Invariant manifolds of parabolic points with nilpotent part

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Abstract

Let F be a planar analytic map having a parabolic fixed point with nilpotent part. Given the normal form of F, we provide an algorithm to compute an approximation up to any order of a stable curve associated with this point. Then, we prove the existence of such a curve as an *a posteriori* result using the parameterization method for invariant manifolds. Concretely, we show that the approximation obtained from the algorithm converges to a parameterization of the invariant curve and we provide the analyticity of the curve in an open set that does not contain the fixed point.

Introduction

Let $U \subseteq \mathbb{R}^2$ be an open set, $0 \in U$, and let $F : U \to \mathbb{R}^2$ be an analytic map with $F(0) = 0$ and

$$DF(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The origin is a fixed point of F of parabolic type and with nilpotent part. This class of maps appear as Poincaré maps at infinity in some problems of celestial mechanics [2].

Via a polynomial change of variables, F can be written in the normal form

$$F(x, y) = \begin{pmatrix} x + y \\ y + x^k(a_k + \cdots + a_r x^{r-k}) + x^{l-1}y(b_l + \cdots + b_r x^{r-l}) \end{pmatrix} + \begin{pmatrix} O(\|(x, y)\|^{r+1}) \\ O(\|(x, y)\|^{r+1}) \end{pmatrix}, \qquad (1)$$

where $a_k \neq 0$, $b_l \neq 0$; $2 \leq k \leq r$, $2 \leq l \leq r$.

We study the existence and properties of invariant manifolds of F associated with the fixed point at the origin.

We look for a parameterization $K(t)$ of an invariant curve of F and for a one-dimensional map $R(t)$ representing the dynamics of F restricted to the invariant curve. The maps K and R must satisfy the equation

$$F \circ K - K \circ R = 0. \qquad (2)$$

We shall consider tree cases [3] in the normal form (1),

- Case 1, $k < 2l - 1$,
- Case 2, $k = 2l - 1$,
- Case 3, $k > 2l - 1$.

Stable and unstable manifolds

Figures 1, 2 and 3 show the stable and unstable manifolds of the fixed point at the origin for each case. The unstable manifolds are obtained as stable manifolds of F^{-1}.

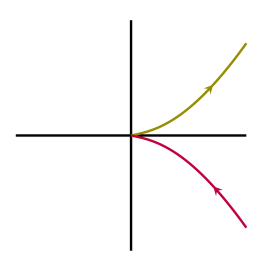

Figure 1: Case 1, with $a_k > 0$.

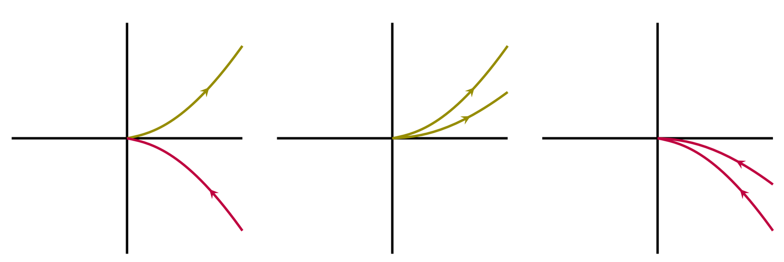

Figure 2: Case 2. Left: $a_k > 0$, center: $a_k < 0$, $b_l > 0$, right: $a_k < 0$, $b_l < 0$.

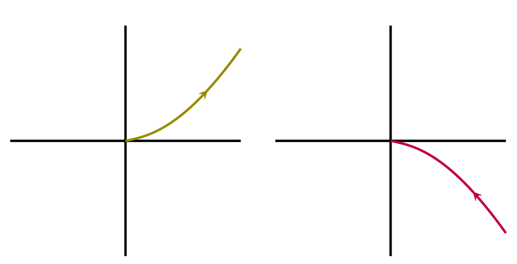

Figure 3: Case 3. Left: $b_l > 0$, right: $b_l < 0$.

The parameterization method

The parameterization method [1] is a theoretical frame to study the existence and properties of several types of invariant manifolds and to provide numerical algorithms to compute approximations of those manifolds.

The approach of the parameterization method is to consider (2) as a functional equation defined by an operator, namely,

$$\mathcal{T}(F, K, R) := F \circ K - K \circ R = 0, \qquad (3)$$

and to study the operator $\mathcal{T}$ in a suitable function space.

The algorithm

The following statements show that one can obtain a polynomial $\tilde{K}$ and a map R that approximate equation (3) up to any order.

