Varieties and normalization of partially integrable analytic differential systems

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Outline of the talk

- Background and the related results.
- The main results.
- Sketch proof of the main results.

For an analytic differential system

$$\frac{dx}{dt} = \dot{x} = Ax + f(x) = F(x), \quad x \in (\mathbb{F}^n, 0),\ \mathbb{F} = \mathbb{C} \text{ or } \mathbb{R}, \tag{1}$$

with

- $A \in M_n(\mathbb{F})$, the set of all $n \times n$ matrices with entries in $\mathbb{F}$,
- $f(x) = o(|x|)$ vector valued analytic function,

we will study:

- the varieties of partial integrability of system (1),
- the existence of analytic normalization,

provided that system (1) is partially integrable.

Related to these two problems, there are classical Poincaré results :

If system (1) is a real planar one, and A has a pair of pure imaginary eigenvalues, it can be written as

$$\dot{u} = -v + p(u,v), \quad \dot{v} = u + q(u,v), \tag{2}$$

Theorem P1. *There exists an analytic function or a formal series* $\Phi(u,v)$ *such that*

$$(-v+p(u,v))\frac{\partial\Phi}{\partial u}+(u+q(u,v))\frac{\partial\Phi}{\partial v}=\sum_{l=m}^{\infty}\alpha_l(u^2+v^2)^l,$$

where $2\le m\in\mathbb{N}$*, and* α_l *are polynomials in the coefficients of* $p(u,v)$ *and* $q(u,v)$*.*

Note, if system (2) is polynomial,

$\Downarrow$ by Hilbert's basis theorem

- $\bigcap_l\{\alpha_l=0\}$ is finitely determined, and
- it is a variety in the space of coefficients of system (2).

Theorem P2. *System* (2) *has a center at the origin*

if and only if

- *it has an analytic first integral in a neighborhood of the origin, and*

  if and only if

- *it is locally analytically equivalent to its Poincaré–Dulac normal form.*

Note,

A similar result for a saddle of planar analytic Hamiltonian systems was obtained by Moser [CPMA 1956].

Recall that system (1) is in Poincaré–Dulac normal form if

- A is in the Jordan normal form
- any monomial $x^m e_j$ in the jth component of $f(x)$ is resonant, i.e. $\langle m, \lambda \rangle = \lambda_j$, where
    - λ is the eigenvalues of A,
    - $x^m = x_1^{m_1} \ldots x_n^{m_n}$ for $x = (x_1, \ldots, x_n)$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$
    - $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$,
    - $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.

Poincaré–Dulac normal form theorem:

If

- A *is in the Jordan normal form, and*
- $f(x)$ *is analytic or a formal series,*

then

- *system* (1) *can be transformed to its Poincaré–Dulac normal form by a near identity transformation* $($*analytic or formal*$)$.

A near identity transformation is the one of the form

$$x = y + \varphi(y)$$

with $\varphi = o(|y|)$.

Theorem P2 was generalized to

- any finite dimensional analytic systems (1) by Zhang [JDE 2013], Llibre et al [BSM 2012]:

Set

$$\mathscr{R}_\lambda := \left\{ m \in \mathbb{Z}_+^n \mid \langle m, \lambda \rangle = 0, \ |m| = m_1 + \ldots + m_n \geq 2 \right\},$$

r_λ : resonant rank, i.e.

maximum number of $\mathbb{Q}_+$–linearly independent elements of $\mathscr{R}_\lambda$.

Theorem A. *Assume that $n \geq 2$ and $\lambda \neq 0$. Then system* (1) *has $n-1$ functionally independent analytic first integrals in $(\mathbb{C}^n, 0)$ if and only if*

- $r_\lambda = n-1$, *and*
- *system* (1) *is analytically equivalent to its PD normal form*

$$\dot{y}_i = \lambda_i y_i (1 + g(y)), \qquad i = 1, \ldots, n,$$

by a near identity analytic normalization, where $g(y)$ is an analytic function of y^m with $m \in \mathscr{R}_\lambda$.

