Inseparable leaves of polynomial submersions

Francisco Braun

Departamento de Matemática
Universidade Federal de São Carlos

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[L. Markus, Trans. Amer. Math. Soc. 1954], following [W. Kaplan, Duke Math. J. 1940 and 1941], proved that it is enough to analyse some special leaves and their configuration in the plane.

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For instance, let X be the system $\dot{x} = -x^2$, $\dot{y} - 1 + 2xy$.

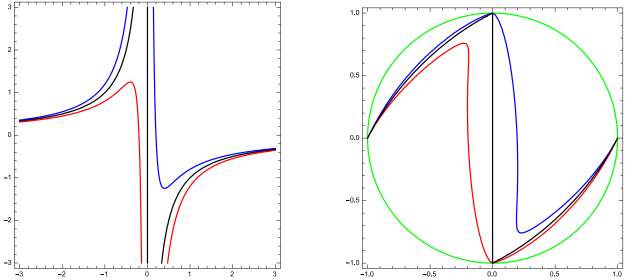

Figure: Some leaves of X

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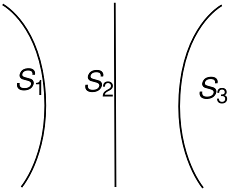

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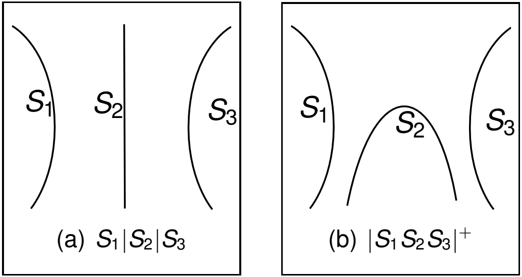

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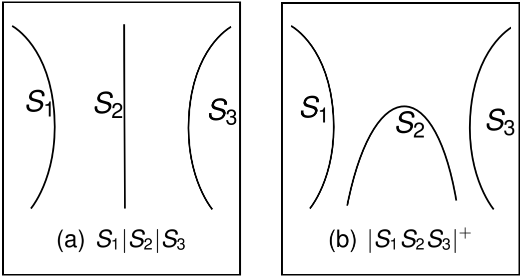

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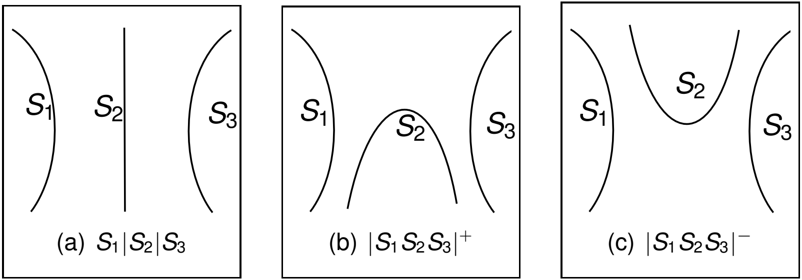
In the cyclic case, we can have a positive cycle, denoted by $|S_1 S_2 S_3|^+$ or a negative one, denoted by $|S_1 S_2 S_3|^-$.



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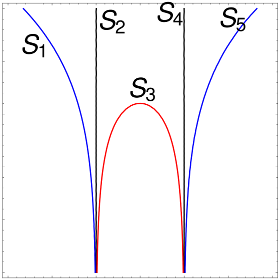

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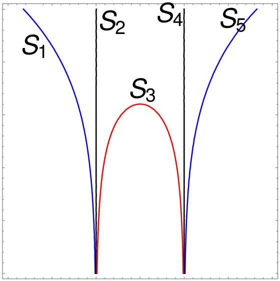

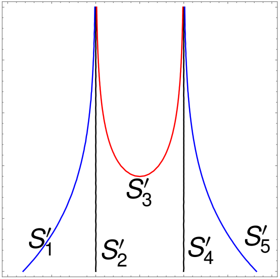

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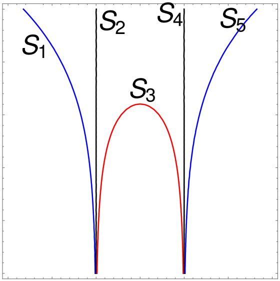

