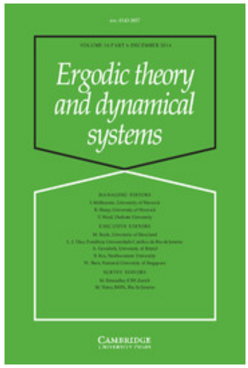

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Algebraic geometry of the center-focus problem for Abel differential equations

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Abstract. The Abel differential equation $y' = p(x)y^3 + q(x)y^2$ with polynomial coefficients p, q is said to have a center on $[a, b]$ if all its solutions, with the initial value $y(a)$ small enough, satisfy the condition $y(a) = y(b)$. The problem of giving conditions on (p, q, a, b) implying a center for the Abel equation is analogous to the classical Poincaré center-focus problem for plane vector fields. Center conditions are provided by an infinite system of 'center equations'. During the last two decades, important new information on these equations has been obtained via a detailed analysis of two related structures: composition algebra and moment equations (first-order approximation of the center ones). Recently, one of the basic open questions in this direction—the 'polynomial moments problem'—has been completely settled in Pakovich and Muzychuk [Solution of the polynomial moment problem. *Proc. Lond. Math. Soc. (3)* **99**(3) (2009), 633–657] and Pakovich [Generalized 'second Ritt theorem' and explicit solution of the polynomial moment problem. *Compositio Math.* **149** (2013), 705–728]. In this paper, we present a progress in the following two main directions: first, we translate the results of Pakovich and Muzychuk [Solution of the polynomial moment problem. *Proc. Lond. Math. Soc. (3)* **99**(3) (2009), 633–657] and Pakovich [Generalized 'second Ritt theorem' and explicit solution of the polynomial moment problem. *Compositio Math.* **149** (2013), 705–728] into the language of algebraic geometry of the center equations. Applying these new tools, we show that the center conditions can be described in terms of composition algebra, up to a 'small' correction. In particular, we significantly extend the results of Briskin, Roytvarf and Yomdin [Center conditions at infinity for Abel differential equations. *Ann. of Math. (2)* **172**(1) (2010), 437–483].

Second, applying these tools in combination with explicit computations, we start in this paper the study of the 'second Melnikov coefficients' (second-order approximation of the center equations), showing that in many cases vanishing of the moments and of these coefficients is sufficient in order to completely characterize centers.

1. *Introduction*

In this paper we consider the Abel differential equation

$$y' = p(x)y^3 + q(x)y^2, \tag{1.1}$$

with polynomial coefficients p, q, on a complex segment $[a, b]$. A solution $y(x)$ of (1.1) is called 'closed' on $[a, b]$ if $y(a) = y(b)$ for the initial element of $y(x)$ around a analytically continued to b along $[a, b]$. Equation (1.1) is said to have a center on $[a, b]$ if any of its solutions $y(x)$, with the initial value $y(a)$ small enough, are closed on $[a, b]$. For p, q polynomials this property depends only on the end points $a, b \in \mathbb{C}$, but not on the continuation path.

Below we shall denote by P, Q the primitives $P(x) = \int_a^x p(\tau)\,d\tau$ and $Q(x) = \int_a^x q(\tau)\,d\tau$.

The center-focus problem for the polynomial Abel equation is to give an explicit, in terms of the coefficients of p and q, necessary and sufficient condition on p, q, a, b for (1.1) to have a center on $[a, b]$. The Smale–Pugh problem is to bound the number of isolated closed solutions of (1.1). While we restrict ourselves to the polynomial case only, there are other important settings of these problems, in particular, with p, q trigonometric polynomials, piecewise-linear or even discontinuous piecewise-constant functions (compare [**1**, **4**, **6**, **11**, **12**, **15**–**17**, **21**–**23**]). The relation of the above problems to the classical Hilbert 16th and Poincaré center-focus problems for plane vector fields is well known (see e.g. [**10**, **14**, **25**, **26**]).

Algebraic Geometry enters the above problems from the very beginning: it is well known that center conditions are given by an infinite system of polynomial equations in the coefficients of p, q, expressed as certain iterated integrals of p, q ('center equations'; see §3 below). The structure of the ideal generated by these equations in an appropriate ring (called the Bautin ideal), specifically, the number of its generators, determines local bifurcations of the closed solutions as p, q vary.

One of the main difficulties in the center-focus and the Smale–Pugh problems is that a general algebraic–geometric analysis of the system of center equations is very difficult because of their complexity and absence of apparent general patterns.

In recent years the following two important algebraic–analytic structures, deeply related to the center equations for (1.1), have been discovered: composition algebra of polynomials and generalized polynomial moments of the form $m_k = \int_a^b P^k(x)q(x)\,dx$ (the last one is a special case of iterated integrals). The use of these structures provides important tools for investigation of the center-focus problem for the Abel equation (see [**1**–**17**, **20**–**23**, **29**] and references therein). In particular, it was shown in [**10**] that center equations are well approximated by the moment equations $m_k = \int_a^b P^k(x)q(x)\,dx = 0$ and in fact coincide with them 'at infinity'. Moment equations, in turn, impose (in many cases) strong restrictions on P and Q, considered as elements of the composition algebra of polynomials (see §4 below). Notice that usually *linear* moment equations $m_k = 0$ are considered, where P is fixed while Q is the unknown. However, consideration of center equations at infinity in [**10**, **20**] and in the present paper leads to a *nonlinear* setting where Q is fixed, while the equations have to be solved with respect to the unknown P.

The following composition condition imposed on P and Q plays a central role in the study of the moment and center equations (see the references above): there exist polynomials $\widetilde{P}$, $\widetilde{Q}$ and W with $W(a) = W(b)$ such that

$$P(x) = \widetilde{P}(W(x)), \quad Q(x) = \widetilde{Q}(W(x)).$$

Being a kind of 'integrability condition', the composition condition implies vanishing of center and moment equations as well as of all the iterated integrals entering the center equations. It is the only *sufficient* center condition known to us for the polynomial Abel equation. Using the interrelation between center and moment equations at infinity, and the composition condition, a rather accurate description of the affine center set for the polynomial Abel equation has been given in [**10**]. Very recently, important results relating center and composition conditions for trigonometric and polynomial Abel equations have been obtained in [**9**, **15**–**17**, **21**–**23**].

These results, as well as some further examples and partial results (see [**3**, **7**, **10**–**13**, **16**, **20**] for the most recent contributions), seem to support the following 'composition conjecture'.

CONJECTURE 1. *The center and composition sets for any polynomial Abel equation coincide.*

This conjecture was originally suggested in [**8**, Conjecture 1.6], together with its extended versions [**8**, Conjectures 1.7 and 1.8], which all remain open. A similar conjecture is known to be false for p, q trigonometric polynomials and $a, b \in \mathbb{R}$ (see [**6**]). However, besides various special cases of polynomial Abel equations, described in papers mentioned above, as well as in [**21**, **27**] and in other publications, an equivalence of the center and (the appropriate) composition conditions holds, for example, for piecewise-constant p and q of a certain special form ('rectangular paths' [**12**], see also [**4**]). As was shown in [**12**], for 'rectangular paths' the equivalence of the center and composition conditions follows from a highly non-trivial result of [**18**], stating (roughly) that the group of transformations of $\mathbb{R}$ generated by translations and positive rational powers is free.

Part of the methods developed in the present paper can be applied to arbitrary coefficients p, q of the Abel equation (1.1). This certainly concerns all the constructions in §2.1 below. In particular, we can apply our methods to p, q trigonometric polynomials, Laurent polynomials or rational functions. The problem is that in the case of rational p, q the consequences of the moments vanishing are much weaker than in the polynomial case, while the presentation is technically much more involved (see [**1**, **34**] and references therein). The same is true for the description of the composition algebra of rational functions, which turns out to be significantly more complicated than for polynomials (compare [**1**, **15**, **16**, **32**, **36**]). So, in the present paper, we restrict ourselves to the polynomial case only. We plan to present our results for rational and trigonometric cases separately.

Now, in [**33**, **35**] essentially a complete description of the *polynomial* moments vanishing has been achieved, as well as of the relevant *polynomial* composition algebra. In particular, explicit necessary and sufficient conditions for vanishing of all the moments m_k have been given there, in terms of certain relations between P, Q, a, b in the composition algebra of polynomials (see §4 below).

Accordingly, one of the main goals of the present paper is to give an algebraic–geometric interpretation of the results of [**31**, **33**, **35**] in the context of the center-focus problem for the polynomial Abel equation, and to apply these results to the study of center conditions. Here we heavily use the fact, found in [**10**], that the moment equations are the restrictions, in a proper 'projective setting', of the center equations to the infinite hyperplane. On this basis we obtain new information on the affine center conditions, significantly extending the results of [**10**].

Another main goal of this paper is to start the investigation of the 'second Melnikov coefficients', which form the second set of the center equations 'at infinity'. We show that in many important cases *vanishing of the moments and of the second Melnikov coefficients implies composition*, and so it is sufficient in order to completely characterize centers.

1.1. *Statement of the main results.* A general form of the results in this paper is the following: as was explained above, the composition set is always a subset of the center set. We show that the composition condition is indeed a good approximation to the center condition, showing that the dimension of the (possibly existing) non-composition components in the center set is small. In various circumstances we provide an upper bound for the dimension of these possible non-composition components, which is significantly smaller than the dimension of the composition center strata. In many cases this bound is zero, so the center set coincides with the composition set up to a finite number of points. The following theorems summarize our main new results on the center configurations for the polynomial Abel equation (1.1). Since there is a one-to-one correspondence between pairs of polynomials p, q and pairs of their primitives P, Q defined above, we shall formulate all our results in terms of P and Q. Below we always assume that Q with $Q(a) = Q(b) = 0$ is fixed, while P varies in the space $\mathcal{P}_d$ of all the polynomials of degree up to d vanishing at a and b.

Let us start with a description of the composition set $\mathrm{COS}_{d,Q}$ of all the polynomials P in $\mathcal{P}_d$, such that P and Q satisfy the composition condition.

THEOREM 1.1. *For $V \subset \mathcal{P}_d$ and for any polynomial Q of degree at most five, the composition set $\mathrm{COS}_{d,Q}$ is a linear subspace in $\mathcal{P}_d$ of dimension at most $[d/2]$. For $6 \le \deg Q \le 89$, the set $\mathrm{COS}_{d,Q}$ is a union of at most two linear subspaces in $\mathcal{P}_d$ and, for $\deg Q \ge 90$, the set $\mathrm{COS}_{d,Q}$ is a union of at most three linear subspaces. The dimension of each of these subspaces is at most $[d/2]$; their double and triple intersections have dimensions at most $[d/6] + 1$ and $[d/90]$, respectively.*

The rest of our results bound the dimension of the non-composition components, i.e. those which are not contained in $\mathrm{COS}_{d,Q}$ (if they exist).

THEOREM 1.2. *Consider equation (1.1) with Q fixed and P varying in the space $\mathcal{P}_d$ of all the polynomials of degree up to d vanishing at a and b. Then the dimension of the non-composition components of the center set of (1.1), if they exist, does not exceed $[d/6] + 2$. In particular, this dimension is of order at most one-third of the maximal dimension of the composition center strata (which is of order $d/2$, being achieved on the composition strata with the right factor $W(x) = (x - a)(x - b)$).*

The main steps in the proof of Theorem 1.2 are the following: we consider the projective compactification $P\mathcal{P}_d$ of $\mathcal{P}_d$ and use the fact, proved in [**10**], that the center equations 'at infinity' become the moment equations. Therefore, to bound the dimensions of the affine non-composition components of the center set CS in the *complex* affine space $\mathcal{P}_d$, it is enough to bound the dimensions of the non-composition components of the moment vanishing set MS 'at infinity' in $P\mathcal{P}_d$. We show that these dimensions do not exceed $[d/6] + 2$, using a complete description of the moment vanishing conditions, obtained in [**33**].

More accurately, we define the set ND of 'non-definite' polynomials which provide non-composition solutions to the moment equations, and bound from above its dimension. Then the following theorem describes an inclusion structure at infinity of the sets we are interested in.

THEOREM 1.3. *For an algebraic set $Y \subset \mathcal{P}_d$, let $\bar{Y}$ denote the intersection of Y with the infinite hyperplane of $P\mathcal{P}_d$. Then, for each irreducible non-composition component A of the affine central set* CS, *we have* $\bar{A} \subset \overline{\mathrm{CS}} \cap \mathrm{ND} \subset \overline{\mathrm{MS}} \cap \mathrm{ND}$. *Consequently,* $\dim A \leq \dim(\overline{\mathrm{MS}} \cap \mathrm{ND}) + 1$.