Theorem P1 was extended to any finite dimensional systems by Romanovski, Xia, Zhang [JDE 2014]

Theorem B. *Let $\mathscr{X}$ be the analytic vector field associated to system* (1). *The following hold.*

(a) $\exists$ *series* $\psi(x)$ *with its resonant monomials arbitrary such that*

$$\mathscr{X}(\psi(x)) = \sum_{\alpha \in \mathfrak{R}_\lambda} p_\alpha x^\alpha, \tag{3}$$

where p_α *are polynomials in the coefficients of* $\mathscr{X}$ *and of the resonant monomials of* ψ*.*

(b) *If the vector field* $\mathscr{X}$ *has* $n-1$ *functionally independent analytic or formal first integrals, then for any* ψ *satisfying* (3), *we have*

$$p_\alpha = 0, \qquad \text{for all} \quad \alpha \in \mathfrak{R}_\lambda. \tag{4}$$

Variety of partial integrability

Now we extend Theorem B to partially integrable systems.

Theorem 1

Assume that

- $\mathfrak{R}_\lambda$ has $d < n$ $\mathbb{Q}_+$–linearly independent elements,
- $\mathscr{X}$ has d functionally independent analytic or formal first integrals.

Then

(a) For any ψ satisfying (3), we have

$$p_\alpha = 0, \qquad \text{for all} \quad \alpha \in \mathfrak{R}_\lambda. \tag{5}$$

Theorem 1 (Continuity)

(b) The vector field $\mathscr{X}$ has d functionally independent first integrals of the form

$$H_1(x) = x^{\alpha_1} + h_1(x), \ldots, H_d(x) = x^{\alpha_d} + h_d(x), \tag{6}$$

with

- $\alpha_1, \ldots, \alpha_d$ $\mathbb{Q}_+$–linearly independent elements of $\mathfrak{R}_\lambda$,
- each $h_j(x)$, $j = 1, \ldots, d$, consisting of nonresonant monomials in x of degree larger than $|\alpha_j|$.

Remark:

- The functions p_α in Theorem 1 are not uniquely defined,
- but they have the same set of zeros for any choice of the resonant coefficients.

Hence,

- we can set the resonant coefficients in $\psi(x)$ equal to zero,
- when system (1) is polynomial, the p_α are uniquely defined polynomials in the parameters of system (1)
- the zero set of these polynomials is an affine variety in the space of parameters of system (1).

Note that

- the number of functionally independent analytic or formal first integrals of $\mathscr{X}$ is less than or equal to the maximal number of $\mathbb{Q}_+$–linearly independent elements of $\mathfrak{R}_\lambda$.

So in the cases of Theorem 1 we call system (1) is

- *partially integrable* if $d < n-1$
- *completely integrable* (for simplicity, integrable) if $d = n-1$.

For this reason, the variety mentioned above is called *variety of partially integrable system* (1).

Remark:

- The methods for proving Theorem B cannot be applied to the proof of Theorem 1.

Because

- the key point in proving Theorem B is that the integrable differential system (1) has the PD normal form

$$\mathrm{diag}(\lambda_1 y_1, \ldots, \lambda_n y_n)(1 + g(y))$$

  with $g(y)$ a scalar function or a scalar formal series.
- here partially integrable systems in general do not have this special PD normal form.

Normalization of partially integrable systems

Set $\lambda = (\lambda', \lambda'')$ with $\lambda' = (\lambda_1, \ldots, \lambda_k)$, $\lambda'' = (\lambda_{k+1}, \ldots, \lambda_n)$ for some $k \in \{2, \ldots, n\}$, and

$$\lambda_j \neq \lambda_l, \quad j \in \{1, \ldots, k\},\ l \in \{k+1, \ldots, n\}.$$

System (1) can be written in

$$\begin{aligned} \dot{x}' &= A'x' + f'(x', x''), \\ \dot{x}'' &= A''x'' + f''(x', x''), \end{aligned} \tag{7}$$

with A' and A'' having respectively the eigenvalues λ' and λ''.

Let $\mathscr{X}^*$ be the analytic vector field associated to system (7).

According to Bibikov [LNM 1979], by definition

- system (7) is in *Poincaré–Dulac normal form on invariant manifold* if
  - $f''(x',0) = 0$ and
  - the Taylor expansion of $f'(x',0)$ consists of resonant monomials.
- system (7) is *formally* $($*analytically*$)$ *equivalent* to its Poincaré–Dulac normal form on invariant manifold if after a near identity formal (analytic) change of coordinates system (7) is transformed to a normal form on invariant manifold.