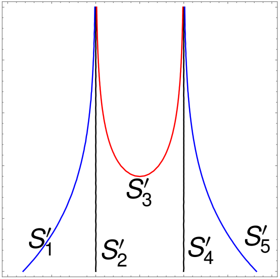

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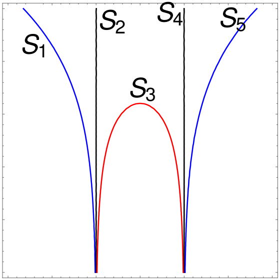

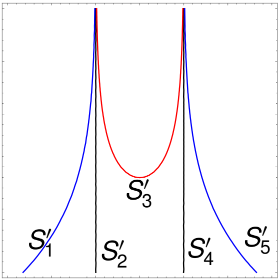

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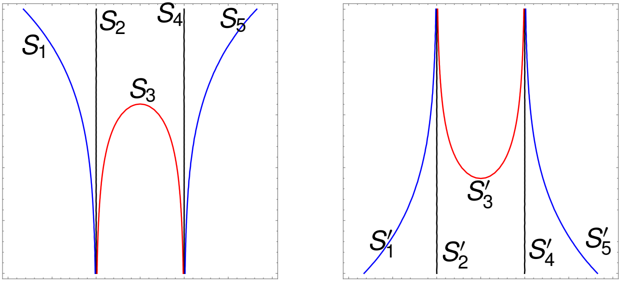

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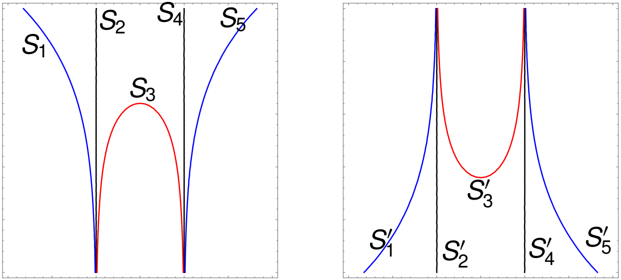

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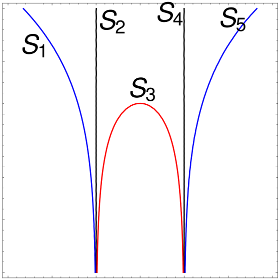

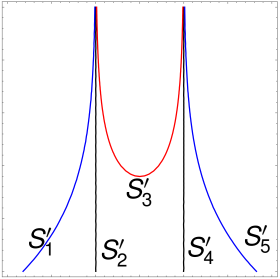

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Inseparable configuration

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(By different chooses of canonical regions, the related inseparable configurations are isomorphic).

Kaplan - Markus result

Theorem

Two polynomial chordal systems X_1 and X_2 are topologically equivalent if and only if the related inseparable configurations $X_1\Sigma$ and $X_2\Sigma$ are isomorphic or anti-isomorphic by a map carrying Σ_1 to Σ_2.

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So the classification problem proposed by Markus depends only on the inseparable configuration of chordal systems.

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We denote this number by $s(n)$.

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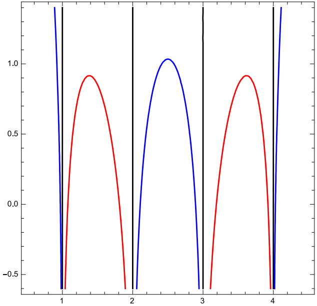
So $s(n) \geq n$.



$$\dot{x} = (x-1)(x-2)(x-3)(x-4),$$
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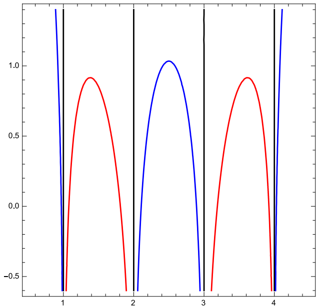

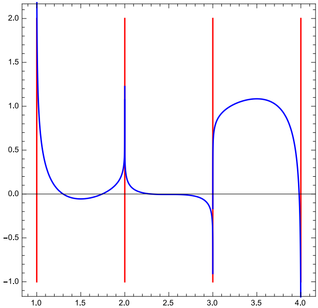

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Therefore $h+e \leq 2n+2$, and hence

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Finally, as consequence of [A. Cima and J. Llibre, Proc. 7th congress dif. eq. app., 1985] and [M. Carbonell and J. Llibre, Publ. Mat., 1989], it follows that $s(3) = 3$.

