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Bifurcation of One Codimension and Structural Stability in Piecewise Smooth Vector Fields via Regularization.

Yagor Romano Carvalho [1],
Claudio Aguinaldo Buzzi [2],
IBILCE/UNESP - São José do Rio Preto, SP - Brazil.

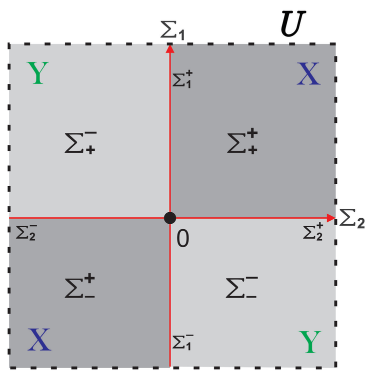
Introduction

This work is part of the text of my Phd thesis, with Prof. Dr. Claudio Aguinaldo Buzzi as advisor and financed by FAPESP. Here we are considering discontinuous vector fields in the plane, with the " cross " as a set of discontinuities. We are interested in studying the regularizations of bifurcations of zero and one codimension, according to an equivalence relation, from the normal forms described in [2] .

Results and Discussions

Let's consider discontinuous fields in the plane $Z = (X, Y)$, given by $Z(x, y) = X(x, y)$ if $xy \geq 0$ and $Z(x, y) = Y(x, y)$ if $xy \leq 0$, where $X, Y : V \subset \mathbb{R}^2 \to \mathbb{R}^2$ and V is a neighborhood of the origin. We are considering the following set of discontinuities, the " cross", $\Sigma = \{(x, y) \in \mathbb{R}^2 \,;\, h(p) = xy = 0\}$, in which $p = (x, y) \in V$ and $h : V \to \mathbb{R}$ such that $h(p) = h(x, y) = xy$. The set of all these fields will be denoted by χ^D. We denote by χ^S the set of all smooth vector fields $X : V \subset \mathbb{R}^2 \to \mathbb{R}^2$ with the C^r topology, and thus we will consider χ^D with product topology. The C^r topology is such that two C^r vector fields , Z_1 e Z_2 are close, if the fields and their derivatives until r order are close in a neighborhood of the origin.

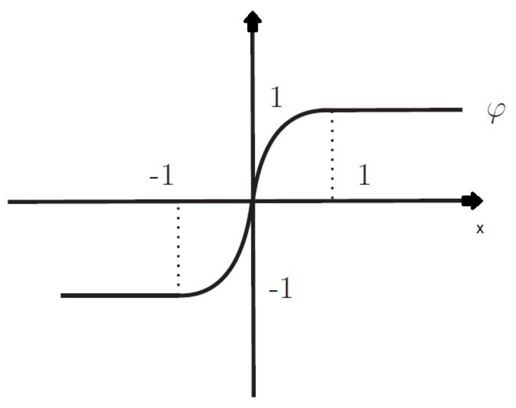
The algebraic manifold $\Sigma = h^{-1}(0)$ divides the open U in the regions $\Sigma^+ = \{p \in U \,;\, h(p) > 0\}$ and $\Sigma^- = \{p \in U \,;\, h(p) < 0\}$. Moreover, we can decompose $\Sigma^\pm$ in $\Sigma^\pm_+ = \Sigma^\pm \cap \{y > 0\}$ and $\Sigma^\pm_- = \Sigma^\pm \cap \{y < 0\}$ which can be seen in the following figure:



Definition 1 *Consider Z, $\widehat{Z} \in \chi^D$, defined in U_0 and $\widehat{U_0}$ neighborhoods of the origin, with discontinuity sets Σ and $\widehat{\Sigma}$, respectively. We say that Z and $\widehat{Z}$ are locally Σ-equivalent if there is a neighborhood V_0 of the origin and a homeomorphism $\sigma : V_0 \to \widehat{V_0}$ which takes the trajectories of Z in trajectories of $\widehat{Z}$, preserving the orientation, and takes the Σ discontinuity set in the $\widehat{\Sigma}$ discontinuity set.*