Set

$$\mathfrak{R}_{\lambda'} = \left\{m' \in \mathbb{Z}_+^k \,|\, \langle m', \lambda' \rangle = 0,\ |m| \geq 2\right\},$$

Theorem 2

Assume that $\lambda' \neq 0$, and

- for any $m = (m', m'') \in \mathfrak{R}_\lambda$ we have $m'' = 0$,
- any component of $m' \in \mathfrak{R}_{\lambda'}$ cannot always vanish.

If $\mathscr{X}^*$ has $k-1$ functionally independent analytic first integrals, then

- $\mathfrak{R}_{\lambda'}$ has exactly $k-1$ $\mathbb{Q}_+$–linearly independent elements,
- $\mathscr{X}^*$ is analytically equivalent to its PD normal form on invariant manifold

$$\begin{aligned} \dot{y}' &= \operatorname{diag}(\lambda_1, \ldots, \lambda_k) y' (1 + q(y')) + g^*(y', y''), \\ \dot{y}'' &= A'' y'' + g''(y', y''), \end{aligned} \tag{8}$$

with $g^*(y', 0) = 0$, $g''(y', 0) = 0$, and all monomials of $q(y')$ resonant.

Remark:

- In Theorem 2, the hyperplane $y'' = 0$ is invariant under the flow of the normal formal system (19).
- Theorem 2 is an extension from integrable systems to partially integrable ones.

Note: in Theorem 2

- $m'' = 0$ means that the resonance of λ only depends on λ'
- any component of $m' \in \mathfrak{R}_{\lambda'}$ does not always vanishes implies that the resonances depend on each component of λ'.

In these senses, the assumption on λ in Theorem 2 is not a restriction.

Proof of Theorem 1

Preparations:

Let

- A_s be the semisimple part of A,
- $\mathscr{X}_s$ be the linear vector field associated to $\dot{x} = A_s x$.

Proposition 1

Assume that

- system (1) is in the PD normal form,
- $\mathscr{X}_s$ has d functionally independent polynomial first integrals.

Then system (1) has d functionally independent formal first integrals $\Leftrightarrow$ it admits all polynomial first integrals of $\mathscr{X}_s$.

This result was obtained by Llibre *et al* [BSM2012].

Structure of the matrix A of the partially integrable system (1):

Write $A = \mathrm{diag}(A', A'')$ with eigenvalues $\lambda = (\lambda', \lambda'')$ such that

(H) for $k = (k', k'') \in \mathfrak{R}_\lambda$ we have that $k'' = 0$ and that any component of k' cannot always vanish.

Proposition 2

Assume

- $\mathscr{X}_s$ admits d functionally independent polynomial first integrals
- $\mathscr{X}$ admits d functionally independent formal first integrals.

Let A have the decomposition (A', A'') satisfying (H). Then A' is diagonal.

Note: the decomposition of A is not restriction.

Sketch proof of Proposition 2

Let

$$\dot{y} = Ay + g(y), \tag{9}$$

be the PD normal form of system (1).

$\Downarrow$

System (9) has also d functionally independent first integrals.
Let $\alpha_1, \ldots, \alpha_d \in \mathfrak{R}_\lambda$ be $\mathbb{Q}_+$–linearly independent elements.

$\Downarrow$

$y^{\alpha_1}, \ldots, y^{\alpha_d}$ are d functionally independent first integrals of $\mathscr{X}_s$.

$\Downarrow$ By Proposition 1

$y^{\alpha_1}, \ldots, y^{\alpha_d}$ are first integrals of system (9).

$\Downarrow$

they are also first integrals of $\dot{y} = Ay$.
Set $y = (y', y'')$, and $\alpha_j = (\alpha_j', \alpha_j'')$ be the corresponding decomposition, $j = 1, \ldots, d$.