Proposition 1 (Case 1)

Let F be as in (1) and assume $k < 2l - 1$ and $a_k > 0$. Then, for all $n \geq 2$, there exists a polynomial $\tilde{K}(t)$ of the form

$$\tilde{K}(t) = \begin{pmatrix} t^2 + \cdots + K_n^x t^n \\ K_{k+1}^y t^{k+1} + \cdots + K_{n+k-1}^y t^{n+k-1} \end{pmatrix},$$

and a map R of the form $R(t) = t + R_k t^k + R_{2k-1} t^{2k-1}$ such that

$$F(\tilde{K}(t)) - \tilde{K}(R(t)) = (O(t^{n+k}),\, O(t^{n+2k-1})).$$

Proposition 2 (Cases 2, 3)

Let F be as in (1) and assume $k \geq 2l - 1$. Then, for all $n \geq 1$, there exists a polynomial $\tilde{K}(t)$ of the form

$$\tilde{K}(t) = \begin{pmatrix} t + \cdots + K_n^x t^n \\ K_l^y t^l + \cdots + K_{n+l-1}^y t^{n+l-1} \end{pmatrix},$$

and a map R of the form $R(t) = t + R_l t^l + R_{2l-1} t^{2l-1}$ such that

$$F(\tilde{K}(t)) - \tilde{K}(R(t)) = (O(t^{n+l}),\, O(t^{n+2l-1})).$$

The proof of Propositions 1 and 2 provide a recursive algorithm to obtain expressions for the coefficients of $\tilde{K}(t)$ and $R(t)$.
For the first coefficients we obtain

- Case 1: $K_{k+1}^y = 2\, R_k = \pm 2 \sqrt{\frac{a_k}{2(k+1)}}$,
- Case 2: $K_l^y = R_l = \frac{b_l \pm \sqrt{b_l^2 + 4 a_k l}}{2l}$,
- Case 3: $K_l^y = R_l = \frac{b_l}{l}$.

The analytical setting

We use here the notation corresponding to case 1 ($k < 2l - 1$), but the setting is analogous for cases 2 and 3.

If $\tilde{K}$ and R are as in Proposition 1, with $R_k < 0$ and $\tilde{K}$ being a polynomial of degree $(n,\, n+k-1)$, to study (3) we may look for a solution Δ of

$$F \circ (\tilde{K} + \Delta) - (\tilde{K} + \Delta) \circ R = 0. \qquad (4)$$

Fixed $\tilde{K}$, the existence of such a solution will provide a stable invariant curve $K = \tilde{K} + \Delta$ of F.

We consider the family of Banach spaces

$$\mathcal{X}_n := \{f : S \to \mathbb{C} \;:\; f \in \mathrm{Hol}(S),\ \|f\|_n = \sup_{z \in S} \frac{|f(z)|}{|z|^n} < \infty\}, \quad n \in \mathbb{N},$$

where S is a small sector in the complex plane, and we define the operators

$$\begin{aligned} \mathcal{S}_n : \mathcal{X}_n &\to \mathcal{X}_n \\ f &\mapsto f - f \circ R, \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{n+1,\, n+k} : \mathcal{X}_{n+1} \times \mathcal{X}_{n+k} &\to \mathcal{X}_{n+k} \times \mathcal{X}_{n+2k-1} \\ \Delta &\mapsto F \circ (\tilde{K} + \Delta) - F \circ \tilde{K} - (DF \circ \tilde{K})\, \Delta, \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{n+1,\, n+k} : \mathcal{X}_{n+1} \times \mathcal{X}_{n+k} &\to \mathcal{X}_{n+k} \times \mathcal{X}_{n+2k-1} \\ \Delta &\mapsto (DF \circ \tilde{K} - I)\, \Delta. \end{aligned}$$

Then (4) can be written as

$$E + \mathcal{L}_{n+1,\, n+k} \Delta + \mathcal{J}_{n+1,\, n+k} \Delta + \mathcal{S}_{n+1,\, n+k} \Delta = 0, \quad \Delta \in B_{n+1,\, n+k}^r, \qquad (5)$$

where $E := F \circ \tilde{K} - \tilde{K} \circ R = O(t^{n+k},\, t^{n+2k-1})$, and $B_{n+1,\, n+k}^r \subseteq \mathcal{X}_{n+1} \times \mathcal{X}_{n+k}$ is a closed ball centered at 0 and of radius r.

The following lemmas are the key tool to obtain the existence of a solution of (5).

