Asymptotic lower bounds on Hilbert numbers using canard cycles

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Given a system of polynomial differential equations

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we ask ourselves how many limit cycles it can have, call this number H.

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The maximum number of limit cycles for vector fields up to degree N is called

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$H(2) \geq 4$ ✓ Shi Song Ling (1979)

$H(3) \geq 13$ ✓ Li, Liu, Yang (2009)

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$H(5) \geq 33$	✓	Giné (2012), Gouveia (2019)

Theorem

There exists a function $\underline{H}\colon \mathbb{N} \to \mathbb{R}^+$ *with the property*

$$\underline{H}(N) = \left(\frac{N^2 \log N}{2(\log 2)}\right)(1 + o(1)) \text{ as } N \to \infty,$$

and a sequence $(N_k)_{k\in\mathbb{N}}$*, with* $N_k \to \infty$ *as* $k \to \infty$ *and for which*

$$H(N_k) \geq \underline{H}(N_k), \qquad \text{for all} \quad k \in \mathbb{N}.$$

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Asymptotic lower bound is comparable to known bounds:

- Christopher, LLoyd (1995)
- Xiong, Han (2014)

Today: Novel approach using singular perturbations

For generalized Liénard systems:

$$\begin{cases} \dot{x} &= y - F(x), \\ \dot{y} &= G(x) \end{cases} \qquad\qquad H_{g\ell}(N) < \infty?$$

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There exists a function $\underline{H}_{g\ell} : \mathbb{N} \to \mathbb{R}^+$ *with the property*

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For ~~generalized~~ classical Liénard systems:

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$H_{c\ell}(N) < \infty$?

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De Maesschalck, Huzak (2015) For $N \geq 6$:

$$H_{c\ell}(N) \geq N - 2.$$

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$$\Downarrow \qquad y = \rho(Y)$$

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There exists a function $\underline{H}\colon \mathbb{N} \to \mathbb{R}^+$ *with the property*

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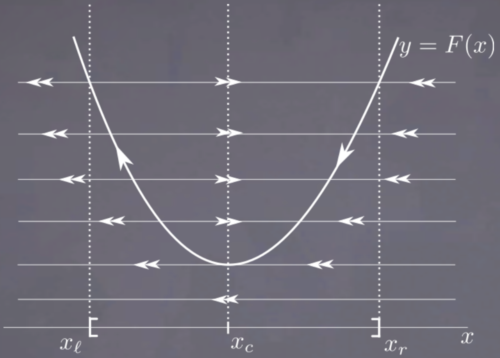
Canard nests



$$\begin{cases} \dot{x} &= y - F(x), \\ \dot{y} &= \varepsilon\, G(x), \end{cases}$$

(i) $F'(x_c) = G(x_c) = 0$,

(ii) $G'(x_c) < 0$ and $F''(x_c) \neq 0$,

(iii) $G(x)F'(x) \neq 0$, for all $x \in [x_\ell, x_r] \setminus \{x_c\}$,

(iv) $F(x_\ell) = F(x_r)$.

A canard nest is defined by (F, G, x_c, x_ℓ, x_r)

Let us consider the canard nest (F, G, x_c, x_ℓ, x_r). Assume that $F''(x_c) > 0$. For any $Y_0 \in (F(x_c), F(x_\ell))$ we consider the singular orbit Γ_{Y_0} formed by the fast part

$$\{(x, y) : y = Y_0, F(x) \leq Y_0\}$$

and the slow part

$$\{(x, y)) : y = F(x), F(x) \leq Y_0\}$$

The singular orbit Γ_{Y_0} will be called a canard cycle.

Similarly in case $F''(x_c) < 0$.

$$X_{\varepsilon,a} : \begin{cases} \dot{x} &= y - F(x), \\ \dot{y} &= \varepsilon(a + G(x)) \end{cases}$$

Theorem (Slow-fast Hopf, Dumortier, Roussarie (1996))

Let Γ_{Y_0} be a canard cycle around a slow-fast Hopf point.
Then there exists a smooth control curve $a = A(\varepsilon)$ with $A(0) = 0$ so that $X_{\epsilon,A(\varepsilon)}$ has a ε-family of periodic orbits γ_ε that tends in Hausdorff sense to Γ_{Y_0} as $\epsilon \to 0$.
Different choices of Y_0 may or may not lead to different control curves, but in any case all such control curves are exponentially close to each other.