So it follows that

$$s(0) = s(1) = 0, \; s(2) = s(3) = 3,$$

and

$$\begin{aligned}
&6 \leq s(4) \leq 8, \\
&9 \leq s(6) \leq 12 \text{ and} \\
&2n - 4 \leq s(n) \leq 2n \text{ if } n = 5 \text{ or } n \geq 7.
\end{aligned}$$

So it follows that

$$s(0) = s(1) = 0, \ s(2) = s(3) = 3,$$

and

$$\begin{aligned} &6 \leq s(4) \leq 8, \\ &9 \leq s(6) \leq 12 \text{ and} \\ &2n - 4 \leq s(n) \leq 2n \text{ if } n = 5 \text{ or } n \geq 7. \end{aligned}$$

In joint work with F. Fernandes, we prove that

$$s(n) \geq 2n - 1, \qquad \text{for all } n \geq 4.$$

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Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial submersion of degree $n + 1$,

Indeed, we prove more...
Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial submersion of degree $n+1$, and consider the chordal Hamiltonian system of degree n, henceforward denoted by H_p:

$$\dot{x} = -p_y(x,y), \quad \dot{y} = p_x(x,y),$$

We define $s_{\mathrm{H}}(n)$ the maximal number of inseparable leaves a chordal Hamiltonian polynomial vector field of degree n can have.

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Theorem

$s_{\mathrm{H}}(n) \geq 2n - 1$ *for all* $n \geq 4$.

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The idea to construct such examples is to use the "blow up" of Γ

$$(x, y) \mapsto (x, x/y).$$

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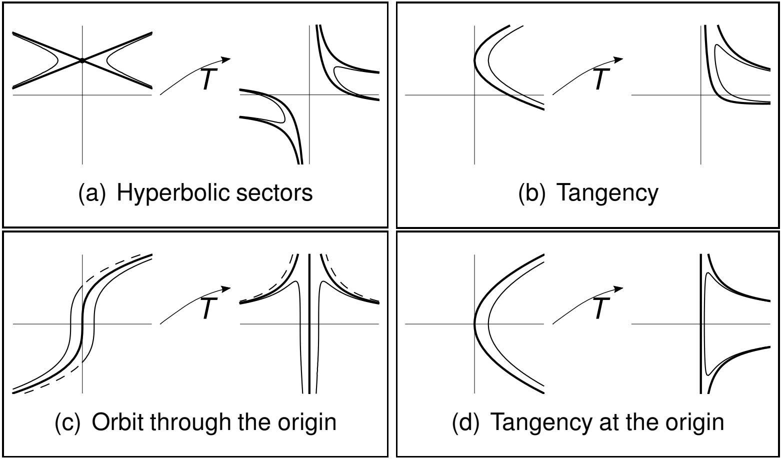

Figure: Some orbits of $H_{\widetilde{p}}$ and H_p.

Lemma

Let $\widetilde{p} : \mathbb{R}^2 \to \mathbb{R}$ *be a submersion away from* Γ*. Then* $p : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$p(x, y) = \widetilde{p}(x, xy),$$

is a submersion in $\mathbb{R}^2$ *if and only if* $\widetilde{p}_x(0, 0) \neq 0$ *and* $\widetilde{p}_y(0, 0) = 0$*.*

Theorem

Let $\widetilde{p}$ and p as above. The following statements hold true:

1. *Each pair of inseparable leaves of $H_{\widetilde{p}}|_{\Gamma^{\complement}}$ induces a pair of inseparable leaves of H_p.*
2. *Any hyperbolic sector of a singular point $(0, y_0)$ of $H_{\widetilde{p}}$ contained in $\Gamma^{\complement} \cup \{(0, y_0)\}$ produces a pair of inseparable leaves of H_p.*
3. *Each leaf of $H_{\widetilde{p}}$, different from Γ, tangent to Γ induces a pair of inseparable leaves of H_p.*
4. *A regular orbit of $H_{\widetilde{p}}$ intersecting Γ in exactly k points induces $k + 1$ orbits of H_p.*
5. *The curve Γ is an orbit of H_p.*
6. *If $y \mapsto \widetilde{p}_y(0, y)$ is not the zero polynomial, then there are two orbits of H_p that are inseparable with Γ.*