Definition 2 *We say that $Z \in \chi^D$ is locally Σ-structurally stable at the origin if there is a neighborhood $U_Z \subset \chi^D$ such that if $\widehat{Z} \in U_Z$ then $\widehat{Z}$ is locally Σ-equivalent to Z around the origin.*

Definition 3 *A class $\mathcal{C}^\infty$ function $\varphi : \mathbb{R} \to \mathbb{R}$ is a transition function if $\varphi(x) = -1$ for $x \leq -1$, $\varphi(x) = 1$ for $x \geq 1$ and $\varphi'(x) > 0$ for $x \in (-1, 1)$.*



A regularization is the approximation of a discontinuous vector field by a continuous vector field. In this work we will consider the double regularization of the "cross" described in [1], which for $Z \in \chi^D$ and $\eta,\ \varepsilon > 0$ will be given by

$$Z^R_{\varepsilon,\eta}(x, y) = \frac{1}{2}\left[\left(1 + \phi\left(\frac{x}{\varepsilon}\right)\psi\left(\frac{y}{\eta}\right)\right)X + \left(1 - \phi\left(\frac{x}{\varepsilon}\right)\psi\left(\frac{y}{\eta}\right)\right)Y\right](x, y), \qquad (1)$$

in which ψ and ϕ are transition functions.

In [2] were classified the fields Σ-structurally stable inχ^D by the following theorem:

Theorem 1 *(Σ-structural stability in χ^D). Let $Z \in \chi^D$, then Z is locally Σ-structurally stable at the origin if, and only if, $Z \in \Sigma^0$. That is, Z satisfies one of the following conditions:*

(A) $X_i(0).Y_i(0) > 0$ for $i = 1, 2$,

(B) $X_i(0).Y_i(0) < 0$ for $i = 1, 2$ and $detZ(0) \neq 0$,

(C) $X_i(0).Y_i(0) > 0$, $X_j(0).Y_j(0) < 0$ for $i, j = 1, 2$, $i \neq j$. Moreover, when Z is transient satisfies $\alpha_Z \neq -1$, where $\alpha_Z = \frac{X_1(0)\cdot Y_2(0)}{X_2(0)\cdot Y_1(0)}$.

Moreover, the subset Σ^0 is open and dense in χ^D, so the local Σ-structural stability is a generic property.

The normal forms of (A), (B) and (C) in the Theorem 1 have been listed in [2], and the normal form of a field means that it is locally Σ-equivalent around the origin to its respective normal form.

Theorem 2 *If Z is structurally stable in χ^D, then $Z^R_{\varepsilon,\eta}$ is structurally stable in χ^S.*

Idea of proof:

We take each normal form of the vector fields with conditions (A), (B) and (C), so we apply the regularization $Z^R_{\varepsilon,\eta}$ as in (1). Then we verify each regularization doesn't have equilibrium points in a neighborhood of the origin.

Once we classify the genetic behavior of $Z \in \Sigma_0$, we will investigate whats happen in the bifurcation set $\chi^D_1 = \chi^D \setminus \Sigma_0$. When $Z \in \chi^D$ isn't locally Σ-structurally stable, we say that Z belongs to the bifurcation set.

Consider Z in the bifurcation set, there is a codimension one singularity in the origin if it's relatively Σ-structurally stable in the induced topology of the bifurcation set, that is, if given the induced topology of χ^D in the bifurcation set there is an open set in the bifurcation set such that every field in that open is locally Σ-equivalent to Z and any unfolding of Z and of the fields in this open are weak equivalents, where two unfoldings are weak equivalent if there is a homeomorphic change of parameter, such that, for each correspondence of the parameter their respective fields are locally Σ -equivalent.