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Questions:

1. Is the periodic orbit a limit cycle?
2. Are there multiple cycles in the nest?

Lemma (Fast relation function)

Associated to a canard nest there exists a smooth fast relation function $L : [x_c, x_r] \to [x_\ell, x_c]$ *such that* $F(L(x)) = F(x)$*. (For* λ*-families of canard nests, both* L *and its the domain may be* λ*-dependent.)*

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Next, we define the slow divergence integral as

$$I(x) = \sigma \int_{L(x)}^{x} \frac{F'(s)^2}{G(s)}\, ds, \qquad \sigma = \operatorname{sign} F''(x_c).$$

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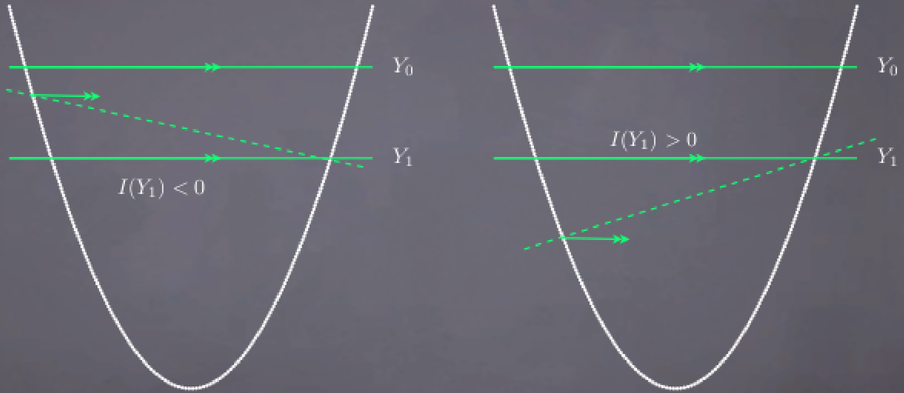
Theorem (Multiple cycles in a canard nest, Dumortier, Roussarie (2001))

Let $Y_0 = F(x_0)$ *for some* $x_0 \in (x_c, x_r)$ *and consider the canard cycle* Γ_{Y_0}. *The orbit* γ_ε *is a uniformly hyperbolic limit cycle when* $I(x_0) \neq 0$.

Furthermore, suppose $I(x_0) \neq 0$ *but I has k simple zeros* $\{x_1, \ldots, x_k\}$ *on the interval* (x_c, x_0) *then* $X_{\epsilon, A(\varepsilon)}$ *has k additional limit cycles.*



Y_0
Y_1
$I(Y_1) < 0$
Y_0
$I(Y_1) > 0$
Y_1

A canard nest for which the slow divergence integral has k simple zeros on (x_c, x_r) hence has the potential to generate $k+1$ limit cycles. We call $(k+1)$ the nest configuration. If I is identically zero (like in the case of a global center), the nest configuration is undefined.

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Lemma (Change of coordinates of canard nests)

If (F, G, x_c, x_ℓ, x_r) is a canard nest, then any smooth change of coordinates $\{x = \rho(X), y = Y + Y_0\}$ for which ρ has no singular values in $[x_\ell, x_r]$ leads to a canard nest

$$(F \circ \rho, \rho' \cdot (G \circ \rho), X_c, X_\ell, X_r)$$

where $\rho(X_c) = x_c, \rho(X_\ell) = x_\ell, \rho(X_r) = x_r$. Furthermore, the nest configuration is retained.

Lemma (Robustness of canard nests)

Let (F, G, x_c, x_ℓ, x_r) be a canard nest with configuration $(k+1)$, $k \geq 0$. For any pair of functions (F_1, G_1) defined on $[x_\ell, x_r]$ and satisfying the hypothesis (i) given above, i.e. $F_1'(x_c) = G_1(x_c) = 0$, there exists, for δ small enough, a perturbation

$$(F + \delta F_1, G + \delta G_1, x_c, x_\ell + o(1), x_r + o(1)),$$

which is a canard nest of configuration at least $(k+1)$.

Theorem

De Maesschalck, Huzak (2015) Given any $n \geq 6$, there exist a polynomial $F(x)$ of degree n and $x_\ell < 0 < x_r$ for which the canard nest $(F(x), -x, 0, x_\ell, x_r)$

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - F(x) \\ \dot{y} & = & -x \end{array} \right.$$

has configuration at least $n - 2$.

Theorem

De Maesschalck, Huzak (2014) Given any $n \geq 6$ and $m \geq 2$, there exist a polynomial $F(x)$ of degree n, a polynomial $G(x)$ of degree m and $x_\ell < 0 < x_r$ for which the canard nest $(F(x), G(x), 0, x_\ell, x_r)$

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - F(x) \\ \dot{y} & = & G(x) \end{array} \right.$$

has configuration at least $2[\frac{n-2}{2}] + [\frac{m}{2}]$.