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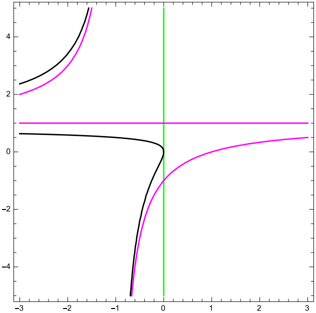

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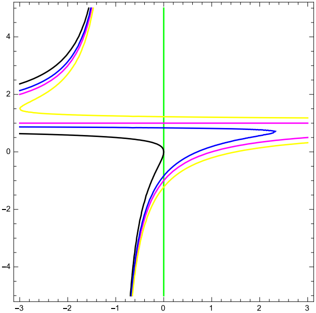

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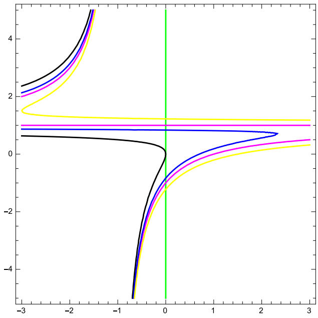

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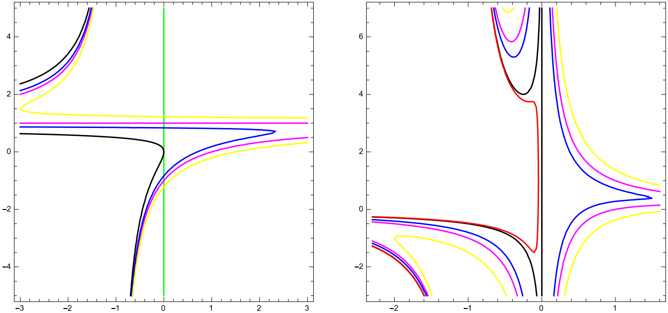

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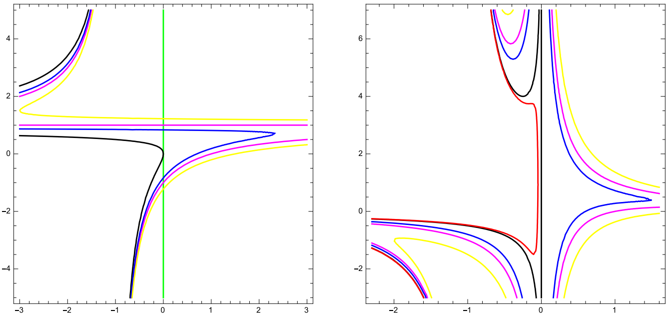

(a) Some leaves of $H_{\widetilde{p}}$

(b) Some leaves with the inseparable configuration of H_p

Let $p(x, y) = \widetilde{p}(x, xy) = (xy-1)^2 x + x^2 y^2$. By the theorem, H_p, of degree 4, has 7 inseparable leaves.

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In our general construction we will always have these 7 inseparable leaves. We will get more by adding tangencies to Γ and saddle points of $H_{\widetilde{p}}$, in different level sets, paying the price of increasing the degree of the system.

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It is simple to see that $\widetilde{p}_x(0,0) \neq 0$ and $\widetilde{p}_y(0,0) = 0$. So p and $\widetilde{p}$ are in the assumptions of our theorem.
So H_p is a chordal Hamiltonian system of even degree $n = 2(k+2)$ if h is constant, and of odd degree $n = 2(k+2) - 1$ if h is not constant.

Lemma

The singular points of $H_{\widetilde{p}}$ *are* $(0, A_i)$, $i = 1, \ldots, u$, $u \leq r$, *where* $A_1, \ldots, A_u$ *are the zeros of* $g(y)$. *Each of them is a saddle point with two separatrices in the region* $x < 0$ *and two separatrices in the region* $x > 0$.

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The separatrices of the saddle $(0, A_i)$ are in the level c_i. So separatrices of different saddles cannot connect to each other. Therefore

$$H_p \text{ has at least } 4u \text{ inseparable leaves,}$$

where u is the number of zeros of $g(y)$.