In many specific cases, Theorem 1.3 allows us to improve the general bound provided by Theorem 1.2. In order to formulate corresponding results, it is convenient to normalize points a and b to be the points $-\frac{\sqrt{3}}{2}$ and $\frac{\sqrt{3}}{2}$, respectively. Further, let $\mathcal{S} \subset \mathcal{P}$ be a subset of all polynomials $Q \in \mathcal{P}$ representable as a sum $Q = S_1(T_2) + S_2(T_3)$, where S_1, S_2 are arbitrary polynomials, while T_2, T_3 are the Chebyshev polynomials of degrees two and three, respectively (notice that the normalization of the interval $[a, b]$ is chosen in such a way that $T_2(a) = T_2(b)$, $T_3(a) = T_3(b)$). Below we show that the dimension of $\mathcal{S} \cap \mathcal{P}_d$ does not exceed $[\frac{2}{3}d] + 1$, so 'most' of the polynomials Q of degree d cannot be represented in the above form.

THEOREM 1.4. *Let P vary in the space $\mathcal{P}_9$. Then, for each fixed $Q \in \mathcal{P} \backslash \mathcal{S}$, the center set of (1.1) consists of a composition set and possibly a finite set of additional points. For an arbitrary fixed Q, the dimension of the non-composition components of the center set of (1.1) in $\mathcal{P}_9$ does not exceed one. For P varying in the space $\mathcal{P}_{11}$ and for an arbitrary fixed Q, the dimension of the non-composition components of the center set of (1.1) does not exceed two.*

The next result heavily relies on computations with the second Melnikov coefficients.

THEOREM 1.5. *Let P vary in the space $\mathcal{P}_9$. Then, for each fixed $Q \in \mathcal{S} \cap \mathcal{P}_9$, which is not a polynomial in T_2 or T_3, the center set of (1.1) consists of a composition set and possibly a finite set of additional points.*

Our last result (Theorem 6.6 in §6 below) concerns the center set in subspaces of polynomials with a special structure. Here we formulate its important particular case. Let U_d consist of all polynomials $P \in \mathcal{P}_d$ such that the degrees of x, appearing in P with the non-zero coefficients, are powers of prime numbers.

THEOREM 1.6. *Let P vary in U_d. Then, for any fixed Q, the center set of (1.1) in U_d consists of a composition set and possibly a finite set of additional points.*

2. *Preliminaries: Poincaré mapping, center equations and composition condition*

2.1. *Poincaré mapping and center equations.* Both the center-focus and the Smale–Pugh problems can be naturally expressed in terms of the Poincaré 'first return' mapping $y_b = G_{[a,b]}(y_a)$ along $[a, b]$. Let $y(x, y_a)$ denote the element around a of the solution $y(x)$ of (1.1) satisfying $y(a) = y_a$. The Poincaré mapping $G_{[a,b]}$ associates to each initial value y_a at a, sufficiently close to zero, the value y_b at b of the solution $y(x, y_a)$ analytically continued along $[a, b]$.

According to the definition above, the solution $y(x, y_a)$ is closed on $[a, b]$ if and only if $G_{[a,b]}(y_a) = y_a$. Therefore, closed solutions correspond to the fixed points of $G_{[a,b]}$, and (1.1) has a center if and only if $G_{[a,b]}(y) \equiv y$. It is well known that $G_{[a,b]}(y)$ for small y is given by a convergent power series

$$G_{[a,b]}(y) = y + \sum_{k=2}^{\infty} v_k(p, q, a, b) y^k. \tag{2.1}$$

Therefore, the center condition $G_{[a,b]}(y) \equiv y$ is equivalent to an infinite sequence of algebraic equations in p and q:

$$v_k(p, q, a, b) = 0, \quad k = 2, 3, \ldots. \tag{2.2}$$

Each $v_k(p, q, a, b)$ can be expressed as a linear combination of certain iterated integrals of p and q along $[a, b]$ (see [**10**] and Theorem 2.1 below).

2.2. *Projective setting and center equations at infinity over fixed Q.* Let $\mathcal{P} = \mathcal{P}_{[a,b]}$ be the vector space of all complex polynomials P satisfying $P(a) = P(b) = 0$, and $\mathcal{P}_d$ the subspace of $\mathcal{P}$ consisting of polynomials of degree at most d. We always shall assume that the polynomials

$$P(x) = \int_a^x p(\tau)\, d\tau, \quad Q(x) = \int_a^x q(\tau)\, d, \tag{2.3}$$

defined above, are elements of $\mathcal{P}$. This restriction is natural in the study of the center conditions, since it is forced by the first two of the center equations (2.2). Since (2.3) provides a one-to-one correspondence between (p, q) and (P, Q), which is an isomorphism of the corresponding vector spaces, in order to avoid cumbersome notation all the results below are formulated in terms of (P, Q).

We shall assume that the points $a \neq b$ are fixed, and usually shall omit a, b from the notation.

From now on we shall assume that $Q \in \mathcal{P}_{d_1}$ is fixed, while P varies in a certain linear subspace V of the space $\mathcal{P}_d$. This restrictive setting significantly simplifies the presentation, although it describes only 'slices' of the center set. The approach of [**10**] and of the present paper can be extended to the full coefficient space of $(P, Q) \in \mathcal{P}_d \times \mathcal{P}_{d_1}$. We consider this extension as an important research direction, but it significantly increases the complexity of the algebraic geometry involved, and is beyond the scope of the present paper. See [**10**] for a comparison of different possible settings of the problem.

Let a subspace $V \subset \mathcal{P}$ be given. We shall consider the projective space PV and the infinite hyperplane HV $\subset$ PV. To construct PV, we introduce an auxiliary variable $\nu \in \mathbb{C}$ and consider the couples (S, ν), $S \in V$, with (S, ν) and $(\lambda S, \lambda\nu)$ identified for any $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The infinite hyperplane HV is defined in PV by the equation $\nu = 0$.

Let us denote by $\hat{v}_k(p, q) = \hat{v}_k(p, q, a, b)$ the 'homogenization' of the center equations $v_k(P, Q, a, b) = 0$ with respect to the variable P. In other words, we multiply each term in v_k by an appropriate degree of an auxiliary variable ν to make v_k homogeneous.

Notice that the center equations can be considered in two ways: as polynomial equations in the coefficients of P, Q, or as symbolic equations, containing 'symbolic iterated integrals' of the form $\int p \int q \int q \ldots$ (which can be interpreted as poly-linear forms, i.e. polynomials, in the symbols p, q). Since each p, q is a linear form in its coefficients, the degrees of the polynomials in both interpretations are the same. Accordingly, the projective space PV and the homogeneous polynomials $\hat{v}_k(p, q) = 0$ can be treated symbolically, until the moment where we have to actually integrate and get the explicit answer.

We call 'center equations at infinity' the restrictions of the homogeneous center equations to the infinite hyperplane HV. They are obtained by putting $\nu = 0$ in the homogeneous equations described above. The following Theorem 2.1 provides a description of the center equations at infinity obtained in [**10**]. We take into account a different order of the polynomials p and q in the Abel equation (1.1) in the present paper and in [**10**].

THEOREM 2.1. [**10**] *For $k = 2, 4, \ldots$ even and $l = (k/2) - 1$, the center equations at infinity over Q are given by vanishing of the generalized moments*

$$v_k^\infty(P, Q) = m_l(P, Q) = \int_a^b P^l(x) q(x)\, dx = 0. \tag{2.4}$$

For k odd, the center equations at infinity over Q are given by vanishing of the coefficients of the 'second Melnikov function'

$$v_k^\infty(P, Q) = D_k(P, Q) = 0, \tag{2.5}$$

represented by integer linear combinations $\sum n_\alpha I_\alpha$, with the sum running over all the iterated integrals in p, q with exactly two appearances of q. Here $\alpha = (\alpha_1, \ldots, \alpha_s)$, with exactly two of α_j equal to 1, and the rest equal to 2, and with $\sum_{j=1}^s \alpha_j = k - 1$. The integrals I_α are defined as

$$I_\alpha = \int_a^b h_{\alpha_1}(x_1)\, dx_1 \left(\int_a^{x_1} h_{\alpha_2}(x_2)\, dx_2 \cdots \left(\int_a^{x_{s-1}} h_{\alpha_3}(x_s)\, dx_s \right) \cdots \right),$$

with $h_1 = q$, $h_2 = p$. The integer coefficients n_α are given as the products $n_\alpha = (-1)^s \prod_{r=1}^s (k - \sum_{j=1}^r \alpha_j)$.

In Proposition 6.1 below, the first four Melnikov equations at infinity $D_k(P, Q) = 0$ are given explicitly.

2.3. *Center, moment and composition sets.* Let us assume that $Q \in \mathcal{P}_{d_1}$ and a subspace $V \subset \mathcal{P}_d$ are fixed. We define the center set $\mathrm{CS} = \mathrm{CS}_{V,Q}$ as the set of $P \in V$ for which equation (1.1) has a center. Equivalently, CS is the set of $P \in V$ satisfying the center equations (2.2). The moment set $\mathrm{MS} = \mathrm{MS}_{V,Q}$ consists of $P \in V$ satisfying the moment equations (2.4).

To introduce the composition set $\mathrm{COS} = \mathrm{COS}_{V,Q}$, we recall the polynomial composition condition defined in [**8**], which is a special case of the general composition condition introduced in [**6**] (for brevity, below we shall use the abbreviation 'CC' for the 'composition condition').

Definition 2.1. Polynomials P, Q are said to satisfy the 'composition condition' on $[a, b]$ if there exist polynomials $\widetilde{P}$, $\widetilde{Q}$ and W with $W(a) = W(b)$ such that P and Q are representable as

$$P(x) = \widetilde{P}(W(x)), \quad Q(x) = \widetilde{Q}(W(x)).$$

The composition set $\mathrm{COS}_{V,Q}$ consists of all $P \in V$ for which P and Q satisfy the composition condition.

It is easy to see that the composition condition implies a center for (1.1), as well as the vanishing of each of the moments and iterated integrals above. So, we have $\mathrm{COS} \subset \mathrm{CS}$, $\mathrm{COS} \subset \mathrm{MS}$.

Define $\overline{\mathrm{CS}}$, $\overline{\mathrm{MS}}$, $\overline{\mathrm{COS}}$ as the intersections of the corresponding affine sets with the infinite hyperplane HV. It follows directly from Theorem 2.1 that the following statement is true.

PROPOSITION 2.1. *We have* $\overline{\mathrm{COS}} \subset \overline{\mathrm{CS}} \subset \overline{\mathrm{MS}}$.

Notice that COS and MS are homogeneous and hence these sets are cones over $\overline{\mathrm{MS}}$, $\overline{\mathrm{COS}}$. However, CS *a priori* may not be homogeneous, and the connection of the affine part CS to $\overline{\mathrm{CS}}$ may be more complicated.

Our main goal will be to compare the affine center set CS with the composition set COS. For this purpose, we shall bound the dimension of the affine non-composition components of CS, analyzing their possible behavior at infinity (§§5 and 6). To obtain these bounds, we first describe the geometry of the composition set COS (§3) and compare the moment set MS and its subset COS (§4).

3. *The structure of the composition set*

The geometry of the composition set reflects the algebraic structure of polynomial compositions, which is well known to provide rather subtle phenomena. In comparison with the classical theory developed by Ritt [**38**], we are interested in what we call below $[a, b]$-compositions, i.e. compositions of polynomials under the requirement that some of the factors take equal values at the points a and b.

3.1. *Elements of Ritt's theory.* Let us recall first some basic facts on polynomial composition algebra, including the classical first and second Ritt theorems [**38**].

Definition 3.1. A polynomial P is called indecomposable if it cannot be represented as $P(x) = R \circ S(x) = R(S(x))$ for polynomials R and S of degree greater than one. A decomposition $P = P_1 \circ P_2 \circ \cdots \circ P_r$ is called maximal if all $P_1, \ldots, P_r$ are indecomposable and of degree greater than one. Two decompositions $P = P_1 \circ P_2 \circ \cdots \circ P_r$ and $P = Q_1 \circ Q_2 \circ \cdots \circ Q_r$, maximal or not, are called equivalent (notation '$\sim$') if there exist polynomials of degree one, μ_i, $i = 1, \ldots, r-1$, such that $P_1 = Q_1 \circ \mu_1$, $P_i = \mu_{i-1}^{-1} \circ Q_i \circ \mu_i$, $i = 2, \ldots, r-1$, and $P_r = \mu_{r-1}^{-1} \circ Q_r$.

