Integrability and Center problem for perturbation of quasi-homogeneous centers.

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Outline

- Introduction and Motivation
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We consider an autonomous system of the form,

$$\dot{\mathbf{x}} \;=\; \mathbf{F}(\mathbf{x}) = (P(\mathbf{x}), Q(\mathbf{x}))^T,\; \mathbf{x} \in \mathbb{R}^2,$$

where $\mathbf{F}$ is an analytic planar vector field defined in a neighborhood of the origin $\mathcal{U} \subset \mathbb{C}^2$ having an equilibrium point at the origin, i.e., $\mathbf{F}(\mathbf{0}) = \mathbf{0}$ and P, Q analytic in $\mathcal{U}$. An equilibrium point is monodromic if there is no orbit tending to it. After we know the equilibrium point is monodromic, the question is, when the equilibrium point is a center? The integrability problem is very interesting as well, and both problems are related.

Quasi-homogeneous Vector Fields

- Let $\mathbf{f} \in \mathcal{P}_k^{\mathbf{t}=(t_1,t_2)}$ be a quasi-homogeneous polynomial of type $\mathbf{t}$ and degree $k \Leftrightarrow f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x,y)$.
- Let $\mathbf{F} \in \mathcal{Q}_k^{\mathbf{t}=(t_1,t_2)}$ be a quasi-homogeneous vector field of type $\mathbf{t}$ and degree $k \Leftrightarrow \mathbf{F} = (P,Q)^T$, where $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$.

Any system can be written as the sum of quasi-homogeneous terms of type $\mathbf{t}$:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots ,$$

where $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ for all $k \in \mathbf{N}$. We are going to consider the conservative-dissipative decomposition:

$$\mathbf{F}_r \quad = \quad \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0,$$

where $\mathbf{D}_0 = (t_1 x, t_2 y)^T$, $h_{r+|\mathbf{t}|} = \frac{1}{r+|\mathbf{t}|}(\mathbf{D}_0 \wedge \mathbf{F}_r)$ and $\mu_r = \frac{1}{r+|\mathbf{t}|} div(\mathbf{F}_r)$.

Sufficient Condition of Monodromy

Our goal is to know when a equilibrium point of $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \cdots$ is a center, but first we should know if it is monodromic. We will focus in the generic cases because when this does not happen, it is a degenerate case.

Theorem

Let be $\mathbf{F}_r = \mathbf{X}_{h_{r+|\mathbf{t}|}} + \mu_r \mathbf{D}_0 \in \mathcal{Q}_r^{\mathbf{t}}$. If $h_{r+|\mathbf{t}|}(x,y) \neq 0$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$ where $\mathcal{U}$ is a neighborhood of the origin then the origin of system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x}) + \cdots$ is monodromic.

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \qquad (1)$$

where $h_{r+|\mathbf{t}|}(x,y) \neq 0$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$.

Necessary condition of center

Let be

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots, \qquad (2)$$

where $h_{r+|\mathbf{t}|}(x, y) \neq 0$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\}$.

Theorem

If the origin of (2) is a center, then the origin of system $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ is also a center.

From now on we assume these systems

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots,$$

where $\dot{\mathbf{x}} = \mathbf{F}_r(\mathbf{x})$ has a center at the origin.

An ilustrative example

$\mathbf{F}_r$ has a center at the origin is a necessary condition but it is not sufficient:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{pmatrix} -4y^3 - 2x^2y \\ 2xy^2 + 6x^5 \end{pmatrix}$$

$\mathbf{F}(\mathbf{x}) = \mathbf{F}_2(\mathbf{x}) + \mathbf{F}_4(\mathbf{x})$ where
$\mathbf{F}_2 = (-4y^3 - 2x^2y, 2xy^2)^T \in \mathcal{Q}_2^{(1,1)}$ and $\mathbf{F}_4 = (0, 6x^5)^T \in \mathcal{Q}_4^{(1,1)}$
The origin of system is monodromic and a center also because $\mathbf{F} = \mathbf{X}_H$, with $H(x,y) = y^4 + x^2y^2 + x^6$, $H(x,y) > 0$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$ and $\mathbf{F}$ is hamiltonian.
On the other hand, $\mathbf{F}_2(\mathbf{x}) = \mathbf{X}_h$ with $h(x,y) = y^2(y^2 + x^2)$ and it is not true that $h(x,y) \neq 0$ for all $(x,y) \in \mathcal{U} \setminus \{(0,0)\}$.