Theorem

De Maesschalck, Dumortier (2011) *For any even degree $m \geq 2$ there exists a polynomial $G_1(x)$ of degree m so that the canard nest $(x^2, -x + \delta G_1(x), 0, -x_{max} + o(1), x_{max} + o(1))$*

$$\begin{cases} \dot{x} &= y - x^2 \\ \dot{y} &= -x + \delta G_1(x) \end{cases}$$

has configuration $(\frac{m}{2})$, for sufficiently small $\delta \neq 0$.

Theorem

De Maesschalck, Dumortier (2011) *For any odd degree $n \geq 3$ there exists a polynomial $F_1(x)$ of degree n so that the canard nest $(x^2 + \delta F_1(x), -x, 0, -x_{max} + o(1), x_{max} + o(1))$*

$$\begin{cases} \dot{x} &= y - x^2 - \delta F_1(x) \\ \dot{y} &= -x \end{cases}$$

has configuration $(\frac{n-1}{2})$, for sufficiently small $\delta \neq 0$.

Canard populations

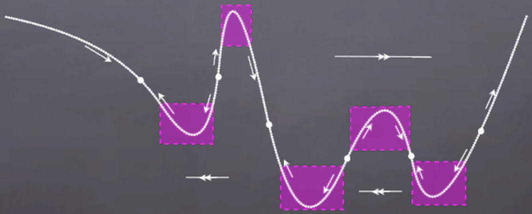
Given a smooth system

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = \varepsilon\, G(x). \end{cases}$$

It is called a canard population on $[x_\ell, x_r]$ when there is a sequence of disjoint subsets $[x_\ell^{(i)}, x_r^{(i)}]$, $i = 1, \ldots, N$ and within $x_c^{(i)}$ such that

$$(F, G, x_c^{(i)}, x_\ell^{(i)}, x_r^{(i)})$$

are canard nests. The population configuration is defined as $(k_1, \ldots, k_N)$, where k_i is the nest configuration of the i-th canard nest.



Proposition

Given a canard population (F, G) *defined on* $[x_\ell, x_r]$ *with* $x_\ell < 0 < x_r$. *Suppose there exists* $z_\ell < x_\ell$ *for which* $G(z_\ell) < 0$ *and* $F'(z_\ell) \neq 0$, *and that there exists* $z_r > x_r$ *for which* $G(z_r) > 0$ *and* $F'(z_r) \neq 0$.

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$$\begin{cases} x = \frac{1}{M}(X^2 - M^2) \\ y = Y + F(-M) \end{cases}$$

leads to a canard population $(\tilde{F}, \tilde{G})$ and an associated vector field

$$\begin{cases} \dot{X} = Y - \tilde{F}(X), \\ \dot{Y} = \varepsilon\, \tilde{G}(X) \end{cases}$$

with

$$\tilde{F}(X) = F\left(\tfrac{1}{M}(X^2 - M^2)\right) - F(-M),$$
$$\tilde{G}(X) = \tfrac{2X}{M} G\left(\tfrac{1}{M}(X^2 - M^2)\right),$$

on some interval $[X_\ell, X_r]$ with $X_\ell < 0 < X_r$.

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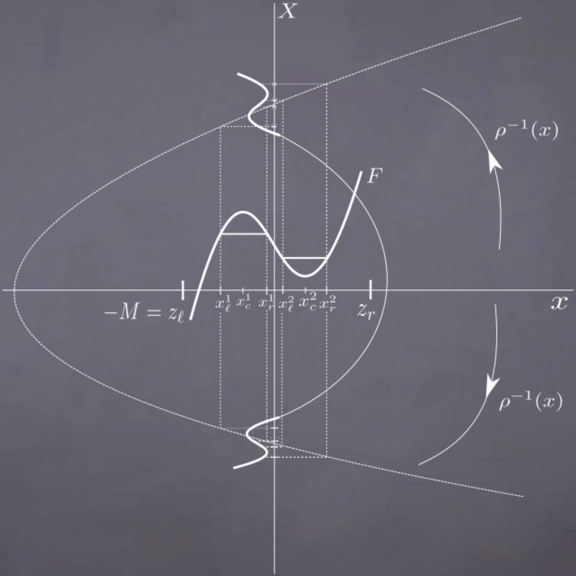
$$\begin{aligned} \tilde{F}(X) &= F\left(\tfrac{1}{M}(X^2 - M^2)\right) - F(-M), \\ \tilde{G}(X) &= \tfrac{2X}{M} G\left(\tfrac{1}{M}(X^2 - M^2)\right), \end{aligned}$$

on some interval $[X_\ell, X_r]$ *with* $X_\ell < 0 < X_r$. *The new canard population has two diffeomorphic copies of each canard nest of the original population, and has an additional canard nest near* $X = 0$.