The first Ritt theorem [**38**] states that any two maximal decompositions of a polynomial P have an equal number of terms, and can be obtained from one another by a sequence of transformations replacing two successive terms $A \circ C$ with $B \circ D$, such that

$$A \circ C = B \circ D. \tag{3.1}$$

Let us mention that decompositions of a polynomial P into a composition of two polynomials, up to equivalence, corresponds in a one-to-one way to imprimitivity systems of the monodromy group G_P of P (see e.g. [**38**] or [**32**]). In their turn, imprimitivity systems of G_P are in a one-to-one correspondence with subgroups A of G_P containing the stabilizer G_ω of a point $\omega \in G$. In particular, for a given polynomial P, the number of its right composition factors W, up to the change $W \to \lambda \circ W$, where λ is a polynomial of degree one, is finite. Below we shall call (with a slight abuse of notation) two right composition factors W and $\lambda \circ W$ of P, where λ is a polynomial of degree one, equivalent, and write $W \sim \lambda \circ W$. We also usually shall write just 'right factor' of P instead of 'compositional right factor'.

The first Ritt theorem reduces the description of maximal decompositions of polynomials to the description of indecomposable polynomial solutions of the equation (3.1). It is convenient to start with the following result [**19**]: if polynomials A, B, C, D satisfy (3.1), then there exist polynomials $U, V, \hat{A}, \hat{B}, \hat{C}, \hat{D}$, where

$$\deg U = \mathrm{GCD}(\deg A, \deg B), \quad \deg V = \mathrm{GCD}(\deg C, \deg D), \tag{3.2}$$

such that

$$A = U \circ \hat{A}, \quad B = U \circ \hat{B}, \quad C = \hat{C} \circ V, \quad D = \hat{D} \circ V \tag{3.3}$$

and

$$\hat{A} \circ \hat{C} = \hat{B} \circ \hat{D}. \tag{3.4}$$

In particular, if $\deg A = \deg B$, then necessarily $A \circ C$ and $B \circ D$ are equivalent as decompositions. More generally, if $\deg B | \deg A$, then there exists a polynomial W such that the equalities

$$A = B \circ W, \quad D = W \circ C$$

are satisfied.

Note that the above result concerning the reduction of (3.1) to (3.4) is equivalent to the statement that the lattice of imprimitivity systems of the monodromy group G of a polynomial P of degree n is isomorphic to a sublattice of the lattice L_n consisting of all divisors of n, where by definition

$$d_1 \wedge d_2 = \mathrm{GCD}(d_1, d_2), \quad d_1 \vee d_2 = \mathrm{LCM}(\mathrm{d}_1, \mathrm{d}_2)$$

(see [**28**]). For example, for the polynomials z^n the corresponding lattices consist of all divisors of n, since, for any $d|n$, the equality $z^n = z^d \circ z^{n/d}$ holds. The same is true for the Chebyshev polynomials T_n, since the equality $T_n(\cos\phi) = \cos n\phi$ implies that $T_n = T_d \circ T_{n/d}$ for any $d|n$. On the other hand, for an indecomposable polynomial P, the corresponding lattice contains only elements 1 and n.

The second Ritt theorem [**38**] states that if A, B, C, D satisfy (3.1) and degrees of A and B as well as of C and D are coprime, then there exist linear polynomials U, V such that (3.3) and (3.4) hold and, up to a possible replacement of $\hat{A}$ by $\hat{B}$ and $\hat{C}$ by $\hat{D}$, either

$$\hat{A} \circ \hat{C} \sim z^n \circ z^r R(z^n), \qquad \hat{B} \circ \hat{D} \sim z^r R^n(z) \circ z^n, \tag{3.5}$$

where $R(z)$ is a polynomial, $r \geq 0$, $n \geq 1$ and $\mathrm{GCD}(n, r) = 1$ or

$$\hat{A} \circ \hat{C} \sim T_n \circ T_m, \qquad \hat{B} \circ \hat{D} \sim T_m \circ T_n, \tag{3.6}$$

where T_n and T_m are the Chebyshev polynomials, $n, m \geq 1$ and $\mathrm{GCD}(n, m) = 1$. In particular, this holds when A, B, C, D solving (3.1) are indecomposable, and the decompositions $A \circ C$ and $B \circ D$ are non-equivalent, since in this case the degrees of polynomials U, V in (3.2) and (3.3) are necessarily equal to one.

Clearly, the second Ritt theorem together with the previous result imply the following statement: if A, B, C, D satisfy (3.1), then there exist polynomials U, V such that (3.2), (3.3) and (3.4) hold and, up to a possible replacement of $\hat{A}$ by $\hat{B}$ and $\hat{C}$ by $\hat{D}$, either (3.5) or (3.6) holds.

3.2. $[a, b]$-*compositions.* Now we return to $[a, b]$-compositions, i.e. compositions of polynomials under the requirement that some of the right factors take equal values at two distinct points a and b.

Definition 3.2. Let a polynomial P satisfying $P(a) = P(b)$ be given. We call a polynomial W a right $[a, b]$-factor of P if $P = \widetilde{P} \circ W$ for some polynomial $\widetilde{P}$ and $W(a) = W(b)$. A polynomial P is called $[a, b]$-indecomposable if $P(a) = P(b)$ and P does not have right $[a, b]$-factors non-equivalent to P itself.

Remark. Notice that any right $[a, b]$-factor of P necessarily has degree greater than one, and that an $[a, b]$-indecomposable P may be decomposable in the usual sense.

PROPOSITION 3.1. *Any polynomial P up to equivalence has a finite number of $[a, b]$-indecomposable right factors W_j, $j = 1, \ldots, s$. Furthermore, each right $[a, b]$-factor W of P can be represented as $W = \widetilde{W}(W_j)$ for some polynomial $\widetilde{W}$ and $j = 1, \ldots, s$.*

Proof. As was mentioned above, up to equivalence there are only finitely many general right factors W of P. In particular, this is true for $[a, b]$-indecomposable right $[a, b]$-factors W_j of P.

Now let W be a right $[a, b]$-factor of P. If it is $[a, b]$-indecomposable, then, by the first part of the proposition, $W = \lambda \circ W_j$ for some $j = 1, \ldots, s$, with λ a linear polynomial. Otherwise, W can be represented as $W = V \circ \hat{W}$, where $\hat{W}$ is a right $[a, b]$-factor of P and $\deg V > 1$. Since $\deg \hat{W} < \deg W$, it is clear that continuing this process we ultimately will find an $[a, b]$-indecomposable right factor W_j of P such that $W = \widetilde{W}(W_j)$. □

An easy consequence of Proposition 3.1 is the following description of the composition set given in [**10**].

PROPOSITION 3.2. *Let W_j, $j = 1, \ldots, s$, be all indecomposable right $[a, b]$-factors of Q. Then the set $\mathrm{COS}_{V,Q}$ is a union of the linear subspaces $L_j \subset V$, $j = 1, \ldots, s$, where L_j consists of all the polynomials $P \in V$ representable as $P = \widetilde{P}(W_j)$, $j = 1, \ldots, s$ for a certain polynomial $\widetilde{P}$.*

It has been recently shown in [**33**] that for any $P \in \mathcal{P}$, the number s of its non-equivalent $[a, b]$-indecomposable right factors can be at most three. Moreover, if $s > 1$, then these factors necessarily have a very special form, similar to what appears in Ritt's description above.

The precise statement is given by the following theorem [**33**, Theorem 5.3].

THEOREM 3.1. *Let complex numbers $a \neq b$ be given. Then, for any polynomial $P \in \mathcal{P}_{[a,b]}$, the number s of its $[a, b]$-indecomposable right factors W_j, up to equivalence, does not exceed three.*

Furthermore, if $s = 2$, then either

$$P = U \circ z^{rn} R^n(z^n) \circ U_1, \quad W_1 = z^n \circ U_1, \quad W_2 = z^r R(z^n) \circ U_1,$$

where R, U, U_1 are polynomials, $r > 0$, $n > 1$, $\mathrm{GCD}(n, r) = 1$ or

$$P = U \circ T_{nm} \circ U_1, \quad W_1 = T_n \circ U_1, \quad W_2 = T_m \circ U_1,$$

where U, U_1 are polynomials, $n, m > 1$, $\mathrm{GCD}(n, m) = 1$.

On the other hand, if $s = 3$, then

$$P = U \circ z^2 R^2(z^2) \circ T_{m_1 m_2} \circ U_1,$$

$$W_1 = T_{2m_1} \circ U_1, \quad W_2 = T_{2m_2} \circ U_1, \quad W_3 = z R(z^2) \circ T_{m_1 m_2} \circ U_1,$$

where R, U, U_1 are polynomials, $m_1, m_2 > 1$ are odd and $\mathrm{GCD}(m_1, m_2) = 1$.

Notice that in all the cases above $U_1(a) \neq U_1(b)$, while $W_j(a) = W_j(b)$.

We are interested in the stratification of the space $\mathcal{P}_d$ of polynomials P of degree d according to the structure of their $[a, b]$-indecomposable right $[a, b]$-factors. Following Theorem 3.1, let us use the following notation for the appropriate strata.

Definition 3.3. Let $\mathrm{DEC}_s^d(a, b) \subset \mathcal{P}_d$ denote the set of polynomials P of degree at most d satisfying $P(a) = P(b) = 0$ and possessing exactly s non-equivalent $[a, b]$-indecomposable right factors. For $s = 1$, we write $\mathrm{DEC}_1^d(a, b) = \mathrm{DEC}_{1,0}^d(a, b) \cup \mathrm{DEC}_{1,1}^d$ (a, b). Here $\mathrm{DEC}_{1,0}^d(a, b)$ consists of polynomials P for which their only indecomposable right factor W is equivalent to P. In turn, $\mathrm{DEC}_{1,1}^d(a, b)$ consists of P for which W is not equivalent to P and hence $\deg W < \deg P$.

As a first consequence of Theorem 3.1, we get upper bounds on the dimensions of the sets $\mathrm{DEC}_s^d(a, b)$ considered as subsets of the complex space $\mathbb{C}^{d-1}$, which we identify with $\mathcal{P}_d$.

PROPOSITION 3.3. *The set* $\mathrm{DEC}^d_{1,0}(a, b)$ *consists of* $[a, b]$*-indecomposable polynomials* $P \in \mathcal{P}_d$ *and its dimension is* $d - 1$. *We have* $\mathrm{DEC}^d_{1,1}(a, b) = \emptyset$ *for* $d \leq 3$, *and* $\dim \mathrm{DEC}^d_{1,1}(a, b) \leq [d/2]$ *for* $d \geq 4$. *Also,* $\mathrm{DEC}^d_2(a, b) = \emptyset$ *for* $d \leq 5$, *and* $\dim \mathrm{DEC}^d_2(a, b) \leq [d/6] + 1$ *for* $d \geq 6$. *And,* $\mathrm{DEC}^d_3(a, b) = \emptyset$ *for* $d \leq 89$, *and* $\dim \mathrm{DEC}^d_3(a, b) \leq [d/90]$ *for* $d \geq 90$.

Proof. Assume that we are given l parametric families of polynomials $\mathcal{S}_r = \{S_r(\tau_r, z)\}$, $r = 1, \ldots, l$, with $\tau_r \in \Sigma_r \subset \mathbb{C}^{n_r}$ being the parameters of $\mathcal{S}_r$. We assume that the degree of the polynomials $S_r(\tau_r, z)$ remains constant and equal to d_r for all the values of the parameters $\tau_r \in \Sigma_r$. Put $\tau = (\tau_1, \ldots, \tau_l)$ and let

$$P_\tau = S_1(\tau_1) \circ S_2(\tau_2) \circ \cdots \circ S_l(\tau_l).$$

The degree of the polynomials P_τ of this form is $d_1 \cdot \cdots \cdot d_l$ and they form a parametric family with the parameters $\tau = (\tau_1, \ldots \tau_l) \in \mathbb{C}^n$, where $n = n_1 + \cdots + n_l$.

The dimension D of the stratum S in $\mathcal{P}$ formed by the polynomials P_τ as above is at most n, and it may be strictly less than n, since the parametric representation as above may be redundant. The requirement that $P_\tau \in \mathcal{P}_d$ is equivalent to $d_1 \cdot \cdots \cdot d_l \leq d$.

So, in order to bound from above the dimensions of the strata $\mathrm{DEC}^d_s(a, b)$, we have to accurately estimate the number $D \leq n_1 + \cdots + n_l$ of free parameters, and the degrees $d_1, \ldots, d_l$ in composition representations of the corresponding polynomials P, provided by Theorem 3.1. We have to take into account the redundancy in the parametric representation, and then to maximize D under the constraint $d_1 \cdot \cdots \cdot d_l \leq d$.