Center conditions

Choosing the change to generalized polar coordinates.

$$\begin{aligned} x &= \rho^{t_1}\mathrm{Cs}(\theta), \\ y &= \rho^{t_2}\mathrm{Sn}(\theta), \end{aligned}$$

where $(\mathrm{Cs}(\theta), \mathrm{Sn}(\theta))$ are the periodic solutions of period $T > 0$, of the initial value problem $(\dot{x}, \dot{y})^T = \mathbf{F}_r(x, y)$, $x(0) = 1$, $y(0) = 0$. System $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$ with a reparemetrization in time $dt = \frac{1}{\rho^r} d\tau$ could be written as

$$\begin{aligned} \dot{\rho} &= \rho \sum_{j=1}^{\infty} R_j(\theta)\rho^j, \\ \dot{\theta} &= 1 + \sum_{j=1}^{\infty} \Psi_j(\theta)\rho^j, \end{aligned}$$

Using power series of ρ, we obtain the **generalized Abel's equation**.

$$\frac{d\rho}{d\theta} = \sum_{i=2}^{\infty} g_i(\theta)\rho^i,$$

Center conditions

$$\frac{d\rho}{d\theta} = \sum_{i=2}^{\infty} g_i(\theta)\rho^i, \tag{3}$$

(3) converges if $|\rho| \ll 1$. Let $\rho(\theta, \rho_0) = \sum_{n\geq 1} a_n(\theta)\rho_0^n$ be the solution of (3) when $\rho(0, \rho_0) = \rho_0$. With this, the Poincaré Map is

$$P(\rho_0) = \rho(T, \rho_0) = \rho_0 + \sum_{n\geq 2} a_n(T)\rho_0^n.$$

Theorem

The origin of $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$ is a center if, and only if, $a_n(T) = 0$ for all $n \geq 2$.

Integrating factor of Abel's Equation

Proposition

There exists a single formal serie $\Phi(\rho,\theta)=\sum_{i=1}^{\infty}\Phi_i(\theta)\rho^i$, *such that* $\frac{1}{1-\Phi(\rho,\theta)}$ *is an integrating factor of Abel's Equation with* $\Phi(\rho,0)\equiv 0$. $\Phi_i(\theta)$, $i\geq 1$, *verifies:*

$$\Phi_1(\theta) = -2\int_0^{\theta} g_2(s)ds, \quad \textit{and if} \ \ i\geq 2,$$

$$\left\{\begin{array}{rcl} \Phi_i'(\theta) & = & -(i+1)g_{i+1}(\theta)+\displaystyle\sum_{j=1}^{i-1}(i+1-2j)\Phi_j(\theta)g_{i+1-j}(\theta), \\ \Phi_i(0) & = & 0. \end{array}\right.$$

.

Integrating factor of Abel's Equation

Proposition

Let $\frac{1}{1-\Phi(\theta,\rho)}$ *be the integrating factor of Abel's Equation then the inverse of Poincaré Map is*

$$P^{-1}(\rho) = \rho + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \sum_{\substack{i_1 \geq 1, \ldots, i_j \geq 1 \\ i_1 + \cdots + i_j = n}} \Phi_{i_1}(T) \cdots \Phi_{i_j}(T) \right) \frac{\rho^{i+1}}{i+1}$$

Theorem

If the origin of $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$ *is a center then* $\Phi_k(T) = 0$ *for all* $k \geq 1$*.*

Improvement of focal constant calculates

Proposition

We suppose that we can write system $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$ *as:* $\dot{\mathbf{x}} = \bar{\mathbf{F}}(\mathbf{x}) + \mathbf{F}(\mathbf{x})$, *where the origin of system* $\dot{\mathbf{x}} = \bar{\mathbf{F}}(\mathbf{x})$ *is a center.*

System	*Abel's Eq.*	*i.i.f*
$\dot{\mathbf{x}} = \bar{\mathbf{F}} + \mathbf{F}$	$\dot{\rho} = \sum_i g_i(\theta)\rho^i$	$1 - \Phi$
$\dot{\mathbf{x}} = \bar{\mathbf{F}}$	$\dot{\rho} = \sum_i \bar{g}_i(\theta)\rho^i$	$1 - \bar{\Phi}$