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Lemma

There is a connected component of the level set $\widetilde{p}^{-1}(c_i)$ containing the point $(0, A_i)$. This curve is tangent to Γ in $(0, A_i)$.

By our theorem, and properties, it follows that

H_p has $2v$ more inseparable leaves,

where v is the number of zeros of h.

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The leaves γ^- and γ^+ are both inseparable (as leaves of $H_{\widetilde{p}}$) to the straight line $y = 1$.

So, from the properties and theorem, it follows that

H_p has 4 more inseparable leaves.

We add the 3 inseparable leaves given by the theorem and properties, and obtain at least

$$4u + 2v + 4 + 3$$

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If f has $k + 1$ zeros, we have $k = u + v$.

In this situation, if h is constant, and so $v = 0$, we have $4k + 7$ inseparable leaves.

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In the first case we can consider all $k \geq 0$ and in the other one we can consider all $k \geq 1$. Therefore $s_{\mathrm{H}}(n) \geq 2n-1$ for all $n \geq 4$.

Does there exists a polynomial f, of degree $k + 1$, with $k + 1$ zeros, satisfying the properties?

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For each $z \in \mathbb{R}$, we define the polynomial

$$f_z(y) = y\Pi_{i=1}^{k}(y - z^i) = \sum_{i=1}^{k} z^{(k-i)(k-i+1)/2} v_i(z) y^{i+1},$$

where $v_i(z)$ are suitable polynomials (Vieta's relations).

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We now consider the polynomials in the variable z

$$C(z^j, z) = \int_0^{z^j} f_z(s)ds = \sum_{i=1}^{k} z^{\tau(i)} \frac{v_i(z)}{i+2},$$

where $\tau(i) = (k-i)(k-i+1)/2 + j(i+2)$.

By considerations on τ and v_i, we can write

$$C(z^j, z) = z^{\tau(k-j)} \left(\frac{(-1)^j}{(k-j+2)(k-j+3)} + z m_j(z) \right),$$

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By taking a transcendental number $z_0 \in \mathbb{R}$, it follows that $C(z_0^j, z_0)$ are pairwise distinct and different from 0.

So the polynomial $f_{z_0}(y)$ satisfies what we wanted.

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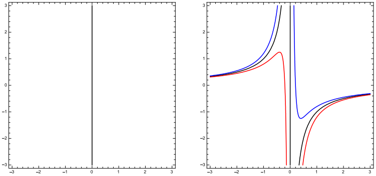
Third question (Markus original question): what are the possible inseparable configurations of chordal polynomial systems of degree n, Hamiltonian and in general?

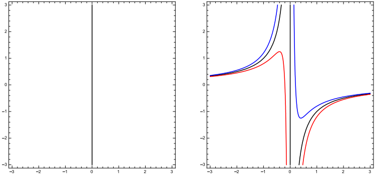
Hamiltonians



(a) $p = x + x^3$ (b) $x(1 + xy)$

Figure: Degree 2

Hamiltonians



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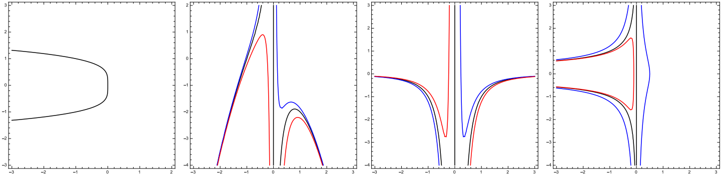
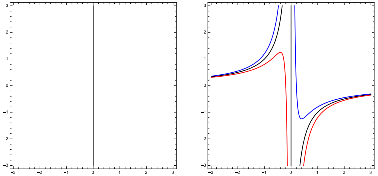


(a) $x + y^4$ (b) $x(1+xy+x^3)$ (c) $x(1 + x^2y)$ (d) $x(1 + xy^2)$

Figure: Degree 3

Hamiltonians



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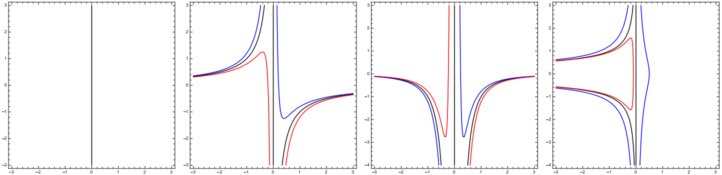

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Thank you!