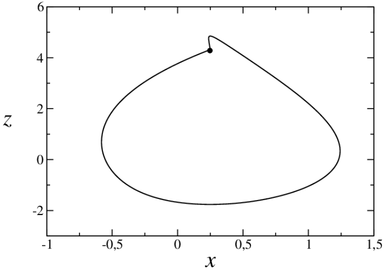
$a = 0, b = 0{,}3$



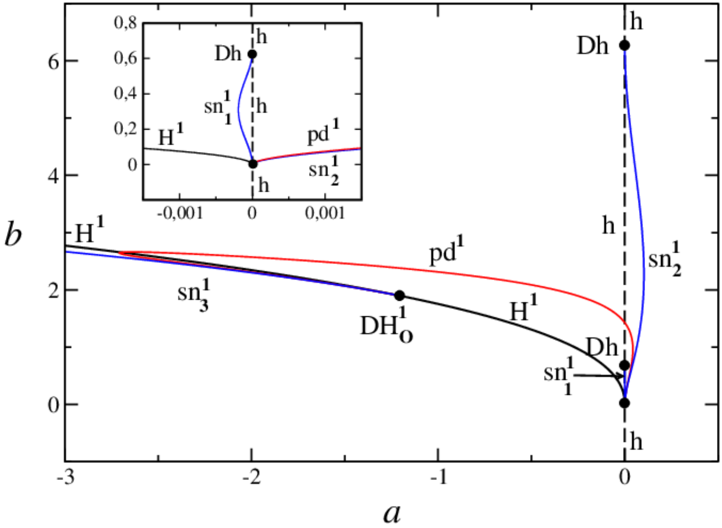
$a = 0, b = 1$



$a = 0, b = 7$



Projection H^1 onto (x, z)- plane

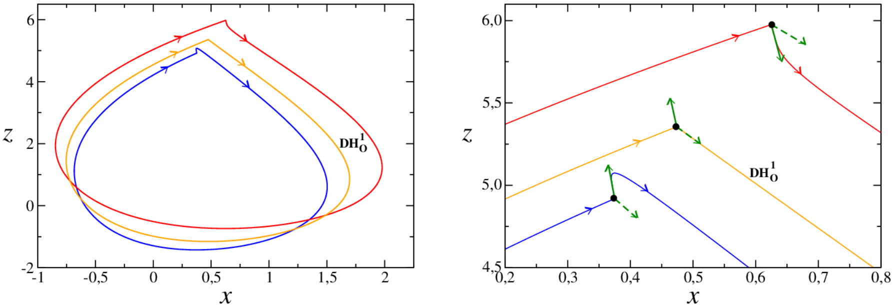


Partial bifurcation set in the (a, b)-plane.

The equilibrium eigenvalues at the point $\mathbf{DH_0^1}$:

$\lambda_s \approx -2{,}68251, \quad \lambda_u \approx 0{,}23082, \quad \lambda_{uu} \approx 1{,}45168.$

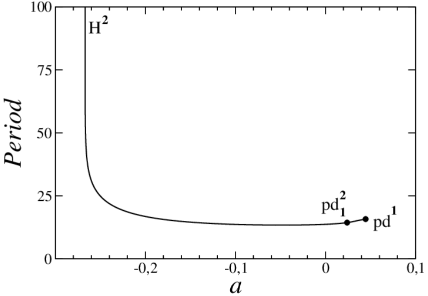
$\alpha \approx 0{,}541, \quad \beta \approx 0{,}086. \qquad \alpha < 1 \rightarrow$ Case **C.**



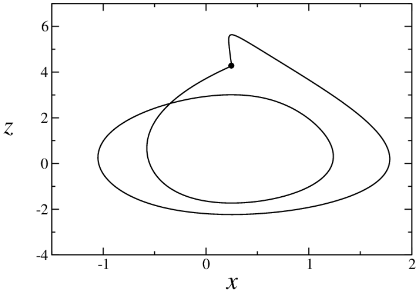
Projection of the homoclinic connection H^1 onto (x, z)-plane for values located on both sides of the point DH_o^1.

Cin or Cout?

We determine the position of the curve of double-period homoclinic connection H^2 with respect the main homoclinic H^1.

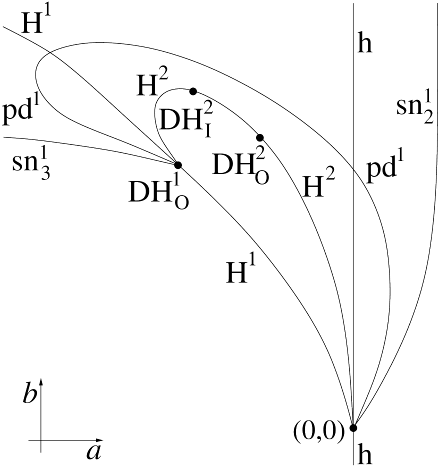


For $b = 1$: bifurcation diagram of the periodic orbit born in pd^1.



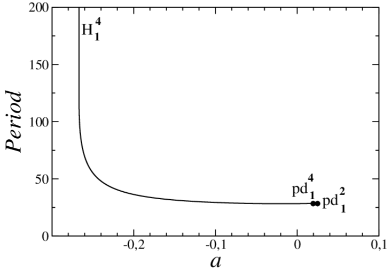
Projection H^2 onto (x, z)- plane

DH_O^2: orbit-flip bifurcation. DH_I^2: inclination-flip bifurcation.