X
$\rho^{-1}(x)$
F
$-M = z_\ell$
x_ℓ^1 x_c^1 x_r^1 x_ℓ^2 x_c^2 x_r^2
z_r
x
$\rho^{-1}(x)$

Proposition

Given a canard population in $[x_\ell, x_r]$ with a N canard nests, defined with polynomials F, G, and in each nest having a k_i nest configuration, $i = 1, \ldots, N$. Then we consider the family of vector fields

$$\left\{ \begin{array}{rcl} \dot{x} & = & y - F(x) \\ \dot{y} & = & \epsilon \left[G(x) + \sum_{j=1}^{N} a_j \prod_{j \neq i} \frac{x - x_c^{(i)}}{x_c^{(j)} - x_c^{(i)}} \right), \end{array} \right.$$

where $(a_1, \ldots, a_N)$ is close to $(0, \ldots, 0)$. There exists a curve in parameter space

$$a_1 = \mathcal{A}_1(\epsilon), \qquad , a_N = \mathcal{A}_N(\epsilon)$$

with $\mathcal{A}_1(0) = \cdots = \mathcal{A}_N(0) = 0$ and along which the above vector field realizes the limit cycle configuration $(k_1, \ldots, k_N)$ as prescribed in all nests.

Proof.
The case for $N = 1$ is just the slow-fast Hopf case. By induction we have chosen

$$a_1 = \mathcal{A}_1(\epsilon, a_k, \dots, a_N), \dots, a_{k-1} = \mathcal{A}_{k-1}(\epsilon, a_k, \dots, a_N)$$

iteratively applying the slow-fast Hopf result.

- all a_i are $O(\epsilon)$ $\implies$ slow divergence integral computations are not affected
- at each induction step process the configuration of limit cycles is the same as in the previous one, but including additional limit cycles corresponding to $a_k = \mathcal{A}_k(\epsilon)$.

$\square$

The center canard nest

In the induction process, at some point we have a canard population

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= \epsilon G(x) \end{cases}$$

with

- $G(x) = -x + O(x^3)$ is odd
- $F(x) = x^2 + O(x^4)$ is even
- Both are polynomials of some degree
- Away from the origin there are several canard nests with some canard configuration
- there is a canard nest of center type near the origin

Robustness lemma: we can perturb the center canard nest without affecting the canard configuration of existing nests!

For a given pair of integers r and s, we introduce the following definition

$$\mathcal{H}_{r,s}(F,G) = \det \begin{pmatrix} h_{s-1} & h_{s-2} & \cdots & h_{s-r} \\ h_s & h_{s-1} & \cdots & h_{s-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{s+r-2} & h_{s+r-3} & \cdots & h_{s-1} \end{pmatrix}$$

where h_k is the $2k$-th Taylor coefficient of the (even) function $H = G/F'$ with the convention that $h_k = 0$ for $k \leq -1$.

Theorem

Let $(F, G, 0, -x_{max}, x_{max})$ be a canard nest of center type (i.e. F is even and G is odd) and let m be an odd integer and $n \geq 4$ be an even integer. Suppose

$$\mathcal{H}_{\frac{n}{2}-1, \frac{m+1}{2}}(F, G) \neq 0.$$

Then there exists an odd polynomial F_1 of degree $n-1$ and an even polynomial G_1 of degree $m-1$ (with $F_1'(0) = G_1(0) = 0$), for which

$$(F(x) + \delta F_1(x), G(x) + \delta G_1(x), 0, -x_{max} + o(1), x_{max} + o(1))$$

is a canard nest with configuration at least $(\frac{m+n-3}{2})$, for sufficiently small $\delta \neq 0$.

Theorem

Let $(F, G, 0, -x_{max}, x_{max})$ be a canard nest of center type (i.e. F is even and G is odd) and let m be an odd integer and $n \geq 4$ be an even integer. Suppose

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is a canard nest with configuration at least $(\frac{m+n-3}{2})$, for sufficiently small $\delta \neq 0$.

$\longrightarrow$ We look for F_1, G_1 for which many simple zeros of

$$I_\delta(x) = \sigma \int_{L_\delta(x)}^{x} \frac{(F + \delta F_1)'(s)^2}{(G + \delta G_1)(s)}\, ds, \qquad \sigma = \text{sign } F''(0).$$

appear.