Definition 4 *We say that the origin is a tangency point if the origin is a tangency point of the vector fields X or Y for Σ_1 or Σ_2, when we consider Σ_1 and $\Sigma_2$separately.*

The bifurcation of one codimension set, that is, the fields Σ-structurally stable in χ^D_1, will be denote by Σ_1. If $Z \in \Sigma_1$ then we must break at most one condition in Theorem 1, that is, and we will break these conditions as minimally as possible.

Definition 5 *We say that $Z \in \Sigma^1_1$ if satisfies following conditions:*

(a) $X_i(0)Y_i(0) < 0$, for $i = 1, 2$;

(b) $detZ(0) = 0$ and $\frac{\partial}{\partial x_j}[detZ(0)] \neq 0$ for $i, j = 1, 2$ with $i \neq j$.

Definition 6 *We say that $Z \in \Sigma^2_1$ if satisfies following conditions:*

(a) $Xi(0)Yi(0) > 0$ and $Xj(0)Yj(0) < 0$, with $X_1(0)X_2(0) < 0$ and $i; j = 1, 2$, $i \neq j$;

(b) $\alpha_Z = -1$, $\beta_Z \neq 0$ and $\eta_Z \neq 0$,

where we have the following expression for the Poincaré's first return applications of Z:

$$\phi_Z(x) = \alpha^2_Z x + (\alpha_Z + \alpha^2_Z)\beta_Z x^2 + \eta_Z x^3 + O(x^4),$$

and O is denoting the large order Landau symbol.

Definition 7 *We say that $Z \in \Sigma^3_1$ if satisfies one of the following conditions:*

(a) $X_i(0) = 0$, $X_j(0) \cdot \frac{\partial}{\partial x_j} X_i(0) \neq 0$ for $i, j = 1, 2$, $i \neq j$ e $Y_i(0) \neq 0$ with $i = 1, 2$;

(b) $Y_i(0) = 0$, $Y_j(0) \cdot \frac{\partial}{\partial x_j} Y_i(0) \neq 0$ for $i, j = 1, 2$, $i \neq j$ e $X_i(0) \neq 0$ with $i = 1, 2$.

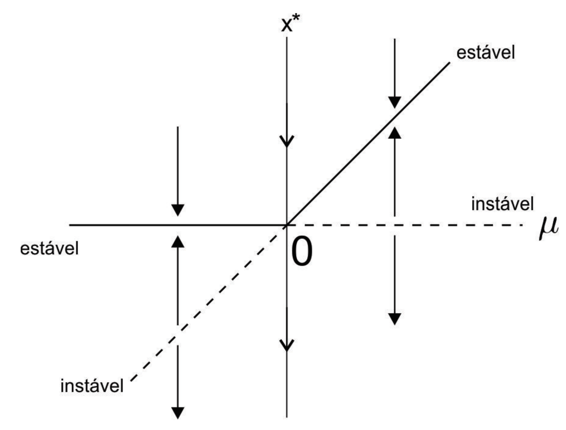
The normal forms of Σ^1_1, Σ^2_1 and Σ^3_1 have been listed in [2], and the normal form of a field means that it is locally Σ-equivalent in bifurcation set around the origin to its respective normal form.

Proposition 1 *If $Z \in \Sigma^2_1 \cup \Sigma^3_1$ then its regularization is structurally stable in a neighborhood of the origin and in this case the regularization doesn't preserve the codimension of the bifurcation.*

Idea of proof:

We take each normal form of the vector fields in Σ^2_1 and Σ^3_1, so we apply the regularization $Z^R_{\varepsilon,\eta}$ as in (1). Then we verify each regularization doesn't have equilibrium points in a neighborhood of the origin.

Transcritical bifurcation is the situation that two equilibrium points situations exist for all values of a parameter, although the stabilities of these equilibrium points are change when the parameter passes through a critical value, and the conditions for its occurrence is given by Sotomayor's Theorem for the transcritical bifurcation.