$\Downarrow$ By assumption (H)

$$y^{\alpha_j} = y'^{\alpha'_j}, \qquad j = 1, \ldots, d,$$

are first integrals of system $\dot{y}' = A'y'$. Let

$$A' = \begin{pmatrix} \lambda_1 & & & \\ \delta_2 & \lambda_2 & & \\ & \ddots & \ddots & \\ & & \delta_\ell & \lambda_\ell \end{pmatrix},$$

with $\delta_s = 0$ or 1, and in case $\delta_s = 1$ we have $\lambda_{s-1} = \lambda_s$. From

$$\langle A'y', \partial_{y'} y'^{\alpha'_j} \rangle = 0, \quad \langle \lambda', \alpha'_j \rangle = 0, \quad j = 1, \ldots, d,$$

$\Rightarrow$ $\delta_s = 0$ for $s = 2, \ldots, \ell$.

$\Rightarrow$ A' is diagonal. $\square$

Proof of Theorem 1

Recall Theorem 1

Assume that

- $\mathfrak{R}_\lambda$ has d $\mathbb{Q}_+$–linearly independent elements,
- $\mathscr{X}$ has d functionally independent formal first integrals.

Then

(a) For any ψ satisfying (3), i.e. $\mathscr{X}(\psi(x)) = \sum_{\alpha\in\mathfrak{R}_\lambda} p_\alpha x^\alpha$, we have

$$p_\alpha = 0, \qquad \text{for all} \quad \alpha \in \mathfrak{R}_\lambda. \tag{10}$$

(b) $\mathscr{X}$ has d functionally independent first integrals of the form

$$H_1(x) = x^{\alpha_1} + h_1(x), \ldots, H_d(x) = x^{\alpha_d} + h_d(x). \tag{11}$$

Proof of (a)

Let $\psi(x)$ be a formal series satisfying (3).

By contrary, assuming that $\exists$ some $m_0 \in \mathfrak{R}_\lambda$ s.t. $v_{m_0} \neq 0$, and

$$\mathscr{X}(\psi(x)) = v_{m_0} x^{m_0} + \sum_{m \in \mathfrak{R},\, |m|=|m_0|, m \neq m_0} v_m x^m + \text{h.o.t.} \tag{12}$$

Set $k_0 = |m_0|$.

$\Downarrow$ By the PD normal form theorem

$\exists$ a near identity transformation, saying

$$x = y + h(y) = H(y), \tag{13}$$

which sends $\mathscr{X}$ to its PD normal form vector field $\mathscr{Y}$, i.e.

$$\dot{y} = Ay + g(y) = G(y).$$

Then

$$\mathscr{X}(\psi) \circ H(y) = v_{m_0} y^{m_0} + \sum_{m \in \mathfrak{R},\, |m|=|m_0|} v_m y^m + \text{h.o.t.} \tag{14}$$

Set

$$W(y) = \psi \circ H(y).$$

Then

$$\mathscr{Y}(W(y)) = (\partial_y H(y))^{-1} \mathscr{X} \circ H(y)(W(y)) = \mathscr{X}(\psi) \circ H(y). \tag{15}$$

Expanding $W(y)$ in power series

$$W(y) = \sum_{\ell=m}^{\infty} W_\ell(y),$$

with W_ℓ's homogeneous polynomials of degree ℓ. Then

- $m \leq k_0$.
- for $\ell < k_0$, W_ℓ consist of resonant monomials.

Furthermore, we get that

- $\mathscr{Y}$ has d functionally independent first integrals.
- $W_m(y)$, $W_{m+1}(y), \ldots, W_{k_0-1}(y)$ are first integrals of $\mathscr{Y}$ by Proposition 1.

So

$$\mathscr{Y}\left(\sum_{\ell=m}^{\infty} W_\ell(y)\right) = \mathscr{Y}\left(\sum_{\ell=k_0}^{\infty} W_\ell(y)\right).$$

This yields

$$\mathscr{Y}_1(W_{k_0}(y)) = v_{m_0} y^{m_0} + \sum_{m\in\mathfrak{R},\, |m|=|m_0|} v_m y^m, \tag{16}$$

where $\mathscr{Y}_1$ is the linear part of $\mathscr{Y}$. Separate

$$W_{k_0}(y) = W_{k_0\mathfrak{r}}(y) + W_{k_0\mathfrak{n}}(y)$$

in the summation of the resonant and nonresonant parts.