As a first step, we derive more explicitly the slow divergence integral of the perturbed vector field: we look at the integrand:

$$
\begin{aligned}
\frac{(F' + \delta F_1')^2}{G + \delta G_1} &= \frac{F'^2 + 2\delta F' F_1'}{G(1 + \delta G_1/G)} + O(\delta^2) \\
&= \frac{(F'^2 + 2\delta F' F_1')(1 - \delta G_1/G)}{G} + O(\delta^2) \\
&= \frac{F'^2}{G} + \delta \frac{2F' F_1' G - F'^2 G_1}{G^2} + O(\delta^2).
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Next we use the equation $F(x)+\delta F_1(x) = F(L_\delta(x)) + \delta F_1(L_\delta(x))$ together with $L_0(x) = -x$ to derive that

$$L_\delta(x) = -x - 2\delta \frac{F_1(x)}{F'(x)} + O(\delta^2).$$

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$$
L_\delta(x) = -x - 2\delta \frac{F_1(x)}{F'(x)} + O(\delta^2).
$$

Hence

$$
\begin{aligned}
I_\delta(x) &= \sigma \int_{-x-2\delta \frac{F_1(x)}{F'(x)}}^{x} \left(\frac{F'(s)^2}{G(s)} + \delta \frac{2F'(s)F_1'(s)G(s) - F'(s)^2 G_1(s)}{G(s)^2} \right) ds + O(\delta^2) \\
&= \sigma\delta \left(\int_{-x}^{x} \frac{2F'(s)F_1'(s)G(s) - F'(s)^2 G_1(s)}{G(s)^2} ds - 2\frac{F_1(x)}{F'(x)} \frac{F'(x)^2}{G(x)} \right) + O(\delta^2).
\end{aligned}
$$

Let us thus define

$$I_1(x) = \int_{-x}^{x} \frac{2F'(s)F_1'(s)G(s) - F'(s)^2 G_1(s)}{G(s)^2} ds - 2\frac{F'(x)F_1(x)}{G(x)}.$$

Hence, simple zeros of I_1 in the open set $(0, x_{max})$ will perturb, for $\delta \neq 0$ small enough to simple zeros of I_δ. This reduces our focus to the study of the zeros of I_1.

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Introducing $H = G/F'$ we rewrite it as

$$\frac{1}{2} I_1'(x)H(x)^2 = F_1'(x)H(x) + H'(x)F_1(x) - G_1(x).$$

Keep in mind that H is smooth (we smoothly extend the domain in the origin), $H(0) \neq 0$, and both H and I_1' are even, so the green equation only contains even terms, starting with degree 2.

$$\frac{1}{2}I_1'(x)H(x)^2 = F_1'(x)H(x) + H'(x)F_1(x) - G_1(x).$$

Now, our claim is the following: for each given

$$P_\lambda(x) = \sum_{k=1}^{\frac{n+m-3}{2}} \lambda_k x^{2k+1},$$

where $\lambda = (\lambda_1, \ldots, \lambda_{\frac{n+m-3}{2}})$, there exists a F_1 and G_1 satisfying the hypotheses of the theorem and there exists an analytic function $R_\lambda(x)$ such that the green equation is satisfied with $I_1(x)$ of the form

$$I_1(x) = P_\lambda(x) + R_\lambda(x)x^{n+m}.$$

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$$I_1(x) = P_\lambda(x) + R_\lambda(x)x^{n+m}.$$

$$\frac{1}{2}\left[P_\lambda(x) + R_\lambda(x)x^{n+m}\right]' H(x)^2 = F_1'(x)H(x) + H'(x)F_1(x) - G_1(x).$$

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$$\frac{1}{2}\left[P_\lambda(x)+R_\lambda(x)x^{n+m}\right]' H(x)^2 = F_1'(x)H(x)+H'(x)F_1(x)-G_1(x).$$

Is equivalent to:

$$\begin{cases} G_1 = j_{m-1}\left[F_1'H + H'F_1 - \frac{1}{2}P_\lambda' H^2\right], \\ S^{m+1}\left[\frac{1}{2}(P_\lambda' + (R_\lambda . x^{n+m})')H^2\right] = S^{m+1}[F_1'H + H'F_1]. \end{cases}$$

j_{m-1}: take the $m-1$ jet around the origin
S^{m+1}: shift series

$$\frac{1}{2}\left[P_\lambda(x)+R_\lambda(x)x^{n+m}\right]' H(x)^2 = F_1'(x)H(x)+H'(x)F_1(x)-G_1(x).$$

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$$\begin{cases} G_1 = j_{m-1}\left[F_1'H + H'F_1 - \frac{1}{2}P_\lambda' H^2\right], \\ j_{n-4}\left[S^{m+1}\left[\frac{1}{2}P_\lambda' H^2\right]\right] = j_{n-4}\left[S^{m+1}[F_1'H + H'F_1]\right]. \\ S^{n+m-1}\left[\frac{1}{2}P_\lambda' H^2 + (R_\lambda.x^{n+m})')H^2\right] = S^{n+m-1}[F_1'H + H'F_1]. \end{cases}$$