Notice that $\mathcal{P}_d = \bigcup_{s=1}^3 \mathrm{DEC}^d_s(a, b)$. Let us now consider the sets $\mathrm{DEC}^d_s(a, b)$ for $s = 1, 2, 3$ case by case. We shall see below that all the strata $\mathrm{DEC}^d_s(a, b)$, besides the stratum $\mathrm{DEC}^d_{1,0}(a, b)$, consisting of $[a, b]$-indecomposable polynomials P, have dimension strictly smaller than $\dim \mathcal{P}_d = d - 1$. Hence, $\dim \mathrm{DEC}^d_{1,0}(a, b) = d - 1$. (This follows immediately also from the fact that $\mathrm{DEC}^d_{1,0}(a, b)$ consists of *generic* polynomials in $\mathcal{P}_d$.)

Now, each $P \in \mathrm{DEC}^d_{1,1}(a, b)$ has a form $P = S_1 \circ S_2$, with $\deg S_1 = d_1 > 1$, $\deg S_2 = d_2 > 1$, since we assume that P possesses a right $[a, b]$-factor S_2, not equivalent to P. In this case $d \geq d_1 d_2$ is at least four, and S_1 and S_2 can be any polynomials of degrees d_1 and d_2 with the only restrictions $S_2(a) = S_2(b)$ and $S_1(S_2(a)) = 0$. Hence, $n_1 = d_1$, $n_2 = d_2$. On the space $\mathbb{C}^{n_1+n_2}$ of the parameters of (S_1, S_2) acts a two-dimensional group Γ of linear polynomials γ. It acts by transforming (S_1, S_2) into $(S_1 \circ \gamma, \gamma^{-1} S_2)$. This action preserves P. Accordingly, we have to maximize $D = d_1 + d_2 - 2$ under the constraint $d_1 d_2 \leq d$. For d even, this maximum is achieved for $d_1 = 2$ or $d_2 = 2$ and it is $d/2$. For d odd, still $d_1 = 2$ or $d_2 = 2$, but the maximum of D is $(d - 1)/2$. Finally, we get $\dim \mathrm{DEC}^d_1(a, b) \leq [d/2]$.

Now let us consider the case $s = 2$. In this case, by Theorem 3.1, we have two options.

The first option is that $P = U \circ z^{rn} R^n(z^n) \circ U_1$, where $U(z), R(z), U_1(z)$ are polynomials, $r > 0$, $n > 1$ and $\mathrm{GCD}(n, r) = 1$, and z^n and $z^r R(z^n)$ take equal values at $U_1(a) \neq U_1(b)$.

Here, denoting the degrees of U, U_1, R by $k, m, l \geq 1$, respectively, we get $\deg P = k \cdot n(r + ln) \cdot m \geq 6$, while the number of the independent parameters, i.e. the dimension of the corresponding strata, is at most $k+l+m-1$ (we take into account the requirements $W_1(a)=W_1(b)$, $W_2(a)=W_2(b)$, $P(a)=P(b)=0$ and the fact that the scaling parameters of U and of R act equivalently on P). So, we have to maximize $k+l+m-1$ under the constraint $k\cdot n(r+ln)\cdot m\leq d$. The variables are integers $k\geq 1$, $l\geq 1$, $m\geq 1$, $r\geq 1$, $n\geq 2$, $\mathrm{GCD}(n,r)=1$.

Let us first fix l, r, n. As above, the maximum of $k+l+m-1$ is attained either for $k=1$, $m=[d/(n(r+ln))]$ or for $k=[d/(n(r+ln))]$, $m=1$. In both cases it is $l+[d/(n(r+ln))]$, and this expression increases as l decreases. So, we can put $l=1$ and so we get $[d/(n(r+n))]+1$. Once more, this expression increases as n, r (which do not enter the maximized sum) decrease. Their minimal possible values are $r=1$, $n=2$ and we get $k+l+m-1=[d/6]+1$.

The second option is that $P=U\circ T_{nm}\circ U_1$, with $n, m>1$, $\mathrm{GCD}(n,m)=1$, and T_m and T_n take equal values at $U_1(a)$ and $U_1(b)$. Denote the degrees of U and U_1 by k and l, respectively. We get $\deg P=klmn\geq 6$, while the number of the independent parameters, i.e. the dimension of the corresponding strata, is at most $k+l-1$ (we take into account the requirements that T_m and T_n take equal values at $U_1(a)$ and $U_1(b)$, and $P(a)=P(b)=0$). By exactly the same reasoning as above, we conclude that the maximal dimension of the corresponding strata is achieved at either $\deg U=1$ or $\deg U_1=1$, and it is at most $[d/mn]$. The minimal possible values for m, n here are 2 and 3, so we get the bound $[d/6]$, which is smaller than the one above.

It remains to consider the case $s=3$. In this case, by Theorem 3.1, we have $P=U\circ z^2R^2(z^2)\circ T_{m_1m_2}\circ U_1$, with U, R, U_1 as above, $m_1, m_2>1$ odd and $\mathrm{GCD}(m_1,m_2)=1$. In addition, T_{2m_1}, T_{2m_2} and $zR(z^2)\circ T_{m_1m_2}$ take equal values at $U_1(a)\neq U_1(b)$.

As above, denoting the degrees of U, U_1, R by k, m, l, respectively, we get $\deg P=k\cdot(4l+2)m_1m_2\cdot m\geq 90$. The number of the independent parameters, i.e. the dimension of the corresponding strata, is here at most $k+l+m-2$ (we take into account, besides the requirements that W_1, W_2, W_3 take equal values at a, b and $P(a)=P(b)=0$, also the fact that the scaling parameters of U and of R act equivalently on P). Maximizing the last expression exactly as above, we conclude that the maximum is achieved for $l=1$, $m_1=3, m_2=5$ and either $k=1, m=[d/((4l+2)m_1m_2)]=[d/90]$ or $m=1, k=[d/90]$. This maximum is equal to $[d/90]$. This completes the proof of Proposition 3.3. □

Based on Proposition 3.3 and Theorem 3.1, we can now give a much more accurate description of the composition set $\mathrm{COS}_{V,Q}$ for $V\subset\mathcal{P}_d$.

THEOREM 3.2. *For $V\subset\mathcal{P}_d$ and for any polynomial Q of degree at most five, the composition set $\mathrm{COS}_{V,Q}$ is a linear subspace L in V with $\dim L\leq[d/2]$. For $6\leq\deg Q\leq 89$, the set $\mathrm{COS}_{V,Q}$ is a union of at most two linear subspaces in V, and for $\deg Q\geq 90$ the set $\mathrm{COS}_{V,Q}$ is a union of at most three linear subspaces. The dimension of each of these subspaces is at most $[d/2]$; their double and triple intersections have dimensions at most $[d/6]+1$ and $[d/90]$, respectively.*

Proof. It is sufficient to consider the case $V=\mathcal{P}_d$. Let W_j, $j=1,\dots,s$ be all the mutually prime right $[a,b]$-factors of Q. By Proposition 3.3, for Q of degree at most five we have $s=1$. For $6\leq\deg Q\leq 89$, we have $s\leq 2$ and for $\deg Q\geq 90$ we have

$s \le 3$. Next, by Proposition 3.2, $\mathrm{COS}_{\mathcal{P}_d,Q}$ is a union of linear subspaces $L_j = \{P \in \mathcal{P}_d, P = \widetilde{P}(W_j)\}$.

Next, notice that if $\deg W_j = d$, then L_j is one dimensional and, if $\deg W_j < d$, then $L_j \subset \mathrm{DEC}^d_{1,1}(a,b) \cup \mathrm{DEC}^d_2(a,b) \cup \mathrm{DEC}^d_3(a,b)$. We also have $L_i \cap L_j \subset \mathrm{DEC}^d_2(a,b) \cup \mathrm{DEC}^d_3(a,b)$, $L_i \cap L_j \cap L_k \subset \mathrm{DEC}^d_3(a,b)$. All the required bounds on the dimensions of L_j now follow directly from Proposition 3.3. □

Remark. In fact, the dimensions of the linear subspaces L_j and of their intersections may be strongly smaller than the bounds in Theorem 3.2. The reason is that in this theorem we do not take into account, for example, the fact that if Q has mutually prime right $[a,b]$-factors W_1, W_2, then their degrees, by Theorem 3.1, cannot both be equal to two. Another reason is that in the setting of Theorem 3.2 the right factors are fixed, while in Proposition 3.3 they are variable, which also decreases the dimensions of the strata of $\mathrm{COS}_{\mathcal{P}_d,Q}$ in comparison with the strata $\mathrm{DEC}^d_s(a,b)$.

4. *Moment vanishing versus composition*

The main result of [**33**] can be formulated as follows.

THEOREM 4.1. *Let P with $P(a) = P(b)$ be given and let W_j, $j = 1, \dots, s$ be all its non-equivalent $[a,b]$-indecomposable right $[a,b]$-factors. Then, for any polynomial Q, all the moments $m_k = \int_a^b P^k(x)q(x)\,dx$, $k \ge 0$, vanish if and only if $Q = \sum_{j=1}^s Q_j$, where $Q_j = \widetilde{Q}_j(W_j)$ for some polynomial $\widetilde{Q}_j$.*

This theorem combined with Theorem 3.1 provides an explicit description for vanishing of the polynomial moments. In order to use it for the study of the moment set, let us introduce the notions of 'definite' and 'codefinite' polynomials.

Definition 4.1. Let $V, V_1 \subset \mathcal{P} = \mathcal{P}_{[a,b]}$ be fixed linear spaces. A polynomial $P \in \mathcal{P}$ is called V_1-definite if, for any polynomial $Q \in V_1$, vanishing of the moments $m_k = \int_a^b P^k(x)q(x)\,dx$, $k \ge 0$, implies the composition condition on $[a,b]$ for P and Q. The set of such P is denoted D_{V_1}.

A polynomial $Q \in \mathcal{P}$ is called V-codefinite if, for any polynomial $P \in V$, vanishing of the moments $m_k = \int_a^b P^k(x)q(x)\,dx$, $k \ge 0$, implies the composition condition on $[a,b]$ for P and Q. The set of such Q is denoted COD_V.

If $V_1 = \mathcal{P}$ or $V = \mathcal{P}$ (with respect to the corresponding P or Q), we call polynomials defined above $[a,b]$-definite or $[a,b]$-codefinite correspondingly, and denote their sets by D or COD.

Definite polynomials have been initially introduced and studied in [**37**]. Some of their properties have been described in [**31**]. The notion of codefinite polynomials is apparently new (although some examples have appeared in [**10**]). Below we give a characterization of definite and codefinite polynomials, but many questions still remain open.

4.1. *Definite polynomials.*

Theorem 4.1 allows us to give a complete description of $[a,b]$-definite polynomials.

THEOREM 4.2. *A polynomial P is $[a, b]$-definite if and only if it has, up to equivalence, exactly one $[a, b]$-indecomposable right factor W.*

Proof. Assume that P has exactly one $[a, b]$-indecomposable right factor W. By Theorem 4.1, for any polynomial Q, vanishing of m_k for all $k \geq 0$ implies that there exists $\widetilde{Q}$ such that $Q = \widetilde{Q}(W)$, so the composition condition on $[a, b]$ is satisfied for P and Q. Hence, by Definition 4.1, P is $[a, b]$-definite.

We assume now that P has two non-equivalent $[a, b]$-indecomposable right factors W_1, W_2, and show that the solution $Q = W_1 + W_2$ cannot be represented in the form $Q = \widetilde{Q}(W)$, where W is an $[a, b]$-right factor of P and $\widetilde{Q}$ is a polynomial (cf. [**30**]). First observe that W_1 and W_2 have different degrees, for otherwise equalities (3.3) imply that W_1 and W_2 are equivalent. Thus, without loss of generality, we may assume that $\deg W_2 > \deg W_1$ and so $\deg Q = \deg W_2$, implying that if $Q = \widetilde{Q}(W)$, then $\deg W | \deg W_2$. Therefore, using (3.3) again, we conclude that $W_2 = U(W)$ for some polynomial U. Furthermore, if $\deg W < \deg W_2$, then we obtain a contradiction with the assumption that W_2 is an $[a, b]$-indecomposable right factor of P. On the other hand, if $\deg W = \deg W_2$, then as above we conclude that W and W_2 are linear equivalent, implying that $W_1 = Q - W_2$ is a polynomial in W_2, in contradiction with the assumption that $\deg W_2 > \deg W_1$. □

Corollaries 4.1–4.2 below were proved in [**31**]. Here we give another proof of these results based on Theorem 4.2 and the second Ritt theorem. We believe that these 'more algebraic' proofs clarify to some extent the structure of definite polynomials, which still presents a lot of open questions (see [**37**]). We also extend a classification of non-definite polynomials whose degree does not exceed nine, given in [**31**], up to degree eleven.