We consider the following system that is in the equivalence class for vector fields in Σ^1_1

$$\overline{Z_\alpha}(\overline{x}, \overline{y}) = \begin{cases} \overline{X_\alpha}(\overline{x}, \overline{y}) = (\overline{X^1_\alpha}(\overline{x}, \overline{y}), \overline{X^2_\alpha}(\overline{x}, \overline{y})) = \begin{pmatrix} \varepsilon\overline{a} - \overline{b}\,\overline{c_2}\,\overline{x} \\ \eta\overline{b} + \eta\overline{a}\alpha \end{pmatrix}, & \overline{x}\,\overline{y} > 0 \\ \overline{Y}(\overline{x}, \overline{y}) = (\overline{Y_1}(\overline{x}, \overline{y}), \overline{Y_2}(\overline{x}, \overline{y})) = \begin{pmatrix} -\varepsilon\overline{a} \\ -\eta\overline{b} + \overline{a}\,\overline{c_1}\,\overline{y} \end{pmatrix}, & \overline{x}\,\overline{y} < 0 \end{cases}, \qquad (2)$$

where $\overline{a} = \pm 1$, $\overline{b} = \pm 1$ e $\overline{c_i} = \pm 1$, $i = 1, 2$.

Now let's apply the regularization given in (1) in the system (2). For $(x, y) \in (-1, 1) \times (-1, 1)$, if $Z(x, y) = (1 + \phi(x)\psi(y))$ and $W(x, y) = (1 - \phi(x)\psi(y))$ we are going to study the equilibrium points of the following equation:

$$g(x, y, \alpha) = 0 \Leftrightarrow BV = \overrightarrow{0}, \qquad (3)$$

where $B(x, y) = \begin{pmatrix} a - bc_2x & -a \\ b + a\alpha & -b + ac_1y \end{pmatrix}$ and $V(x, y) = \begin{pmatrix} Z \\ W \end{pmatrix}$, which is the same that find the equilibrium points of the regularization of the system (2).

Theorem 3 *Consider α a real parameter sufficiently close to the origin, $B(x, y)$ given in (3) and $(x, y) \in (-1, 1) \times (-1, 1)$. Then the regularization for $Z \in \Sigma^1_1$, after deletion of the positive parameters ε and η, can be described as*

(i) if $detB(x, y) \neq 0$ then the regularization of $Z \in \Sigma^1_1$ is structurally stable, in this case regularization doesn't preserve the codimension of the bifurcation;

(ii) if $detB(x, y) = 0$, we suppose that the transition function ϕ passes through the origin such that

$$\phi''(0) \neq \frac{\phi'(0)}{ac_1}\left[bc_1c_2 + 4\phi'(0)\psi'\left(\frac{-\alpha}{c1}\right)(b + a\alpha)\right], \qquad (4)$$

then there is a bifurcation point (x_0, y_0, α_0) such that the regularization of $Z \in \Sigma^1_1$ has a transcritical bifurcation. In this case the regularization preserves the codimension of the bifurcation, but doesn't preserve the fact of bifurcation be generic.

Idea of proof:

We take normal form of the vector fields in Σ^1_1, so we apply the regularization $Z^R_{\varepsilon,\eta}$ as in (1).

(i) So we verify that the regularization doesn't have equilibrium points in a neighborhood of the origin.

(ii) So we verify that the regulazrization satisfies the hypotheses of the Sotomayor's Theorem for the transcritical bifurcation.

Main References

[1] D. Panazzolo, P. R. Silva, *Regularization of Discontinous Foliations: Blowing Up and Sliding Conditions via Fenichel Theory*, Journal of Differential Equations, Volume 263, 12, 2017, 8362-8390.

[2] J. Larrosa, *Generic Bifurcations in Planar Filippov Systems*, Doctoral Thesis, IMECC-UNICAMP, 2015.

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[1] Doctorate student; FAPESP's scholarship process n^o 2016/00242-2 and BEPE/FAPESP's scholarship process n^o 2018/05098-2 -; E-mail: yagor.carvalho@hotmail.com

[2] Mathematics department, IBILCE/UNESP, (BRAZIL) Email: claudio.buzzi@unesp.br