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remains to solve the yellow equation w.r.t. F_1.

$$H(x) = \sum_{k=0}^{\frac{m+n-1}{2}} h_k x^{2k} + O(x^{m+n+1}), \qquad F_1(x) = \sum_{\ell=1}^{\frac{n}{2}-1} f_\ell x^{2\ell+1},$$

and write the left-hand side of the yellow equation as a polynomial

$$W(x) = \sum_{i=0}^{\frac{n}{2}-2} w_i x^{2i}.$$

Lemma

Let $r = (n/2) - 1$ and $s = (m+1)/2$. The yellow equation is a linear system

$$\begin{pmatrix} h_{s-1} & h_{s-2} & \cdots & h_{s-r} \\ h_s & h_{s-1} & \cdots & h_{s-r+1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{s+r-2} & h_{s+r-3} & \cdots & h_{s-1} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = \begin{pmatrix} w_0/(2s+1) \\ w_1/(2s+3) \\ \vdots \\ w_{r-1}/(2s+2r-1) \end{pmatrix}$$

Let us now finish the proof of the theorem. We have shown that, given any P_λ, there exist choices of (F_1, G_1) so that the orange equation is satisfied:

$$I_1(x) = P_\lambda(x) + R_\lambda(x)x^{n+m}.$$

Hence, we rewrite it as

$$I_1(x) = x^3 \bar{I}_1(x^2), \qquad \bar{I}_1(t) = \bar{P}_\lambda(t) + \bar{R}_\lambda(t)t^{\frac{n+m-3}{2}}$$

where

$$\bar{P}_\lambda(t) = \sum_{k=0}^{\frac{n+m-3}{2}-1} \lambda_{k+1} t^k = \sum_{k=0}^{r+s-2} \lambda_{k+1} t^k,$$

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$$\bar{P}_\lambda(t) = \sum_{k=0}^{\frac{n+m-3}{2}-1} \lambda_{k+1}t^k = \sum_{k=0}^{r+s-2} \lambda_{k+1}t^k,$$

$\implies$ zeros of slow divergence integral follow from catastrophe theory: I_1 can have $r+s-2$ simple zeros on $\{t > 0\}$.

Theorem

Let $(F, G, 0, -x_{max}, x_{max})$ be a canard nest of center type (i.e. F is even and G is odd) and let m be an odd integer and $n \geq 4$ be an even integer. Suppose

$$\mathcal{H}_{\frac{n}{2}-1,\frac{m+1}{2}}(F, G) \neq 0.$$

Then there exists an odd polynomial F_1 of degree $n-1$ and an even polynomial G_1 of degree $m-1$ (with $F_1'(0) = G_1(0) = 0$), for which

$$(F(x) + \delta F_1(x), G(x) + \delta G_1(x), 0, -x_{max} + o(1), x_{max} + o(1))$$

is a canard nest with configuration at least $(\frac{m+n-3}{2})$, for sufficiently small $\delta \neq 0$.

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Theorem

Let $(F,$ …

even an …

even int …

Lemma (Robustness of canard nests)

Let (F, G, x_c, x_ℓ, x_r) *be a canard nest with configuration* $(k+1)$, $k \geq 0$. *For any pair of functions* (F_1, G_1) *defined on* $[x_\ell, x_r]$ *and satisfying the hypothesis (i) given above, i.e.* $F_1'(x_c) = G_1(x_c) = 0$, *there exists, for* δ *small enough, a perturbation*

$$(F + \delta F_1, G + \delta G_1, x_c, x_\ell + o(1), x_r + o(1)),$$

which is a canard nest of configuration at least $(k+1)$.

$\Uparrow$

Then th …

polynon …

$(F(x)$ …

is a canard nest with configurat… *t least* $(\frac{m+n-3}{2})$, *for sufficiently small* $\delta \neq 0$. *If additional constraints on* F_1' *and* G_1 *are imposed in the form that* $F_1'(x_j) = G_1(x_j) = 0$ *at several distinct (nonzero) points* x_j, $j = 1, \ldots, J$, *then the results still hold with a reduced configuration of at least* $(\frac{m+n-3}{2}) - 2J$.

Let us now discuss the case $J \geq 1$:

- $F_1'(x_j)$ depend linearly on coefficients of F_1
- coefficients depend linearly on LHS of yellow equation
- LHS depends linearly on λ
- same thing for $G_1(x_j)$.

$\Rightarrow$ we have $2J$ linear constraints on the parameter space $(\lambda_1, \ldots, \lambda_{r+s-1})$.