COROLLARY 4.1. *Let p be a prime. Then each polynomial P of degree p^s, $s \geq 1$, is $[a, b]$-definite for any $a, b \in \mathbb{C}$, $a \neq b$.*

Proof. Indeed, since imprimitivity systems of G_P form a sublattice of L_{p^s} (see §3.1), if W_1, W_2 are arbitrary right factors of P, then either W_1 is a polynomial in W_2 or W_2 is a polynomial in W_1. Therefore, such P cannot have two non-equivalent $[a, b]$-indecomposable right factors. □

COROLLARY 4.2. *If at least one of the points a and b is not a critical point of a polynomial P, then P is $[a, b]$-definite.*

Assume that P is not $[a, b]$-definite and let W_1, W_2 be its nonlinear equivalent $[a, b]$-indecomposable right factors. Then the second Ritt theorem implies that there exist polynomials of degree one, μ_1, μ_2, and polynomials U, W such that either

$$P = U \circ z^{rs} R^n(z^n) \circ W, \quad W_1 = \mu_1 \circ z^n \circ W, \quad W_2 = \mu_2 \circ z^s R(z^n) \circ W, \tag{4.1}$$

where R is a polynomial and $\mathrm{GCD}(s, n) = 1$, or

$$P = U \circ T_{nm} \circ W, \quad W_1 = \mu_1 \circ T_n \circ W, \quad W_2 = \mu_2 \circ T_m \circ W, \tag{4.2}$$

where T_n, T_m are the Chebyshev polynomials and $\mathrm{GCD}(n, m) = 1$. Furthermore, since W_1, W_2 are $[a, b]$-indecomposable and non-equivalent, the inequality $W(a) \neq W(b)$ holds. In particular, $n > 1$, since $W_1(a) = W_1(b)$.

It is easy to see that if (4.1) holds, then the equalities

$$W_1(\widetilde{a}) = W_1(\widetilde{b}), \quad W_2(\widetilde{a}) = W_2(\widetilde{b}),$$

where

$$\widetilde{W}_1 = z^n, \quad \widetilde{W}_2 = z^s R(z^n), \quad \widetilde{a} = W(a), \quad \widetilde{b} = W(b),$$

taking into account the equality $\mathrm{GCD}(s, n) = 1$, imply that the number $\widetilde{a}^n = \widetilde{b}^n$ is a root of the polynomial R. It follows now from the first formula in (4.1) by the chain rule that both a and b are critical points of P.

If (4.2) holds, then, taking into account the identity

$$T_l \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right) \circ z^l \tag{4.3}$$

and the equality $\mathrm{GCD}(m, n) = 1$, it is easy to see that there exist $\alpha, \beta \in \mathbb{C}$ such that

$$\widetilde{a} = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right), \quad \widetilde{b} = \frac{1}{2}\left(\beta + \frac{1}{\beta}\right), \quad \alpha^n = \beta^n, \quad \alpha^m = \frac{1}{\beta^m}, \tag{4.4}$$

where as above $\widetilde{a} = W(a)$, $\widetilde{b} = W(b)$. Furthermore, $\alpha^2 \neq 1$. Indeed, otherwise the equalities

$$\bar{\alpha}^n = \bar{\beta}^n, \quad \bar{\alpha}^m = \frac{1}{\bar{\beta}^m},$$

where $\bar{\alpha} = \alpha^2$, $\bar{\beta} = \beta^2$, taking into account the equality $\mathrm{GCD}(m, n) = 1$, imply that $\beta^2 = 1$. Since $\widetilde{a} \neq \widetilde{b}$, this yields that either $\widetilde{a} = -1$, $\widetilde{b} = 1$ or $\widetilde{a} = 1$, $\widetilde{b} = -1$. On the other hand, since $\mathrm{GCD}(m, n) = 1$, without loss of generality we may assume that m is odd, implying that $T_m(\widetilde{a}) \neq T_m(\widetilde{b})$ for such $\widetilde{a}$ and $\widetilde{b}$ since $T_m(-1) = -1$, $T_m(1) = 1$. Similarly, $\beta^2 \neq 1$. Finally, observe that equalities (4.4) yield that $\alpha^{mn} = \pm 1$, $\beta^{mn} = \pm 1$, implying that

$$T_{mn}(\alpha) = \pm 1, \quad T_{mn}(\beta) = \pm 1. \tag{4.5}$$

In order to finish the proof, observe that the equality $T_n(\cos\phi) = \cos n\phi$ implies easily that the polynomial T_n has exactly two critical values ± 1 and that the only points in the preimage $T_n^{-1}\{\pm 1\}$ which are not critical points of T_n are the points ± 1. Therefore, the equalities (4.5), taking into account that $\alpha \neq \pm 1$, $\beta \neq \pm 1$, imply that α and β are critical points of T_{mn} and hence critical points of P by the chain rule.

Theorem 4.2 combined with the second Ritt theorem allows us, at least in principle, to describe explicitly all the non-definite polynomials up to a given degree. In particular, the following statement holds.

THEOREM 4.3. *For given $a \neq b$, non-definite polynomials $P \in \mathcal{P}_{11}$ appear only in degrees six and 10 and have, up to change $P \to \lambda \circ P$, where λ is a polynomial of degree one, the following form.*

(1) *$P_6 = T_6 \circ \tau$, where T_6 is the Chebyshev polynomial of degree six and τ is a polynomial of degree one transforming a, b into $-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}$.*

(2) $P_{10} = z^2 R^2(z^2) \circ \tau$, *where* $R(z) = z^2 + \gamma z + \delta$ *is an arbitrary quadratic polynomial satisfying* $R(1) = 0$, *i.e.* $\gamma + \delta = -1$, *and* τ *is a polynomial of degree one transforming* a, b *into* $-1, 1$.

Proof. First of all, observe that if in Ritt's second theorem (§3.1 above) the degree of one of polynomials satisfying (3.4) is two, then solutions (3.6) may be written in the form (3.5). Indeed, for odd n the equality

$$T_n(z) = zE_n(z^2) \tag{4.6}$$

holds for some polynomial E_n. Furthermore, $T_2 = \theta \circ z^2$, where $\theta = 2z - 1$ and hence

$$zE_n(z^2) \circ \theta \circ z^2 = T_n \circ T_2 = T_2 \circ T_n = \theta \circ T_n^2 = \theta \circ zE_n^2(z) \circ z^2.$$

Since the last equality implies the equality

$$zE_n(z^2) \circ \theta = \theta \circ zE_n^2(z),$$

we conclude that

$$T_n = \theta \circ zE_n^2(z) \circ \theta^{-1}, \quad T_{2n} = \theta \circ z^2 E_n^2(z^2). \tag{4.7}$$

Therefore, the equality

$$T_n \circ T_2 = T_2 \circ T_n$$

may be written in the form

$$(\theta \circ zE_n^2(z) \circ \theta^{-1}) \circ (\theta \circ z^2) = (\theta \circ z^2) \circ zE_n(z^2). \tag{4.8}$$

Now we are ready to prove the theorem.

Since each integer i, $2 \le i < 11$, distinct from 6 or 10 is either a prime or a power of a prime, it follows from Corollary 4.1 that P is $[a, b]$-definite unless $\deg P = 6$ or $\deg P = 10$. It follows now from the second Ritt theorem and the remark above that if $\deg P = 10$, then P has the form given above. Similarly, if $\deg P = 6$, then $P = z^2 R^2(z^2) \circ \tau$, where R is a polynomial satisfying $R(1) = 0$. However, since in this case the degree of R equals one, up to change $P \to \lambda \circ P \circ \tau$, we obtain a unique polynomial $P = T_6$. □

Let $V, V_1 \subset \mathcal{P}$ be fixed linear spaces. Let us denote by ND_{V,V_1} the set of polynomials $P \in V$ non-definite with respect to V_1. In particular, for $V = \mathcal{P}_d$, $V_1 = \mathcal{P}$, we denote the corresponding set by ND_d. If V_1 is a line spanned by a fixed $Q \in \mathcal{P}$, we write ND_{V,V_1} as $\mathrm{ND}_{V,Q}$.

PROPOSITION 4.1. *For each* $V_1 \subset \mathcal{P}$ *and* $V \subset \mathcal{P}_d$, *we have* $\mathrm{ND}_{V,V_1} \subset \mathrm{ND}_d$. *The dimension of* ND_d *does not exceed* $[d/6] + 1$.

Proof. The conclusion is immediate: any polynomial non-definite with respect to a smaller subspace is non-definite with respect to a larger one. By Theorem 4.2, the set ND_d consists of all $P \in \mathcal{P}_d$ which have $s \ge 2$ mutually $[a, b]$-prime right $[a, b]$-factors. Hence, $\mathrm{ND}_d \subset \bigcup_{s \ge 2} \mathrm{DEC}_s^d(a, b)$. By Proposition 3.3, we have $\dim \mathrm{ND}_d \le [d/6] + 1$. This completes the proof. □

4.2. *Codefinite polynomials.* Let $[a, b]$ and a subspace $V \subset \mathcal{P}_{[a,b]}$ be given.

THEOREM 4.4. *A polynomial Q is not V-codefinite if and only if there exists a polynomial $P \in V$ (necessarily non-definite) with a complete collection of $[a, b]$-indecomposable right factors $W_1, \ldots, W_s$, $s \geq 2$, such that:*
(1) *the polynomial Q can be represented as $Q = \sum_{j=1}^{s} S_j(W_j)$;*
(2) *no one of $W_1, \ldots, W_s$ is a right $[a, b]$-factor of Q.*

Proof. By Definition 4.1, a polynomial Q is not V-codefinite if and only if there exists a polynomial $P \in V$ such that all the moments $m_k = \int_a^b P^k(x)q(x)\,dx$, $k \geq 0$, vanish while P and Q do not satisfy the composition condition. Clearly, if such P exists it cannot be definite. Furthermore, by Theorem 4.1, the polynomial Q can be represented as a sum $Q = \sum_{j=1}^{s} S_j(W_j)$. Finally, since P and Q do not satisfy the composition condition, no one of $W_1, \ldots, W_s$ can be an $[a, b]$-right factor of Q.

In the opposite direction, assume that $P \in V$ as required exists. Since Q possesses a representation $Q = \sum_{j=1}^{s} S_j(W_j)$, where $W_1, \ldots, W_s$ are right $[a, b]$-factors of P, we conclude (by linearity of the moments in Q) that all the moments m_k, $k \geq 0$, vanish. Furthermore, since $W_1, \ldots, W_s$ is a complete collection of right $[a, b]$-factors of P, the second assumption implies that P and Q do not satisfy the composition condition. Hence, Q is not V-codefinite. □

Definition 4.2. For $V \subset \mathcal{P}$, we define the set $\mathcal{S}_{V,d} \subset \mathcal{P}_d$ as the set of polynomials $Q \in \mathcal{P}_d$ which can be represented as $Q = \sum_{j=1}^{s} S_j(W_j)$, where $W_1, \ldots, W_s$ are all $[a, b]$-indecomposable right factors of a certain non-definite $P \in V$. The set $\mathcal{S}_V$ is the union $\bigcup_d \mathcal{S}_{V,d}$.

By Theorem 4.4, in order to describe explicitly all V-codefinite polynomials up to degree d, we have first to describe the set $\mathcal{S}_{V,d}$ and then to describe those $Q \in \mathcal{S}_{V,d}$ for which no one of $W_1, \ldots, W_s$ is a right $[a, b]$-factor of Q. Both these questions in their general form turn out to be rather tricky, and we provide here only very partial results.

Let us stress that the role of Theorem 4.5 below and of the rest of the results in this section is to describe in detail the set of *non-composition* solutions of the moment equations. This will prepare an application, in §6 below, of the second set of the center equations at infinity, according to Theorem 2.1: those provided by the vanishing of the second Melnikov function. We expect that together the two sets of equations always imply the composition condition (compare Conjecture 2 in §6 below).

To make formulas easier, without loss of generality we shall assume that $[a, b]$ coincides with $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$.