Using the decomposition $A = \text{diag}(A', A'')$, we have

$$W_{k_0\mathfrak{r}}(y) = W_{k_0\mathfrak{r}}(y').$$

By Proposition 2 it follows that A' is diagonal. So

$$\mathscr{Y}_1(W_{k_0\mathfrak{r}}(y)) = \langle A'y', \partial_{y'} W_{k_0\mathfrak{r}}(y')\rangle \equiv 0.$$

This forces that

$$\mathscr{Y}_1(W_{k_0}(y)) = \mathscr{Y}_1(W_{k_0\mathfrak{n}}(y)) = L(W_{k_0\mathfrak{n}}(y)),$$

By the spectrum of the linear operator L, we get

- $L(W_{k_0\mathfrak{n}}(y))$ is either identical zero or consists of nonresonant monomials.

Whereas the right hand side of (16) are nonvanishing resonant monomials.

This contradiction implies that $p_m \equiv 0$ in (3) for all $m \in \mathfrak{R}_\lambda$.

Statement (a) follows.

Proof of (b).

System (1) is transformed to its PD normal form $\mathscr{Y}$, i.e.

$$\begin{aligned} \dot{y}' &= A'y' + g'(y', y'')), \\ \dot{y}'' &= A''y'' + g''(y', y''), \end{aligned} \tag{17}$$

through $x = y + h(y) = H(y)$, where

$$A' = \text{diag}(\lambda_1, \ldots, \lambda_\ell).$$

Since $\mathfrak{R}_\lambda$ has exactly d $\mathbb{Q}_+$–linearly independent elements, let $\alpha_1, \ldots, \alpha_d$ be its linearly independent elements. Then

$$\varphi_j(y) = y^{\alpha_j}, \qquad j = 1, \ldots, d,$$

are d functionally independent first integrals of $\mathscr{Y}_s$, the semisimple part of $\mathscr{Y}_1$.

By Proposition 1, it follows that φ_j's are first integrals of $\mathscr{Y}$.

The transformation $x = H(y)$ from $\mathscr{X}$ to $\mathscr{Y}$ is near identity shows

$$\varphi_j \circ H^{-1}(x) = x^{\alpha_j} + \text{h.o.t.}, \qquad j = 1, \ldots, d,$$

are d functionally independent first integrals of $\mathscr{X}$.

For proving that $\mathscr{X}$ has d functionally independent first integrals satisfying the nonresonant conditions, we can choose d functionally independent functions in a neighborhood of the origin of the form

$$V_j(x) = x^{\alpha_j} + v_j(x), \qquad j = 1, \ldots, d, \tag{18}$$

with $v_j(x) = o(|x|^{|\alpha_j|})$ and its resonant monomials arbitrary, such that

$$\mathscr{X}(V_j(x)) = \sum_{k \in \mathfrak{R}} w_k^{(j)} x^k, \qquad j = 1, \ldots, d,$$

where $w_k^{(j)}$ are polynomials in the coefficients of those monomials in $\mathscr{X}$ and of $v_j(x)$ of degrees less than $|k|$.

$\mathscr{X}$ has d functionally independent first integrals,

$\Downarrow$ by statement (a)

$$w_k^{(j)} \equiv 0 \qquad \text{for all } k \in \mathfrak{R} \text{ and } j \in \{1, \ldots, d\}.$$

$\Downarrow$

$w_k^{(j)}$ are independent of the resonant monomials of $v_j(x)$.

$\Downarrow$

We can choose $v_j(x)$ without resonant monomials.

$\Downarrow$

$\mathscr{X}$ has d functionally independent first integrals

$$V_j(x) = x^{\alpha_j} + v_j(x), \qquad j = 1, \ldots, d,$$

with $v_j(x)$ consisting of nonresonant monomials.

$\Downarrow$

Statement (b) and consequently the theorem follows. $\square$

Proof of Theorem 2

Recall Theorem 2

Assume: $\lambda' \neq 0$, and the resonance of λ depends only on λ'.

If $\mathscr{X}^*$ has $k-1$ functionally independent analytic first integrals,

then

- $\mathfrak{R}_{\lambda'}$ has exactly $k-1$ $\mathbb{Q}_+$–linearly independent elements,
- the vector field $\mathscr{X}^*$ is analytically equivalent to its PD normal form on invariant manifold

$$\begin{aligned} \dot{y}' &= \mathrm{diag}(\lambda_1, \ldots, \lambda_k) y' (1 + q(y')) + g^*(y', y''), \\ \dot{y}'' &= A'' y'' + g''(y', y''), \end{aligned} \tag{19}$$

with $g^*(y', 0) = 0$, $g''(y', 0) = 0$, monomials of $q(y')$ resonant.