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$\Rightarrow$ we have $2J$ linear constraints on the parameter space $(\lambda_1, \ldots, \lambda_{r+s-1})$.
Elementary catastrophe under linear constraints

Lemma
Let $f(t,\mu) = \sum_{k=0}^{K-1} \mu_k t^k + O(t^K)$ be smooth in (t,μ) near $(0,0)$, linear w.r.t. μ, and let M be a linear subspace of $\mathbb{R}^K$ of codimension L. Then, inside any open neighborhood W of $\mu = 0$ and T of $t = 0$, there exist choices of $\mu \in M \cap W$ for which $f(t,\mu)$ has $K - L - 1$ simple zeros on $\{t > 0\} \cap T$.

Theorem

Let $(F, G, 0, -x_{max}, x_{max})$ *be a canard nest of center type (i.e.* F *is even and* G *is odd) and let* m *be an odd integer and* $n \geq 4$ *be an even integer. Suppose*

$$\mathcal{H}_{\frac{n}{2}-1,\frac{m+1}{2}}(F, G) \neq 0.$$

Then there exists an odd polynomial F_1 *of degree* $n-1$ *and an even polynomial* G_1 *of degree* $m-1$ *(with* $F_1'(0) = G_1(0) = 0$*), for which*

$$(F(x) + \delta F_1(x), G(x) + \delta G_1(x), 0, -x_{max} + o(1), x_{max} + o(1))$$

is a canard nest with configuration at least $(\frac{m+n-3}{2})$*, for sufficiently small* $\delta \neq 0$*. If additional constraints on* F_1' *and* G_1 *are imposed in the form that* $F_1'(x_j) = G_1(x_j) = 0$ *. . .*

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Then there exists an odd polynomial F_1 of degree $n-1$ and an even polynomial G_1 of degree $m-1$ (with $F_1'(0) = G_1(0) = 0$), for which

$$(F(x) + \delta F_1(x), G(x) + \delta G_1(x), 0, -x_{max} + o(1), x_{max} + o(1))$$

is a canard nest with configuration at least $(\frac{m+n-3}{2})$, for sufficiently small $\delta \neq 0$. If additional constraints on F_1' and G_1 are imposed in the form that $F_1'(x_j) = G_1(x_j) = 0 \ldots$

Lemma

Let (F, G) be as above and such that $\mathcal{H}_{r,s}(F, G) = 0$. Then

$$\mathcal{H}_{r,s}(F, G + \mu x^{2s-1}(1 + O(x))) \neq 0, \qquad \forall \mu \neq 0 \text{ small enough.}$$

The induction process

Consider

$$\left\{ \begin{array}{lcl} \dot{x} & = & y - F_i(x) \\ \dot{y} & = & \epsilon G_i(x) \end{array} \right.$$

with

$$F_0(x) = x^2 + a_3 x^3 + \cdots + a_{n_0} x^{n_0}, G_0(x) = -x$$

Perturb to

$$\bar{F}_0 = F_0, \qquad \bar{G}_0(x) = -x + \delta x^{n_0 - 1}$$

Repeat in iteration:

- Do the doubling step
- Perturb to obtain condition $\mathcal{H}$ if necessary.
- Create extra cycles in the central nest

Let $n_i = \deg F_i$, $m_i = \deg G_i$, $c_i = \#$ nests, $k_i =$ total canard configuration.
As a consequence, we examine the following system of recursions:

$$\begin{cases} n_{i+1} &= 2n_i, \\ m_{i+1} &= 2m_i + 1, \\ c_{i+1} &= 2c_i + 1, \\ k_{i+1} &= 2k_i + (n_i + m_i - 5c_i - 1), \end{cases}$$

with n_0 arbitrary, $m_0 = n_0 - 1$, $c_0 = 1$, $k_0 = n_0 - 2$. It is easily solved:

$$\begin{cases} n_i &= n_0 2^i, \\ m_i &= n_i - 1, \\ c_i &= 2^{i+1} - 1, \\ k_i &= (1+i)n_i - \frac{5i-1}{2} c_i - \frac{5i+5}{2}. \end{cases}$$

In view of obtaining the result on Generalized Liénards, let $N = n_i = n_0 2^i$. Then $c_i = 2\frac{N}{n_0} - 1$, which makes that

$$k_i = (1+i)N - (5i-1)\left(\frac{N}{n_0} - \frac{1}{2}\right) - \frac{5i+5}{2}$$
$$= iN\frac{n_0 - 5}{n_0} + N\frac{n_0 + 1}{n_0} - 3.$$

We define N_i as the outcome of the sequence $(n_i)_i$ at step i, with $n_0 = i$, i.e. $N_i = i2^i$. Then

$$k_i = iN_i\frac{i-5}{i} + N_i\frac{i+1}{i} - 3 = iN_i(1+o(1)), \qquad i \to \infty.$$