THEOREM 4.5. *Let $V = \mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$. Then the set $\mathcal{S}_{V,d}$ is a vector space consisting of all polynomials $Q \in \mathcal{P}_d$ representable as $Q = S_1(T_2) + S_2(T_3)$ for some polynomials S_1, S_2. Furthermore, the dimension $\mathcal{S}_{V,d}$ is equal to* $[(d + 1)/2] + [(d + 1)/3] - [(d + 1)/6]$. *In particular, this dimension does not exceed* $[\frac{2}{3}d] + 1$.

For $d \leq 4$, the space $\mathcal{S}_{V,d}$ coincides with $\mathcal{P}_d$ and, starting with $d = 5$, this space is always a proper subset of $\mathcal{P}_d$. We have $\mathcal{S}_{V,5} = \mathcal{P}_4 \subset \mathcal{P}_5$ and $\mathcal{S}_{V,6}$ is the subspace in $\mathcal{P}_6$ consisting of all the polynomials Q of the form $Q = Q_1 + \alpha T_3$ with Q_1 even of degree *at most six. We have $\mathcal{S}_{V,7} = \mathcal{S}_{V,6}$, while the set $\mathcal{S}_{V,8}$ is the subspace in $\mathcal{P}_8$ consisting of all the polynomials Q of the form $Q = Q_1 + \alpha T_3$ with Q_1 even of degree at most eight. The set $\mathcal{S}_{V,9}$ is the subspace in $\mathcal{P}_9$ consisting of all the polynomials Q of the form $Q = Q_1 + \alpha T_3 + \beta T_3^3$ with Q_1 even of degree at most eight.*

Proof. By Theorem 4.3, the only non-definite polynomials in $V = \mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ are scalar multiples of T_6. Further, $T_6 = T_2 \circ T_3 = T_3 \circ T_2$ has exactly two right $[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$-factors T_2 and T_3. This proves the first claim of Theorem 4.5.

Next, observe that

$$\mathbb{C}[T_n] \cap \mathbb{C}[T_m] = \mathbb{C}[T_l], \tag{4.9}$$

where $l = \operatorname{lcm}(n, m)$. Indeed, if P is contained in $\mathbb{C}[T_n] \cap \mathbb{C}[T_m]$, then there exist polynomials A, B such that

$$P = A \circ T_n = B \circ T_m$$

and, in order to show that there exists a polynomial U such that $P = U \circ T_l$, one can use the second Ritt theorem. However, such a proof is more difficult than it seems, since it requires an analysis of the possibility provided by (3.5) (see e.g. [**33**, Lemma 4.1]). It is more convenient to observe that identity (4.3) implies that the function

$$F = P \circ \frac{1}{2}\left(z + \frac{1}{z}\right) = A \circ \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) = B \circ \frac{1}{2}\left(z^m + \frac{1}{z^m}\right)$$

is invariant with respect to both the groups D_{2n} and D_{2m}, where D_{2s} is the dihedral group generated by the transformations $z \to 1/z$ and $z \to e^{2\pi i/s} z$. Therefore, F remains invariant with respect to the group $< D_{2n}, D_{2m} >= D_{2l}$ generated by D_{2n} and D_{2m}, implying that there exists a rational function U such that

$$F = U \circ \frac{1}{2}\left(z^l + \frac{1}{z^l}\right).$$

Since

$$U \circ \frac{1}{2}\left(z^l + \frac{1}{z^l}\right) = U \circ T_l \circ \frac{1}{2}\left(z + \frac{1}{z}\right),$$

we conclude that $P = U \circ T_l$, and it is easy to see that U actually is a polynomial.

Denote by $U_{d,n}$ the subspace of $\mathbb{C}[T_n]$ consisting of all polynomials of degree $\le d$. By the remark above, we have $U_{d,n} \cap U_{d,m} = U_{d,l}$. This implies that

$$\begin{aligned}\dim \mathcal{S}_{V,d} &= \dim U_{d,2} \oplus U_{d,3} - 2 = \dim U_{d,2} + \dim U_{d,3} - \dim U_{d,6} - 2 \\ &= \left[\frac{d+1}{2}\right] + \left[\frac{d+1}{3}\right] - \left[\frac{d+1}{6}\right] - 1 \le \left[\frac{2}{3} d\right] + 1.\end{aligned}$$

To get an explicit description of $\mathcal{S}_{V,d}$ for $d \le 9$, we shall use the following simple lemma, which is used also in §6. Consider polynomials $\hat{T}_2(x) = 2x^2 - \frac{3}{2}$, $\hat{T}_3(x) = T_3(x) = 4x^3 - 3x$ and $\hat{T}_6 = \hat{T}_2 \circ \hat{T}_3$. Our polynomials $\hat{T}_j$, $j = 2, 3, 6$, differ from the usual Chebyshev polynomials only in a constant term, chosen in such a way that $\hat{T}_j$ vanish at the points $-\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$. In the representation,

$$Q = S_1(\hat{T}_2) + S_2(\hat{T}_3), \quad S_1, S_2 \in \mathcal{P}. \tag{4.10}$$

The polynomial Q belongs to $\mathcal{P}_d$, so it vanishes at the points $-\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$. Hence, we can assume that both S_1 and S_2 do not have constant terms. Next we notice that all the even polynomials Q in $\mathcal{P}_d$, and only them, are representable as $Q = S_1(\hat{T}_2)$.

Let $\deg S_1 = m$, $\deg S_2 = n$.

LEMMA 4.1. *Let $Q \in \mathcal{P}_d$ be represented via (4.10). Then the polynomials S_1 and S_2 in (4.10) can be chosen in such a way that S_2 is odd and* $\max(2m, 3n) \le d$.

Proof. It is enough to consider only odd polynomial S_2. Indeed, it is immediate that all the even polynomials, and only them, are representable as $S_1(\hat{T}_2)$. Since for l even $\hat{T}_3^l = x^l(4x^2-3)^l$ is an even polynomial (and it is odd for l odd), all the even degrees in S_2 can be omitted.

Under this assumption, the odd degree n of S_2 must satisfy $3n \le d$. Indeed, otherwise the odd degree $3n$ of $S_2(\hat{T}_3)$ would be larger than d, and the highest degree term in this polynomial could not cancel with the terms of $S_1(\hat{T}_2)$. By the same reason, assuming that S_2 is odd, we conclude that $2m \le d$. Indeed, otherwise the even degree $2m$ of $S_1(\hat{T}_2)$ would be larger than d, and the highest degree term in this polynomial could not cancel with the terms of $S_2(\hat{T}_2)$. □

Application of Lemma 4.1 completes the proof of Theorem 4.5. □

THEOREM 4.6. *A polynomial of the form $Q = S_1(T_2) + S_2(T_3)$, where S_1, S_2 are non-zero polynomials, has T_2 (respectively T_3) as its right factor if and only if S_2 is a polynomial in T_2 (respectively S_1 is a polynomial in T_3).*

Proof. Indeed, assume, say, that $S_1(T_2) + S_2(T_3) = R(T_2)$ for some polynomial R. Then, by (4.9) there exists a polynomial F such that

$$S_2 \circ T_3 = F \circ T_6 = F \circ T_2 \circ T_3,$$

implying that $S_2 = F \circ T_2$. □

COROLLARY 4.3. *Let $V = \mathcal{P}_{9,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$. A polynomial $Q \in \mathcal{P}_{8,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ is not V-codefinite if and only if it can be represented in the form*

$$Q = R + \alpha T_3, \quad \alpha \in \mathbb{C}, \tag{4.11}$$

where $\alpha \neq 0$, and $R \in \mathcal{P}_{8,[-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}]}$ is an even polynomial distinct from $\beta T_6 + \gamma$, $\beta, \gamma \in \mathbb{C}$.

Proof. By the above results, if $P \in \mathcal{P}_8$ is not codefinite, it can be represented in the form $Q = S_1(T_2) + S_2(T_3)$, where $\deg S_1 \le 4$ and S_1 is not a linear polynomial in T_3, while $\deg S_2 \le 2$ and S_2 is not a linear polynomial in T_2. Since S_2 can be represented in the form $\delta T_2 + \alpha z + \kappa$, where $\delta, \alpha, \kappa \in \mathbb{C}$, we conclude that such Q can be represented in the form

$$Q = \widetilde{S}_1(T_2) + \alpha T_3, \tag{4.12}$$

where $\deg \widetilde{S}_1 \le 4$. Furthermore, $\alpha \neq 0$, since otherwise Q is a polynomial in T_2, and $\widetilde{S}_1$ is not a linear polynomial in T_3, since otherwise Q is a polynomial in T_3. Therefore, since $\mathbb{C}[T_2] = \mathbb{C}[z^2]$ and $T_3 \in \mathcal{P}_8$, the polynomial P admits the representation (4.11).

In the other direction, it follows from (4.11) that (4.12) holds, where $\alpha \neq 0$ and $\widetilde{S}_1 \neq \beta T_3 + \gamma$, $\beta, \gamma \in \mathbb{C}$, implying that Q is not codefinite. □

4.3. *Polynomials with a special structure.* Let $\mathcal{R} = \{r_1, r_2, \ldots\}$ be a set of prime numbers, finite or infinite. Define $U(\mathcal{R})$ as a subset of $\mathcal{P}$ consisting of polynomials $P = \sum_{i=0}^{N} a_i x^i$ such that for any non-zero coefficient a_i, the degree i is either coprime with each $r_j \in \mathcal{R}$ or it is a power of some $r_j \in \mathcal{R}$. Similarly, define $U_1(\mathcal{R})$ as a subset of $\mathcal{P}$ consisting of polynomials Q such that for any non-zero coefficient a_i of Q, all prime factors of i are contained in $\mathcal{R}$. In particular, if $\mathcal{R}$ coincides with the set of all prime numbers, then $U(\mathcal{R})$ consists of polynomials in $\mathcal{P}$ whose degrees with non-zero coefficients are powers of primes, while $U_1(\mathcal{R}) = \mathcal{P}$.

THEOREM 4.7. *Let $\mathcal{R} = \{r_1, r_2, \ldots\}$ be fixed. Then, for any $a \neq b$, each polynomial $P \in U(\mathcal{R})$ is $[a, b]$-definite and, in particular, it is $[a, b]$-definite with respect to $U_1(\mathcal{R})$, and each $Q \in U_1(\mathcal{R})$ is $[a, b]$-codefinite with respect to $U(\mathcal{R})$.*

Proof. We show that vanishing of all the moments $m_k = \int_a^b P^k(x) q(x)\, dx$ for $P \in U(\mathcal{R})$ and $Q \in U_1(\mathcal{R})$ implies the composition condition. By the construction, the degree of any $Q \in U_1(\mathcal{R})$ is the product of certain prime numbers in $\mathcal{R}$. By [**31**, Corollary 4.3], vanishing of the moments implies that the degrees of P and Q cannot be mutually prime. Hence, deg P is divisible by one of r_j. But then by the definition this degree must be a power of r_j. Finally, it was shown in [**31**] (see also §3.2.1 above) that polynomials P with deg P a power of a prime number are definite. Hence, vanishing of the moments m_k implies the composition condition for P, Q on $[a, b]$. □

4.4. *The moment and the composition sets.* Using the information on definite and codefinite polynomials provided above, we now can describe more accurately the interrelation between the moment and the composition sets.

Let $V, V_1 \subset \mathcal{P}$ be fixed linear spaces. As above, ND_{V,V_1} is the set of polynomials $P \in V$ non-definite with respect to V_1.

THEOREM 4.8. *For each $Q \in V_1$, we have $\mathrm{MS}_{V,Q} = \mathrm{COS}_{V,Q} \cup N$, where N is contained in $\mathrm{ND}_{V,V_1} \subset \mathrm{ND}$. In particular, for $V \subset \mathcal{P}_d$ and any Q, the dimension of N is at most* $[d/6] + 1$.

Proof. If $P \in \mathrm{MS}_{V,Q}$ but P is not in $\mathrm{COS}_{V,Q}$, then P is not definite with respect to V_1 and hence it belongs to ND_{V,V_1}, which is always a subset of ND. If $V \subset \mathcal{P}_d$, then $P \in \mathrm{ND}_d$ and the bound on the dimension follows from Proposition 4.1. □

Example [**10**]. Let $[a, b] = [-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]$. Put $Q = (T_2 + T_3)$ and consider $V = \mathcal{P}_6$. Then the moment set $\mathrm{MS}_{V,Q}$ contains exactly two components: the composition component $\mathrm{COS}_{V,Q} = \{P = R(T_2 + T_3)\}$, with R any polynomial of degree two, and the non-composition component $\mathcal{T} = \{P = \alpha T_6, \alpha \in \mathbb{C}\}$. Here $\mathcal{T}$, in fact, coincides with $\mathrm{ND}_{V,Q}$.