Let be $\hat{g}_i = g_i - \bar{g}_i$, $i \geq 2$ *and* $\hat{\Phi}_i(\theta) = \Phi_i(\theta) - \bar{\Phi}_i(\theta)$, $i \geq 0$. *Then*

$$\Phi_n(T) = \hat{\Phi}_n(T) \quad \forall n \geq 1.$$

Corollary

Case k-jet of F center *We suppose that* $\dot{\mathbf{x}} = \bar{\mathbf{F}} = \sum_{j=0}^{k-1} \mathbf{F}_{r+j}$, $k > 1$, *is a center, where* $\bar{\mathbf{F}}$ *are the* k*-first terms from system* $\dot{\mathbf{x}} = \mathbf{F}_r + \cdots$. $\hat{\Phi}_i(\theta)$ *are defined as*

$$\hat{\Phi}_i(\theta) = 0, \quad if \quad i \leq k-2,$$

$$\begin{cases} \hat{\Phi}_i'(\theta) = -(i+1)\hat{g}_{i+1}(\theta) + \displaystyle\sum_{j=k}^{i-1}(i+1-2j)\hat{\Phi}_j(\theta)g_{i+1-j}(\theta) + \\ \displaystyle\sum_{j=1}^{i-k}(i+1-2j)\bar{\Phi}_j(\theta)\hat{g}_{i+1-j}(\theta), \quad if \quad i \geq k-1, \\ \hat{\Phi}_i(0) = 0, \quad if \quad i \geq k-1., \end{cases}$$

Aplications: Type $\mathbf{t} = (2,3)$

Already studied by [T. Liu, F. Li, Y. Liu, S.Li, J.Wang] in Nonlinear Analysis: Real World Appl. (2019).

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^5 \end{pmatrix} + \begin{pmatrix} a_{50}x^5 + a_{22}x^2y^2 \\ b_{41}x^4y + b_{13}xy^3 \end{pmatrix} \qquad (4)$$

The Normal-Form of system (4) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^5 \end{pmatrix} + \left(\alpha_8^{(1)}x^4 + \alpha_8^{(2)}xy^2\right)\mathbf{D}_0 + \mathbf{X}_{\beta_9 x^4 y^2} + \alpha_9 x^3 y \mathbf{D}_0$$
$$+\alpha_{10}x^2y^2\mathbf{D}_0 + \alpha_{11}x^4y\mathbf{D}_0 + (\alpha_{12}^{(1)}h + \alpha_{12}^{(2)}x^3y^2)\mathbf{D}_0 \cdots,$$

where $\mathbf{D}_0 = (2x, 3y)^T$, $h = \frac{1}{4}y^4 + \frac{1}{6}x^6$.

$$\bar{\mathbf{F}}(\mathbf{x}) = \begin{pmatrix} -y^3 \\ x^5 \end{pmatrix} + \alpha_8^{(2)}xy^2\mathbf{D}_0 + \mathbf{X}_{\beta_9 x^4 y^2} + \alpha_9 x^3 y \mathbf{D}_0 + \alpha_{12}^{(2)}x^3y^2\mathbf{D}_0 \cdot$$

$$
\begin{array}{rcl}
\hat{\Phi}_1(T) & = & -2\alpha_8^{(1)} \int_0^T \mathrm{Cs}^4(s)ds, \;\; \alpha_8^{(1)} = \frac{1}{13}(5a_{50}+b_{41}), \\
\hat{\Phi}_1(T) & = & 0 \Leftrightarrow b_{41} = -5a_{50}, \\
\hat{\Phi}_2(T) & = & 0, \\
\hat{\Phi}_3(T) & = & -4\alpha_{10} \int_0^T \mathrm{Cs}^2(s)\mathrm{Sn}^2(s)ds, \\
\alpha_{10} & = & \frac{1}{35} a_{50}(2a_{22}+3b_{13})(5a_{22}-3b_{13}),
\end{array}
$$

$$
\begin{array}{rcl}
\hat{\Phi}_3(T) & = & 0 \Leftrightarrow \left\{ \begin{array}{l} a_{50} = 0 \;\; \text{then is Rx-reversible.} \\ a_{22} = -3/2b_{13} \;\; \text{then is the hamiltonian case.} \\ a_{22} = 3/5b_{13} \end{array} \right. \\
\hat{\Phi}_4(T) & = & 0, \\
\hat{\Phi}_5(T) & = & a_{22}^2 a_{50}^3 A, \;\; \text{with} \;\; A \simeq 175{,}8365.
\end{array}
$$

Aplications: Type $\mathbf{t} = (2,3)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^5 \end{pmatrix} + \begin{pmatrix} a_{50}x^5 + a_{22}x^2y^2 \\ b_{41}x^4y + b_{13}xy^3 \end{pmatrix} \tag{5}$$

Theorem

The origin of system (5) is a center if a only if one of the following conditions holds,

a) $b_{41} = a_{50} = 0$ *(Rx-reversible).*

b) $b_{41} + 5a_{50} = 2a_{22} - 3b_{13} = 0$ *(Hamiltonian case).*

Theorem

The origin of system (5) is analytically integrable if, and only if, the system is orbitally equivalent to $\dot{\mathbf{x}} = (-y^3, x^5)^T + \mathbf{X}_{\beta_9 x^4 y^2}$*, i.e., if it is verified* $b_{41} + 5a_{50} = 2a_{22} - 3b_{13} = 0$ *(Hamiltonian case).*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \begin{pmatrix} a_{80}x^8 + a_{32}x^3y^2 \\ b_{23}x^2y^3 + b_{71}x^7y \end{pmatrix}. \qquad (6)$$