Noting that $\log N_i = \log(i2^i) = \log i + i\log 2 = (i\log 2)(1+o(1))$, we obtain

$$k_i = \frac{N_i \log N_i}{\log 2}(1+o(1)), \qquad i \to \infty.$$

This way we obtain the result on generalized Liénards

Theorem

There exists a function $\underline{H}_{g\ell} \colon \mathbb{N} \to \mathbb{R}^+$ *with the property*

$$\underline{H}_{g\ell}(N) = \left(\frac{N \log N}{\log 2}\right)(1 + o(1)) \text{ as } N \to \infty,$$

and a sequence $(N_k)_{k \in \mathbb{N}}$, *with* $N_k \to \infty$ *as* $k \to \infty$ *and for which*

$$H_{g\ell}(N_k) \geq \underline{H}_{g\ell}(N_k), \qquad \text{for all} \quad k \in \mathbb{N}.$$

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Theorem

There exists a function $\underline{H} \colon \mathbb{N} \to \mathbb{R}^+$ *with the property*

$$\underline{H}(N) = \left(\frac{N^2 \log N}{2(\log 2)} \right) (1 + o(1)) \text{ as } N \to \infty,$$

and a sequence $(N_k)_{k \in \mathbb{N}}$, *with* $N_k \to \infty$ *as* $k \to \infty$ *and for which*

$$H(N_k) \geq \underline{H}(N_k), \qquad \text{for all} \quad k \in \mathbb{N}.$$

We start with a canard population

$$\begin{cases} \dot{x} & = & y - F(x) \\ \dot{y} & = & \epsilon G(x) \end{cases}$$

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conclude: "big almost constant" perturbations preserve canard configuration

$$\left\{\begin{array}{rcl}\dot{x} & = & y-F(x)\\ \dot{y} & = & \epsilon\Delta(y)G(x)\end{array}\right.$$

where $\Delta(0)>0$ and $\sup_{|y|\leq 1}|\Delta(y)-\Delta(0)|<\delta$

We consider a new system

$$\left\{ \begin{array}{rcl} \dot{x} & = & \cancel{\rho'(Y)}(\rho(Y) - F(x)) \\ \dot{Y} & = & \epsilon G(x) \end{array} \right.$$

where ρ is a polynomial of degree N that is yet to be constructed. We will ensure though that $Y \mapsto y = \rho(Y)$ covers the interval $[-1, 1]$ N times.

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Let $y \mapsto Y = \alpha_i(y)$ be the inverses, $i = 1, \ldots, N$. Transforming the orange system via the coordinate change $Y = \alpha_i(y)$ gives

$$\begin{cases} \dot{x} &= y - F(x) \\ \dot{y} &= \epsilon\rho'(\alpha_i(y))G(x) \end{cases}$$

If we ensure that $\sup_{|y|\leq 1} |\rho'(\alpha_i(y)) - \rho'(\alpha_i(0))| < \delta$ and $\rho'(\alpha_i(0)) > 0$, for each value of i, then the orange system has N copies of canard populations in the yellow system, each with configuration k.

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$\rightarrow$ only possible for half of the indexes i.

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Define

$$\rho(Y) = \frac{1}{b}\rho_0(\delta bY/M).$$

Clearly ρ has N simple roots $Y_i \frac{M}{b\delta}$, and it covers the interval $[-1, 1]$ in any root-centered interval with radius $\frac{Ma}{b\delta}$. Finally

$$\rho'(Y) - \rho'(Y_i \frac{M}{b\delta}) = \frac{\delta}{M}(\rho_0'(\delta bY/M) - \rho_0'(Y_i)),$$

so it is bounded in absolute value by δ.

We return to

$$\begin{cases} \dot{x} &= \rho(Y) - F(x) \\ \dot{Y} &= \epsilon G(x) \end{cases}$$

The number of canard nests is $N/2$ times the number of canard nests of the original Liénard system. We then proceed to consider

$$\begin{cases} \dot{x} &= \rho(y) - F(x) \\ \dot{y} &= \epsilon \left[G(x) + P(x, y) \right), \end{cases}$$

with

$$P(x, y) = \sum_{k,\ell} p_{k\ell}(a_{ij}) x^k y^\ell$$

We use two-dimensional interpolation theory on rectangular grids and proceed by induction to find control curves $a = a_{ij}(\epsilon)$ along which the canard configuration is realized.

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$$\Longrightarrow \frac{N}{2} \times O\left(\frac{N \log N}{\log 2}\right) \text{ hyperbolic limit cycles.} \qquad \square$$

Thank you for the attention.