Our description of codefinite polynomials in §4.2 produces the following result on the moment and composition sets.

COROLLARY 4.4. *Let $V \subset \mathcal{P}$ and let $V_1 \subset \mathcal{P}_d$ be such that $V_1 \cap \mathcal{S}_{V,d} = \{0\}$, in the notation of Definition 4.2. Then, for each $Q \in V_1$, we have $\mathrm{ND}_{V,V_1} = \emptyset$ and $\mathrm{MS}_{V,Q} = \mathrm{COS}_{V,Q}$.*

Proof. By Theorem 4.4 and via Definition 4.2, each $Q \in V_1$, $Q \neq 0$ is codefinite with respect to V. Consequently, each $P \in V$ is definite with respect to such Q. Application of Theorem 4.8 completes the proof. □

In the situation of §4.3, we get the following result.

COROLLARY 4.5. *For a fixed set $\mathcal{R}$ of prime numbers, let $V = U(\mathcal{R})$, $V_1 = U_1(\mathcal{R})$, in the notation of §4.3. Then, for each $Q \in V_1$, we have* $\mathrm{MS}_{V,Q} = \mathrm{COS}_{V,Q}$.

Proof. The result follows directly from Theorems 4.8 and 4.7. □

5. *Center set near infinity*

Let a polynomial Q and a linear subspace $V \subset \mathcal{P}_d$ be fixed. In this section we analyze the structure of the center set $\mathrm{CS}_{V,Q}$ at and near the infinite hyperplane HV, as compared to the moment and composition sets $\mathrm{MS}_{V,Q}$ and $\mathrm{COS}_{V,Q}$. By Proposition 2.1, we have at infinity $\overline{\mathrm{COS}} \subset \overline{\mathrm{CS}} \subset \overline{\mathrm{MS}}$.

An important fact is that for each *definite* $P_0 \in \overline{\mathrm{CS}}$, there is an entire projective neighborhood U of P_0 in PV where CS and COS coincide.

THEOREM 5.1. *Let $P_0 \in \overline{\mathrm{CS}}_{V,Q}$ be a definite polynomial. Then:*

(1) *in fact, $P_0 \in \overline{\mathrm{COS}}_{V,Q}$;*

(2) *there exists a projective neighborhood U of P_0 in PV such that $\mathrm{CS}_{V,Q} \cap U = \mathrm{COS}_{V,Q} \cap U$;*

(3) $\mathrm{CS}_{V,Q} \cap U$ *is a linear space defined by vanishing of the linear parts of the center equations. In particular,* CS *is regular in U and its local ideal is generated by the center equations.*

Proof. From the inclusion $\overline{\mathrm{CS}} \subset \overline{\mathrm{MS}}$, we get $P_0 \in \overline{\mathrm{MS}}_{V,Q}$. Since the polynomial P_0 is definite by the assumptions, moments vanishing for this polynomial implies composition, so in fact $P_0 \in \overline{\mathrm{COS}}_{V,Q}$.

In homogeneous coordinates (P, ν) in PV near P_0, put $P = P_0 + P_1$, $P_1 \in V$. By [**10**, Proposition 7.2], the only non-zero linear terms in the expansions of the homogenized center equations around the point (0, 0) in variables P_1, ν are given by the following linear functionals in P_1:

$$L_k(P_1) = -(k-3)\int_a^b P_0^{k-4}(x)q(x)P_1(x)\,dx, \quad k = 4, 5, \ldots. \tag{5.1}$$

Denote by $L \subset V$ the subspace defined by the linear equations $L_k(P_1) = 0$, $k = 4, 5, \ldots$. Let us show first that $L \subset \mathrm{COS}_{V,Q}$. Consider a certain polynomial $P_1 \in L$. Since P_0 is definite, vanishing of $L_k(P_1)$ implies the composition condition for $P_0(x)$ and $S(x) = \int_a^x P_1(\tau)q(\tau)\,d\tau$. Since, being definite, P_0 has only one $[a, b]$-prime right composition $[a, b]$-factor W, we conclude that $S = \widetilde{S}(W)$. By the same reason, from $P_0 \in \mathrm{COS}_{V,Q}$ it follows that $Q = \widetilde{P}(W)$. Now [**10**, Lemma 7.3] implies that $P_1 = \widetilde{P}_1(W)$, i.e. $P_1 \in \mathrm{COS}_{V,Q}$ and hence $L \subset \mathrm{COS}_{V,Q}$.

It follows that all the center equations vanish on L, which is the zero set of their linear parts. Now we are in a situation of [**10**, Lemma 7.4] (Nakayama lemma in commutative algebra see for example [**24**, Ch. 4, Lemma 3.4]). The conclusion is that $\mathrm{CS} = L = \mathrm{COS}$ in a neighborhood of P_0, and the local ideal of this set is generated by the center equations. This completes the proof of Theorem 5.1. □

6. *Main results*

Let $a \neq b$ be fixed. Below we denote by $\widetilde{T}_j$ the transformed Chebyshev polynomials $\widetilde{T}_j = T_j \circ \mu$, μ being a linear polynomial transforming the couple (a, b) to the couple $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$.

Let linear subspaces V, $V_1 \subset \mathcal{P}_{[a,b]}$ and a polynomial $Q \in V_1$ be fixed. The affine center set $\mathrm{CS}_{V,Q}$ always contains the composition set $\mathrm{COS}_{V,Q}$. In this section we provide an upper bound for the dimensions of affine *non-composition* components in CS. As above, $\mathrm{ND}_{V,V_1} \subset \mathrm{ND}$ denotes the set of V_1 non-definite polynomials in V. For each affine algebraic set $A \subset V$, let $\bar{A}$ denote the intersection of A with the infinite hyperplane HV.

THEOREM 6.1. *For each irreducible non-composition component A of the affine central set* $\mathrm{CS}_{V,Q}$, *we have* $\bar{A} \subset \overline{\mathrm{CS}}_{V,Q} \cap \mathrm{ND} \subset \overline{\mathrm{MS}}_{V,Q} \cap \mathrm{ND}$. *Consequently,* $\dim A \leq \dim(\overline{\mathrm{MS}}_{V,Q} \cap \mathrm{ND}) + 1$. *In particular, for any polynomial Q and $V \subset \mathcal{P}_d$, the dimension of A cannot exceed* $[d/6] + 2$.

Proof. We always have $\bar{A} \subset \overline{\mathrm{CS}} \subset \overline{\mathrm{MS}}$. Now, if $P_0 \in \bar{A}$, then P_0 cannot be definite. Indeed, otherwise there would exist a neighborhood U of P_0 provided by Theorem 5.1, where $A \cap U \subset \mathrm{COS} \cap U$. Since A is irreducible, this would imply that $A \subset \mathrm{COS}$, which contradicts the assumption that A is a non-composition component of CS. Thus, $\bar{A} \subset \overline{\mathrm{MS}}_{V,p} \cap \mathrm{ND}$. Now, since the infinite hyperplane HV has codimension one in the projective space PV, for each A we have $\dim A \leq \dim \bar{A} + 1$. Application of Proposition 4.1 completes the proof. □

Notice that the dimension of the composition components of CS may be of order $d/2$, while by Theorem 6.1 the dimension of the non-composition components is of order at most $d/6$. To our best knowledge, this is the first general bound of this form for the polynomial Abel equation.

COROLLARY 6.1. [**10**] *Let $V = \mathcal{P}_5$. Then, for any Q, the center set* $\mathrm{CS}_{V,Q}$ *consists of a composition set with possibly a finite number of additional points.*

Proof. By Theorem 4.3, there are no non-definite polynomials in $V = \mathcal{P}_5$. So, the set $\overline{\mathrm{MS}}_{V,Q} \cap \mathrm{ND}$ is empty and its dimension is -1. □

COROLLARY 6.2. *Let $V = \mathcal{P}_9$. Then, for any Q, the center set* $\mathrm{CS}_{V,Q}$ *consists of a composition set with possibly a finite number of additional curves.*

Proof. By Theorem 4.3, the only non-definite polynomials in $V = \mathcal{P}_9$ are scalar multiples of $\widetilde{T}_6$. So, the set $\overline{\mathrm{MS}}_{V,Q} \cap \mathrm{ND}$ consists at most of one point and its dimension is at most 0. □

COROLLARY 6.3. *Let $V = \mathcal{P}_{11}$. Then, for any Q, the center set* $\mathrm{CS}_{V,Q}$ *consists of a composition set with possibly a finite number of additional two-dimensional components.*

Proof. Theorem 4.3 describes non-definite polynomials in $V = \mathcal{P}_{11}$. We see that the set $\overline{\mathrm{MS}}_{V,Q} \cap \mathrm{ND}$ consists at most of a finite number of points and a one-dimensional component, and its dimension is at most 1. □

Notice that the bounds of Corollaries 6.1–6.3 are more accurate than the general bound of Theorem 4.3.

Recall that by Definition 4.2 the set $\mathcal{S}_V$ consists of all Q which can be represented as $Q = \sum_{j=1}^{s} S_j(W_j)$, where $W_1, \ldots, W_s$ are all $[a, b]$-indecomposable right factors of a certain $P \in V$.

THEOREM 6.2. *Let $V \subset \mathcal{P}$ and let $Q \in \mathcal{P} \backslash \mathcal{S}_V$. Then the center set $\mathrm{CS}_{V,Q}$ consists of a composition set with possibly a finite number of additional points. In particular, this is true for $V = \mathcal{P}_9$ and any Q not representable as $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$.*

Proof. This result follows directly from Theorem 4.3 and Corollary 4.4. The case $V = \mathcal{P}_9$ is covered by Theorem 4.5. However, since Theorem 6.2 is one of the central results of this paper, we give its short independent proof. We show that the moment set $\mathrm{MS}_{V,Q}$ does not contain non-definite polynomials. Indeed, for each non-definite $P \in V$, vanishing of the moments $m_k = \int_a^b P^k(x)q(x)\,dx$ implies that $Q \in \mathcal{S}_V$, by Definition 4.2. But, by our assumptions, $Q \in \mathcal{P} \backslash \mathcal{S}_V$. Therefore, P is not in $\mathrm{MS}_{V,Q}$. Application of Theorem 6.1 completes the proof. □

We expect that the result of Theorem 6.2 can be significantly extended. In particular, we expect that the following statement is true.

Statement 6.1. Let $V \subset \mathcal{P}$. Assume that either $Q \in \mathcal{P} \backslash \mathcal{S}_V$ or $Q \in \mathcal{S}_V$, and it is not V-codefinite. Then the center set $\mathrm{CS}_{V,Q}$ consists of a composition set with possibly a finite number of additional points.

Closely related to Statement 6.1 is the following result.

CONJECTURE 2. *For polynomials P, Q, vanishing of all the moments $m_k(P, Q)$ and of all the second Melnikov coefficients $D_j(P, Q)$ (see Theorem 2.1) implies the composition condition.*

THEOREM 6.3. *Conjecture 2 implies Statement 6.1.*

Proof. Assume, as in Statement 6.1, that either $Q \in \mathcal{P} \backslash \mathcal{S}_V$ or $Q \in \mathcal{S}_V$, and it is not V-codefinite. The first case is treated in Theorem 6.2. In the second case we still show that the center set at infinity $\overline{\mathrm{CS}}_{V,Q}$ does not contain non-definite polynomials. Assume, in contradiction, that $P \in \overline{\mathrm{CS}}_{V,Q}$ is non-definite, and let $W_1, \ldots, W_s$, $s \geq 2$, be all the $[a, b]$-indecomposable right factors of P. According to Theorem 2.1, P satisfies the equations $m_k(P, Q) = 0$ and $D_j(P, Q) = 0$. By the first set of these equations, $Q = \sum_{j=1}^{s} S_j(W_j)$ and, by the second set and by Conjecture 2, we conclude that one of W_j is a right factor of Q. Now, according to Theorem 4.4, Q is V-codefinite, in contradiction with the assumptions. This completes the proof. □

Our next result confirms Conjecture 2 and hence Statement 6.1 for deg P, deg $Q \leq 9$.

THEOREM 6.4.

(1) *Conjecture 2 is valid for* $\deg P, \deg Q \leq 9$, *i.e. vanishing of all the moments* $m_k(P, Q)$ *and of four initial second Melnikov coefficients* $D_j(P, Q)$ *implies the composition condition for such* P, Q.