The Normal-Form of system (6) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \mathbf{X}_{\beta_{14}x^8y} + \left(\alpha_{14}^{(1)}x^2y^2 + \alpha_{14}^{(2)}x^7\right)\mathbf{D}_0 + \mathbf{X}_{\beta_{15}x^6y^2} + \alpha_{15}x^5y\mathbf{D}_0 + (\alpha_{16}^{(1)}x^8 + \alpha_{16}^{(2)}x^3y^2)\mathbf{D}_0 + \mathbf{X}_{\beta_{17}x^7y^2} + \alpha_{17}x^6y\mathbf{D}_0 + \alpha_{18}x^4y^2\mathbf{D}_0 + \mathbf{X}_{\beta_{19}x^8y^2} + \alpha_{19}x^7y\mathbf{D}_0 + \cdots,$$

where $\mathbf{D}_0 = (2x, 5y)^T$, $h = \frac{1}{4}y^4 + \frac{1}{10}x^9$.

$$\bar{\mathbf{F}}(\mathbf{x}) = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \mathbf{X}_{\beta_{14}x^4y^2} + \alpha_{14}^{(2)}x^7\mathbf{D}_0 + \alpha_{15}x^5y\mathbf{D}_0 + \alpha_{16}^{(2)}x^3y^2\mathbf{D}_0 + \mathbf{X}_{\beta_{17}x^7y^2} + \mathbf{X}_{\beta_{19}x^8y^2} + \alpha_{19}x^7y\mathbf{D}_0 + \cdots,$$

$$
\begin{aligned}
\hat{\Phi}_1(T) &= -2\alpha_{14}^{(1)} \int_0^T \mathrm{Cs}^2(s)\mathrm{Sn}^2(s)ds, \;\; \alpha_{14}^{(1)} = a_{32} + b_{23}),\\
\hat{\Phi}_1(T) &= 0 \Leftrightarrow b_{23} = -a_{32},\\
\hat{\Phi}_2(T) &= 0,\\
\hat{\Phi}_3(T) &= -4\alpha_{16}^{(1)} \int_0^T \mathrm{Cs}^8(s)ds,\\
\alpha_{16}^{(1)} &= -\tfrac{3}{253} b_{23}(8a_{80} + b_{71})(3a_{80} - b_{71}),
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Phi}_3(T) &= 0 \Leftrightarrow \begin{cases} b_{23} = 0 \;\; \text{then is Rx-reversible.} \\ b_{71} = -8a_{80} \;\; \text{then is the hamiltonian case.} \\ b_{71} = 3a_{80} \end{cases}\\
\hat{\Phi}_4(T) &= 0,\\
\hat{\Phi}_5(T) &= a_{80}^2 b_{23}^3 A \;\; \text{with} \;\; A \simeq 1{,}582.
\end{aligned}
$$

Aplications: Type $\mathbf{t} = (2,5)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^9 \end{pmatrix} + \begin{pmatrix} a_{80}x^8 + a_{32}x^3y^2 \\ b_{23}x^2y^3 + b_{71}x^7y \end{pmatrix}. \tag{7}$$

Theorem

The origin of system (7) is a center if a only if one of the following conditions holds,

a) $a_{32} = b_{23} = 0$ *(Rx-reversible).*

b) $b_{23} + a_{32} = b_{71} + a_{80} = 0$ *(Hamiltonian case).*

Theorem

The origin of system (7) is analytically integrable if, and only if, the system is orbitally equivalent to $\dot{\mathbf{x}} = (-y^3, x^9)^T + \mathbf{X}_{\beta_{14}x^8y} + \mathbf{X}_{\beta_{15}x^6y^2} + \mathbf{X}_{\beta_{17}x^7y^2} + \mathbf{X}_{\beta_{19}x^8y^2}$*, i.e., if it is verified* $b_{23} + a_{32} = b_{71} + a_{80} = 0$ *(Hamiltonian case).*

Conclusions

The use of the following techniques applied successively improve the calculations.

a) Normal Form.
b) Integrating factor.
c) The origin of a k-jet of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ is a center, (decomposition $\dot{\mathbf{x}} = \bar{\mathbf{F}}(\mathbf{x}) + \mathbf{F}(\mathbf{x})$).

Thank you for your attention