Lemma 1. Given $n \in \mathbb{N}$, the operator $\mathcal{S}_n : \mathcal{X}_n \to \mathcal{X}_n$ has a bounded right inverse, $\mathcal{S}_n^{-1} : \mathcal{X}_n \to \mathcal{X}_{n-k+1}$, and $\|\mathcal{S}_n^{-1}\| < (\rho^{k-1} + \frac{1}{\nu} \frac{k-1}{n-k+1})$, where ρ and ν are constants and ρ can be chosen to be arbitrarily small.

Lemma 2. The family of linear operators $\{\mathcal{L}_{n+1,\, n+k}\}_{n \geq 1}$ is uniformly bounded.

Lemma 3. The family of nonlinear operators $\{\mathcal{J}_{n+1,\, n+k}\}_{n \geq 1}$ is uniformly Lipschitz.

Existence of the stable manifold

By Lemmas 1, 2 and 3, equation (5) can be written as

$$\Delta = \Phi_{n+1,\, n+k} \Delta, \quad \Delta \in B_{n+1,\, n+k}^r, \qquad (6)$$

where

$$\Phi_{n+1,\, n+k} \Delta := -\mathcal{S}_{n+k,\, n+2k-1}^{-1} (E + \mathcal{L}_{n+1,\, n+k} \Delta + \mathcal{J}_{n+1,\, n+k} \Delta).$$

It follows that

- $\Phi_{n+1,\, n+k} : B_{n+1,\, n+k}^r \to B_{n+1,\, n+k}^r$ is a contraction mapping provided that n is sufficiently large.
- The Banach Fixed Point Theorem provides the existence of a unique solution $\Delta \in B_{n+1,\, n+k}^r$ of equation (6).
- There exists a solution $K = \tilde{K} + \Delta$ of equation (4), which defines an analytic stable curve of F associated with the fixed point at the origin.
- We obtain the existence of the invariant curve K as an *a posteriori* result. The polynomial $\tilde{K}$ given by the algorithm approximates a curve which is indeed an invariant manifold of F.

Theorem 1 (Case 1)

Let $F : U \to \mathbb{R}^2$ be an analytic map as given in (1), with $k < 2l - 1$ and $a_k > 0$. Then, there exists $\rho > 0$ and an analytic function $K : (0, \rho) \to \mathbb{R}^2$, and a map of the form $R(t) = t + R_k t^k + R_{2k-1} t^{2k-1}$, $R_k < 0$, such that $F(K(t)) = K(R(t))$, $t \in (0, \rho)$.

Moreover, if $\tilde{K} : \mathbb{R} \to \mathbb{R}^2$ is a polynomial of degree $(n, n+k-1)$, with $n > k$, such that $F(\tilde{K}(t)) - \tilde{K}(R(t)) = (O(t^{n+k}),\, O(t^{n+2k-1}))$, then there exists $\rho > 0$ and a unique analytic function $K : (0, \rho) \to \mathbb{R}^2$ such that $F(K(t)) = K(R(t))$, and $K(t) - \tilde{K}(t) = (O(t^{n+1}),\, O(t^{n+k}))$.

Theorem 2 (Cases 2, 3)

Let $F : U \to \mathbb{R}^2$ be an analytic map as given in (1), with $k \geq 2l - 1$.

For case 2, assume $a_k > 0$ or $b_l > 0$.
For case 3, assume $b_l < 0$.

Then, there exists $\rho > 0$ and an analytic function $K : (0, \rho) \to \mathbb{R}^2$, and a map of the form $R(t) = t + R_l t^l + R_{2l-1} t^{2l-1}$, $R_l < 0$, such that $F(K(t)) = K(R(t))$, $t \in (0, \rho)$.

Moreover, if $\tilde{K} : \mathbb{R} \to \mathbb{R}^2$ is a polynomial of degree $(n, n+l-1)$, with $n > l - 1$, such that $F(\tilde{K}(t)) - \tilde{K}(R(t)) = (O(t^{n+l}),\, O(t^{n+2l-1}))$, then there exists $\rho > 0$ and a unique analytic function $K : (0, \rho) \to \mathbb{R}^2$ such that $F(K(t)) = K(R(t))$, and $K(t) - \tilde{K}(t) = (O(t^{n+1}),\, O(t^{n+l}))$.

- Note that in general we can not expect the curve K to be analytic at the origin. Explicit counterexamples are given in [3].
- For other signs of the parameters a_k and b_k, we may obtain invariant manifolds in other regions of the phase space, depending on whether k and l are even or odd.

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