Its proof needs normal form on invariant manifold, see Bibikov [LNM1979]

Theorem C

If for $q' \in \mathbb{Z}_+^k$ with $|q'| \geq 2$ the following holds

$$\langle q', \lambda' \rangle - \lambda_\ell \neq 0, \quad \ell \in \{k+1, \ldots, n\},$$

then there exists a formal change of coordinates

$$x = \begin{pmatrix} x' \\ x'' \end{pmatrix} = H(y) = y + h(y') = \begin{pmatrix} y' + h'(y') \\ y'' + h''(y') \end{pmatrix}, \qquad (20)$$

which sends system (7) to its PD normal form on invariant manifold, where

- nonresonant monomials of $h'(y')$ and monomials of $h''(y')$ are uniquely determined,
- resonant monomials in $h'(y')$ can be arbitrary.

Convergence of transformation in Theorem C.

Theorem D

Assume that

- $\langle q', \lambda' \rangle - \lambda_\ell \neq 0$ for $q' \in \mathbb{Z}_+^k$ with $|q'| \geq 2$ and $\ell \in \{k+1, \ldots, n\}$;
- $\exists\ \varepsilon > 0$ such that for any nonresonant monomials $y'^{q'} e_j$ of the normalization (20) from system (7) to its normal form on invariant manifold

$$\dot{y}' = A'y' + g'(y', y''), \quad \dot{y}'' = A''y'' + g''(y', y''),$$

  it holds that $|\langle q', \lambda' \rangle - \lambda_k| > \varepsilon, \quad k \in \{1, \ldots, n\}$.
- $A'y' + g'(y', 0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)y'(1 + q(y'))$.

If system (7) is analytic, then the normalization (20) is analytic.

Proof of Theorem 2.

Step 1: proving that the partially integrable system (7) is formally equivalent to its PD normal form on invariant manifold

$$\dot{y}' = A'y' + g'(y', y''), \quad \dot{y}'' = A''y'' + g''(y', y''), \tag{21}$$

via the transformation (20)

By Theorem C, we only need to prove

$$\langle q', \lambda' \rangle \neq \lambda_j, \quad \text{for } q' \in \mathbb{Z}_+^k,\ |q'| \geq 2, \text{ and } j \in \{k+1, \ldots, n\}. \tag{22}$$

By contrary, if $\exists\ q_0' \in \mathbb{Z}_+^k$ with $|q_0'| \geq 2$ and $\ell_0 \in \{k+1, \ldots, n\}$ such that

$$\lambda_{\ell_0} = \langle q_0', \lambda' \rangle.$$

Since $\mathfrak{R}_{\lambda'}$ has elements with its jth component nonvanishing for $j \in \{1, \ldots, k\}$,

$\Downarrow$

$\exists\, q' \in \mathfrak{R}_{\lambda'}$ with $q'_0 \prec q'$ such that

$$\langle q', \lambda' \rangle = 0.$$

Separate $q' = q'_0 + \widetilde{q}'$, we have

$$0 = \langle q', \lambda' \rangle = \langle \widetilde{q}', \lambda' \rangle + \lambda_{\ell_0}$$

for some $\widetilde{q}' \in \mathbb{Z}_+^k$ with $|\widetilde{q}'| \geq 1$, a contradiction.

$\Downarrow$

The (22) is verified. By Theorem C, step 1 is down.

Step 2: Proving that the partially integrable systems (7) has the special normal form given in Theorem 2.

System (7) has $k-1$ functionally independent analytic first integrals by assumption

$\Downarrow$

Its normal form system (21) has $k-1$ functionally independent formal first integrals.

$\Downarrow$ by Theorem 1

System (7) has the first integrals of the form

$$V_1(x) = x^{\alpha_1} + v_1(x), \ldots, V_{k-1}(x) = x^{\alpha_{k-1}} + v_{k-1}(x),$$

with $\alpha_j \in \mathfrak{R}_\lambda$ for $j = 1, \ldots, k-1$ $\mathbb{Q}_+$–linearly independent, and $v_j(x) = o(|x|^{|\alpha_j|})$.