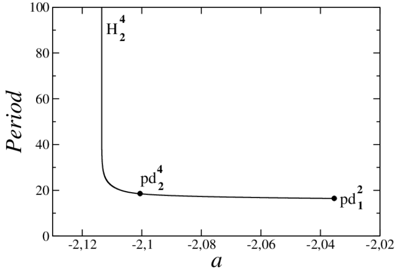


Partial bifurcation set in a vicinity of the homoclinic connections H_1 y H_2.

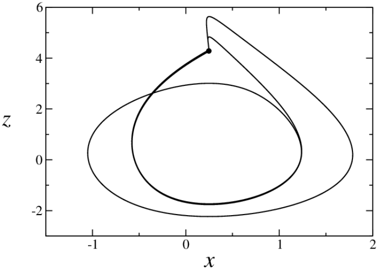
ARE THESE ORBIT-FLIP AND INCLINATION-FLIP DEGENERATIONS ALSO PRESENT IN THE CURVES OF QUADRUPLE-PULSE HOMOCLINIC CONNECTIONS THAT EXIST IN RELATION TO THE DEGENERACIES DH_O^1,DH_O^2 AND DH_I^2?



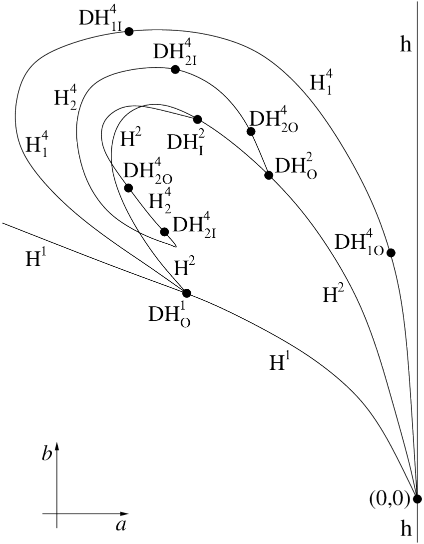
For $b = 1$



For $b = 2{,}4$



Projection H^4 onto (x, z)- plane



Qualitative partial bifurcation set of the homoclinic connections H^1, H^2, H^4_1 and H^4_2.

This suggests the existence, apart from the principal homoclinic H^1, of an infinite succession $\{\mathtt{H}^{2^n}_{2^{(n-1)}}\}$ for $n \geq 1$ of homoclinic connections, specifically $2^{(n-1)}$ homoclinic connections of 2^n-pulse.

$n = 1 \rightarrow \mathrm{H}^2$

$n = 2 \rightarrow \mathrm{H}^4_1$ and H^4_2

$n = 3 \rightarrow \mathrm{H}^8_1$, H^8_2, H^8_3 and H^8_4

...

The first of them, $\mathtt{H}^{2^n}_1$, joins the point $(0, 0)$ with the point DH^1_O, and there are two degeneracies $\mathtt{H}^{2^n}_{1O}$ and $\mathtt{H}^{2^n}_{1I}$ of the previous type on it.

Each of the remaining homoclinic connection curves $\mathtt{H}^{2^n}_2$, $\mathtt{H}^{2^n}_3$, ..., $\mathtt{H}^{2^n}_{2^{(n-1)}}$ have four degenerations, two inclination-flip points $\mathtt{H}^{2^n}_I$ and two orbit-flip points $\mathtt{H}^{2^n}_O$.

In total there are $2(2^n - 1)$ degeneracy points, half of them are of inclination-flip type and the other half are orbit-flip bifurcations.

CONCLUSIONS:

- By means of analytical and numerical tools we detect complex bifurcation behavior in a two-parameter quadratic three-dimensional system with only six terms and two nonlinearities.

- The organizing center, in a region of the parameter plane where the only equilibrium has three real eigenvalues, is a homoclinic flip bifurcation of the inward twist case C_{in}, case where this orbit changes from orientable to non-orientable, being the lowest codimension for a homoclinic bifurcation to a real saddle equilibrium that implies chaotic behavior with strange attractors.

- This family that we have studied will provide the opportunity to complete the study of the case C_{in}.

[1] A. Algaba, M.C. Domínguez-Moreno, M. Merino,A.J.Rodríguez-Luis: Study of a simple 3D quadratic system with homoclinic flip bifurcations of inward twist case C_{in}, Commun Nonlinear Sci Numer Simulat (2019) 77,pp. 324-337.

[2] E.J. Doedel, A.R. Champneys, F. Dercole, T. Fairgrieve, Y. Kuznetsov, B.E. Oldeman, R. Paffenroth, B. Sandstede, X. Wang, C. Zhang, Auto07-P: Continuation and bifurcation software for ordinary differential equations (with HomCont), Technical report, Concordia University, 2010.

[3] Gamero E , Freire E , Ponce E . Normal forms for planar systems with nilpotent linear part. In: Seydel R, Schneider FW, Küpper T, Troger H, editors. Bifurcation and chaos: analysis, algorithms, applications, 97. Basel: Birkhäuser; 1991. p. 123-7 . International Series of Numerical Mathematics

[4] Sandstede B. Constructing dynamical systems having homoclinic bifurcation points of codimension two.J Dyn Differ Equ 1997;9:269–88.