REAL PLANAR QUADRATIC SYSTEMS

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Summary of works with
J. Llibre (UAB), D. Schlomiuk (Montreal), A. Carlucci/R. Oliveira (Sao Carlos) & N. Vulpe (Moldova)

Castro Urdiales, 2010

MAIN QUESTION

HOW MANY PHASE PORTRAITS OF PLANAR REAL QUADRATIC DIFFERENTIAL SYSTEMS EXISTS?

ARE YOU KIDDING???

HAVE SOMEONE HAVE ALREADY SOLVED 16TH HILBERT'S PROBLEM?

- A) YES
- SOMEONE DESERVES ABEL PRIZE!!!
- B) NO
- Please, reformulate your question

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REFORMULATED MAIN QUESTION

- HOW MANY PHASE PORTRAITS OF PLANAR REAL QUADRATIC DIFFERENTIAL SYSTEMS EXISTS ...
- MODULO LIMIT CYCLES?

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2. Systems with a 3rd order weak focus (Artes, Llibre, Schlomiuk, 1984 & 2004)
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To avoid intersections, there are 3 possible ways:

1. According number of finite singularities (stopped at $1 \sim 2$).
2. According number of finite multiplicity (stopped at $2 \sim 3$).
3. According codimension (currently running at 2).

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In 2008 J. C. Artés, J. Llibre and N. Vulpe published "Singular points of quadratic systems: A complete classification in the coefficient space $\mathbb{R}^{12}$" at International J. of Bifurcation and Chaos. All topological combinations of finite singularities were classified using invariant polynomials. Also focus were distinguished from nodes with those tools.

WHY INVARIANT POLYNOMIALS?

Assume we have a quadratic system with 4 finite singularities. We have the four determinants of the Jacobian matrices at those points: δ_1, δ_2, δ_3 and δ_4.

Assume one can get expressions which capture:

1. $J_1 = \delta_1\delta_2\delta_3\delta_4$;
2. $J_2 = \delta_1\delta_2\delta_3 + \delta_1\delta_2\delta_4 + \delta_1\delta_3\delta_4 + \delta_2\delta_3\delta_4$;
3. $J_3 = \delta_1\delta_2 + \delta_1\delta_3 + \delta_1\delta_4 + \delta_2\delta_3 + \delta_2\delta_4 + \delta_3\delta_4$;
4. $J_4 = \delta_1 + \delta_2 + \delta_3 + \delta_4$;

If one can obtain these expressions, he will get the discriminants of the polynomial $x^4 - J_4x^3 + J_3x^2 - J_2x + J_1$, and their signs will be invariant upon any affine change of coordinates. There is no need of normal forms. They work on the 12-parameter space of coefficients.

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With the use of these invariants, the Sibirschi School in Chisinau (Moldavia) has been able to obtain invariants for every imaginable geometric feature related to quadratic systems.

1. How many finite singularities;
2. How many infinite singularities;
3. How many multiple singularities;
4. How many weak singularities and their order (foci and/or saddles);
5. How many nilpotent singularities and their type;
6. How is the tangential behavior around every singularity;
7. Distinction between different nodes;
8. Existence of centers;
9. Isochronicity of centers;
10. Existence of invariant straight lines;
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SIBIRSCHI SCHOOL

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We have defined this way of classifying singularities as *Geometrical Classification*. A Geometrical equivalence class has been defined and the singularities (finite and infinite) are classified according to it.

NOTATION

A notation system to describe all these singularities and their geometrical properties has been developed.

1. $s, s^{(3)}, n, n^d;\ S, N^\infty, N^\infty$
2. $f, f^{(3)}$, ©, ©; $\overline{\binom{0}{2}} SN, S$
3. $s, n^d, f^{(2)}$; $\overline{\binom{1}{1}} SN$, ©, ©
4. $c^{\odot}$, ©, ©; $\widehat{\binom{1}{2}}\ P_{\curlywedge} E P_{\curlywedge} - H, S$
5. \$, \$, $\widehat{cp}_{(2)};\ N^f, N^f, N^f$
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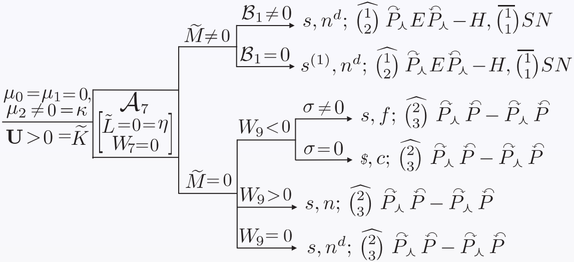
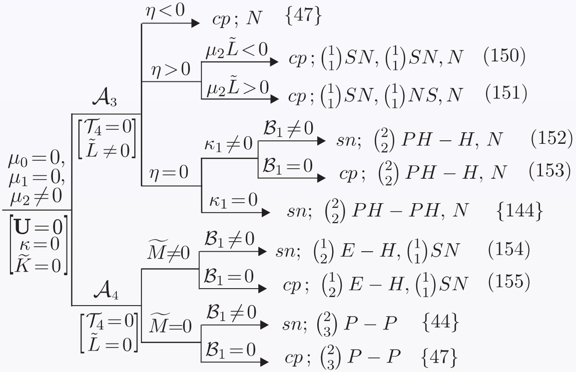
3. $s, n^d, f^{(2)};\ \overline{\binom{1}{1}} SN, \textcircled{c}, \textcircled{c}$
4. $c^{\odot}, \textcircled{c}, \textcircled{c};\ \widehat{\binom{1}{2}}\ \overset{\curvearrowright}{P}_{\curlywedge} E \overset{\curvearrowleft}{P}_{\curlywedge} - H, S$
5. $\$, \$, \widehat{cp}_{(2)};\ N^f, N^f, N^f$
6. $s, \$, n, n^*;\ S,\ N^{\infty},\ N^*$

BIFURCATION DIAGRAM (1765 configurations)



The full diagram takes 70 pages to be written.

REDUCTION TO TOPOLOGICAL (208 configurations)



This "only" needs 13 pages.

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2. Many configurations have a single possible phase portrait;
3. Only some few configurations have more than 10 phase portraits;
4. 22 easy to complete;
5. 29 on work;
6. 814 phase portraits up to now, 231 of them from those uncomplete 29 families;
7. maximum of phase portraits in one completed configuration (up to now): 97;

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FINAL BET

1. Please, make your bet.
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7. THANKS FOR YOUR COLLABORATION.

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