Let $\alpha_j = (\alpha_j', \alpha_j'')$ according to the decomposition of λ.

$\Downarrow$

$\alpha_j'' = 0$, and so

$$x^{\alpha_j} = x'^{\alpha_j'}, \qquad j = 1, \ldots, k-1.$$

$\Downarrow$ by transformation (20) near identity

System (21) has the functionally independent first integrals

$$W_j(y) := V_j \circ H(y) = y'^{\alpha_j'} + w_j(y', y''), \qquad j = 1, \ldots, k-1.$$

$\Downarrow$ by $W_1(y), \ldots, W_{k-1}(y)$ functionally independent

$\Downarrow$ by $y'^{\alpha_1'}, \ldots, y'^{\alpha_{k-1}'}$ functionally independent

$W_1(y', 0), \ldots, W_{k-1}(y', 0)$ are functionally independent.

The above proofs show that

$$F_1(y') := W_1(y',0), \ldots, F_{k-1}(y') := W_{k-1}(y',0)$$

are functionally independent first integrals of system (21)$|_{y''=0}$, i.e.

$$\dot{y}' = A'y' + g'(y',0), \tag{23}$$

which is in th PD normal form

$\Downarrow$ By Zhang [JDE2008]

First integrals $F_1(y'), \ldots, F_{k-1}(y')$ of (23) are resonant.

$\Downarrow$

$\mathscr{Y}_s(F_j(y')) = 0$ for $j = 1, \ldots, k-1$, where

$$\mathscr{Y}_s = \lambda_1 y_1 \frac{\partial}{\partial y_1} + \ldots + \lambda_k y_k \frac{\partial}{\partial y_k}.$$

This shows that

- the $k-1$ linearly independent gradient vector fields $\nabla F_1(y'), \ldots, \nabla F_{k-1}(y')$ are orthogonal to both vector fields $\mathscr{Y}_s$ and $\mathscr{Y}'^*$ associated to (23).

$\Downarrow$ in k dimensional space

$\mathscr{Y}_s$ and $\mathscr{Y}'^*$ must be parallel. That is

$$\mathscr{Y}'^* = (1+q(y'))\mathscr{Y}_s,$$

with $q(y')$ an analytic function or a formal series

$\Downarrow$

$A' = \mathrm{diag}(\lambda_1, \ldots, \lambda_k)$.

$\Downarrow$

The normal form system (21) has the desired form.

Step 3. Convergence of the normalization

By assumption $\Rightarrow \exists\ k-1\ \mathbb{Q}_+$–linearly independent elements $\alpha_j' = (\alpha_{j1},\ldots,\alpha_{jk}) \in \mathfrak{R}_{\lambda'}$ for $j=1,\ldots,k-1$ such that

$$\alpha_{j1}\lambda_1 + \ldots + \alpha_{jk}\lambda_k = 0, \quad j=1,\ldots,k-1.$$

$\Downarrow$ without loss of generality

$$\lambda_s = \frac{p_s}{p}\lambda_1, \quad s=2,\ldots,k, \tag{24}$$

with $p\in\mathbb{N}$ and $p_s\in\mathbb{Z}_+$ for $s=2,\ldots,k$.

$\Downarrow$

$\lambda_1\neq 0$, and $\exists\ \sigma>0$ such that if

$$\langle q',\lambda'\rangle - \lambda_\ell \neq 0, \text{ for } q'=(q_1,\ldots,q_k)\in\mathbb{Z}_+^k, \ell\in\{1,\ldots,n\},$$

then

$$|\langle q',\lambda'\rangle - \lambda_\ell| \geq \sigma, \quad q'\in\mathbb{Z}_+^k,\ \ell\in\{1,\ldots,n\}.$$

This follows from

$$|\langle q',\lambda'\rangle-\lambda_\ell|=\left|\frac{(q_1p+q_2p_2+\ldots+q_kp_k)\lambda_1-p\lambda_\ell}{p}\right|. \qquad (25)$$

and some calculations.

$\Downarrow$ by the three steps, and Theorem D

The normalization (20) is convergence.

$\Downarrow$

The theorem is proved. $\square$

谢　谢!

Thanks for your attention!