(2) *Consequently, for* $V = \mathcal{P}_9$ *and for any* Q *of degree up to nine not of the form* $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$, *or of this form, but such that neither* $\widetilde{T}_2$ *nor* $\widetilde{T}_3$ *are the right composition factors of* Q, *the center set* $\mathrm{CS}_{V,Q}$ *consists of a composition set with possibly a finite number of additional points.*

Proof. By Theorem 6.3, the first part of Theorem 6.4 implies its second part. So, let polynomials P, Q with $\deg P, \deg Q \leq 9$ be given. If $P \neq \alpha\widetilde{T}_6$, it is definite and hence already vanishing of all the moments $m_k(P, Q)$ implies the composition condition for P, Q. Consider now the case $P = \widetilde{T}_6$. Here vanishing of $m_k(P, Q)$ implies that Q has a form $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$ for some polynomials S_2 and S_3. By Lemma 4.1, we conclude that S_1, S_2 can be written in the form $S_1(T) = \sum_{i=1}^{4} c_i T^i$, $S_2(T) = \sum_{i=1}^{2} \alpha_i T^{2i-1}$. Now we use the second set of the center equations at infinity: $D_j(P, Q) = 0$. □

PROPOSITION 6.1. *The first four equations at infinity,* $D_j(P, Q) = 0$, *given in Theorem 2.1 can be written as*

$$\begin{aligned} D_1(P, Q) &= \int_a^b Q^2 p = 0, \\ D_2(P, Q) &= \int_a^b Q^2 P p = 0, \\ D_3(P, Q) &= 2\int_a^b Q^2 P^2 p + \int_a^b Q(t)P(t)p(t)\,dt \int_a^t Q(\tau)p(\tau)\,d\tau = 0, \\ D_4(P, Q) &= \int_a^b Q^2 P^3 p + \int_a^b Q(t)P^2(t)p(t)\,dt \int_a^t Q(\tau)p(\tau)\,d\tau = 0. \end{aligned} \tag{6.1}$$

Proof. The proof is based on rather lengthy computations. We shall use the following result from [**10**].

THEOREM 6.5. [**10**, Theorem 2.2] *Any iterated integral* I_α *with* $m_0 + m_1 + m_2$ *appearances of* p *and exactly two appearances of* q *after* m_0 *and* m_1 *appearances of* p, *respectively, can be transformed via integration by parts to the sum of the iterated integrals of the following form:*

$$I_\alpha = \sum_{i=0}^{m_1} \frac{(-1)^{m_0+m_1-i}}{m_0!m_2!i!(m_1-i)!} \int_a^b P^{m_0+i}(x)q(x)\,dx \int_a^x P^{m_1+m_2-i}(t)q(t)\,dt. \tag{6.2}$$

Below we present calculations of the Melnikov coefficients at infinity $D_1(P, Q)$ and $D_3(P, Q)$. The coefficients D_2 and D_4 are obtained in a similar way. Let us recall that

$$P(a) = P(b) = Q(a) = Q(b) = 0, \tag{6.3}$$

while from Theorem 2.1 for $k = 4, 6, 8, 10$ it follows that

$$\int_a^b P^i q = 0, i = 1, \ldots, 4. \tag{6.4}$$

Case 1. Let $k=5$. Then the corresponding center equation at infinity is given by $D_1=\sum n_\alpha I_\alpha=0$, where the sum runs over all the indices $\alpha=(\alpha_1,\ldots,\alpha_s)$ with exactly two appearances of 1, and with $\sum_{j=1}^{s}\alpha_j=k-1=4$. Hence, we have exactly one appearance of 2, and $s=3$.

Now, for $\alpha_1=(1,1,2)$, we have $n_{\alpha_1}=-12$, $m_0=m_1=0$, $m_2=1$ and hence

$$\begin{aligned}I_{\alpha_1}&=\int_a^b q(x_1)\,dx_1\int_a^{x_1}q(x_2)\,dx_2\int_a^{x_2}p(x_3)\,dx_3\\&=\left[Q(x_1)\cdot\int_a^{x_1}q\int_a^{x_2}p\right]\Bigg|_{x_1=a}^{b}-\int_a^b Qq\,dx_1\int_a^{x_1}p\\&=-\frac{1}{2}\left[Q^2(x_1)\cdot\int_a^{x_1}p\right]\Bigg|_{x_1=a}^{b}+\frac{1}{2}\int_a^b Q^2p=\frac{1}{2}\int_a^b Q^2p.\end{aligned}$$

For $\alpha_2=(1,2,1)$, we have $n_{\alpha_2}=-8$, $m_0=0$, $m_1=1$, $m_2=0$ and

$$\begin{aligned}I_{\alpha_2}&=\int_a^b q\int_a^{x_1}p\int_a^{x_2}q=\left[Q(x_1)\cdot\int_a^{x_1}p\int_a^{x_2}q\right]\Bigg|_{x_1=a}^{b}-\int_a^b Qp\int_a^{x_1}q\\&=-\int_a^b Q(x_1)p(x_1)(Q(x_1)-Q(a))\,dx_1=-\int_a^b Q^2p.\end{aligned}$$

For $\alpha_3=(2,1,1)$, we have $n_{\alpha_3}=-6$, $m_0=1$, $m_1=0$, $m_2=0$ and

$$\begin{aligned}I_{\alpha_3}&=\int_a^b p\int_a^{x_1}q\int_a^{x_2}q=\int_a^b p\int_a^{x_1}Qq\\&=\frac{1}{2}\int_a^b(Q^2(x_1)-Q^2(a))p(x_1)\,dx_1=\frac{1}{2}\int_a^b Q^2p.\end{aligned}$$

Then $D_1=\sum_{i=1}^{3}n_{\alpha_i}I_{\alpha_i}=(-12\cdot\frac{1}{2}+8-6\cdot\frac{1}{2})\int_a^b Q^2p=\int_a^b Q^2p$.

Case 2. For $k=7,9,11$, we shall use expressions (6.2)–(6.4), which will allow us to present somewhat lengthy calculations in a more systematic way. For $k=7$, the calculations easily provide the answer $D_2=\int_a^b Q^2Pp=0$, so we concentrate on the case $k=9$. The center equation at infinity is given by

$$\sum n_\alpha I_\alpha=0 \tag{6.5}$$

with $\sum_{j=1}^{s}\alpha_j=k-1=8$, and with exactly two appearances of 1. Hence, we have exactly three appearances of 2, and $s=5$.

For $\alpha_1=(1,1,2,2,2)$, we have $n_{\alpha_1}=-8\cdot7\cdot5\cdot3$, $m_0=m_1=0$, $m_2=3$. Then, by (6.2), we have $I_{\alpha_1}=\frac{1}{3!}\int_a^b q(x)\,dx\int_a^x P^3q$. Integrating by parts once more, we get $I_{\alpha_1}=\frac{1}{4}\int_a^b Q^2P^2p$.

In a similar way, expressions (6.2)–(6.4) and some additional integrations by part allow us to express all I_α through $J_1=\int_a^b Q^2P^2p$ and $J_2=\int_a^b QPp\int_a^x Qp$.

For $\alpha_2=(1,2,1,2,2)$*:* $n_{\alpha_2}=-8\cdot6\cdot5\cdot3$, $m_0=0$, $m_1=2$, $m_2=2$ and

$$I_{\alpha_2}=-\frac{1}{2}\int_a^b q\int_a^x P^3q+\frac{1}{2}\int_a^b Pq\int_a^x P^2q=\cdots=-\frac{1}{2}J_1-J_2.$$

For $\alpha_3 = (1, 2, 2, 1, 2)$: $n_{\alpha_3} = -8 \cdot 6 \cdot 4 \cdot 3$, $m_0 = 0$, $m_1 = 2$, $m_2 = 1$ and

$$I_{\alpha_3} = \frac{1}{2}\int_a^b q \int_a^x P^3 q - \int_a^b Pq \int_a^x P^2 q + \frac{1}{2}\int_a^b P^2 q \int_a^x Pq = \cdots = 3J_2.$$

For $\alpha_4 = (1, 2, 2, 2, 1)$: $n_{\alpha_4} = -8 \cdot 6 \cdot 4 \cdot 2$, $m_0 = 0$, $m_1 = 3$, $m_2 = 0$ and

$$\begin{aligned} I_{\alpha_4} = -\frac{1}{6}\int_a^b q \int_a^x P^3 q + \frac{1}{2}\int_a^b Pq \int_a^x P^2 q - \frac{1}{2}\int_a^b P^2 q \int_a^x Pq \\ + \frac{1}{6}\int_a^b P^3 q \int_a^x q = \cdots = -2J_2. \end{aligned}$$

In the same way, for the remaining six transpositions $\alpha_5 = (2, 1, 1, 2, 2)$, $\alpha_6 = (2, 1, 2, 1, 2)$, $\alpha_7 = (2, 1, 2, 2, 1)$, $\alpha_8 = (2, 2, 1, 1, 2)$, $\alpha_9 = (2, 2, 1, 2, 1)$, $\alpha_{10} = (2, 2, 2, 1, 1)$, we obtain the following values of (n_α, I_α):

$$\begin{gathered} \left(-7 \cdot 6 \cdot 5 \cdot 3, -\tfrac{1}{4}J_1 + J_2\right), \quad \left(-7 \cdot 6 \cdot 4 \cdot 3, J_1 - 4J_2\right), \quad \left(-7 \cdot 6 \cdot 4 \cdot 2, 3J_2\right), \\ \left(-7 \cdot 5 \cdot 4 \cdot 3, -\tfrac{1}{4}J_1 + J_2\right), \quad \left(-7 \cdot 5 \cdot 4 \cdot 2, -\tfrac{1}{2}J_1 - J_2\right), \quad \left(-7 \cdot 5 \cdot 3 \cdot 2, \tfrac{1}{4}J_1\right). \end{gathered}$$

Substituting these expressions for m_α and I_α into (6.5), we finally obtain $-2(2J_1 + J_2)$, so, after omitting a non-zero coefficient -2, we get $D_3 = 2J_1 + J_2 = 2\int Q^2P^2p + \int QPp \int Qp$.

The last equation in (6.1), for $k = 11$, is obtained in a completely similar way. □

The following results describe the application of these four equations to the specific combinations of Chebyshev polynomials representing Q. To simplify the numeric coefficients, we assume here that $[a, b] = [0, 1]$ and so $\widetilde{T}_2(x) = x(x-1)$, $\widetilde{T}_3(x) = x(x-1)(2x-1)$. We also notice that $\widetilde{T}_6 = \widetilde{T}_3^2 = \widetilde{T}_2^2 + 4\widetilde{T}_2^3$.

In all the calculations below, $P = \widetilde{T}_6$ is fixed, while $Q = S_1(\widetilde{T}_2) + S_2(\widetilde{T}_3)$ with $S_1(T) = \sum_{i=1}^4 c_i T^i$, $S_2(T) = \sum_{i=1}^2 \alpha_i T^{2i-1}$ is variable. We substitute these P and Q into the equations (6.1) of Proposition 6.1 and get a system of algebraic equations with respect to the complex unknowns $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2$.

It is convenient to introduce the expressions $L_k = \int_0^1 S_1(\widetilde{T}_2)\widetilde{T}_3^k \, d\widetilde{T}_6$, which are linear forms in c_1, c_2, c_3, c_4. Using these expressions, we can rewrite equations (6.1) as

$$\begin{aligned} \alpha_1 L_1 + \alpha_2 L_3 &= 0, \\ \alpha_1 L_3 + \alpha_2 L_5 &= 0, \\ \tfrac{16}{15}\alpha_1 L_5 + \tfrac{36}{35} L_7 &= 0, \\ \tfrac{25}{21}\alpha_1 L_7 + \tfrac{49}{45} L_9 &= 0. \end{aligned} \tag{6.6}$$

PROPOSITION 6.2. *The expressions L_k, $k = 1, 3, 5, 7, 9$, can be written explicitly as the following linear forms in the coefficients c_1, c_2, c_3, c_4 of the polynomial $S_1(T)$:*

$$L_1 = -\frac{8}{13} \cdot \frac{(5!)^2}{11!}\left(-13c_1 + 4c_2 - c_3 + \frac{4}{17}c_4\right),$$
$$L_3 = -\frac{3}{14 \cdot 9} \cdot \frac{(8!)^2}{17!}\left(-\frac{38}{3}c_1 + 4c_2 - c_3 + \frac{16}{69}c_4\right),$$
$$L_5 = -\frac{4}{33 \cdot 25} \cdot \frac{(11!)^2}{23!}\left(-\frac{25}{2}c_1 + 4c_2 - c_3 + \frac{20}{87}c_4\right),$$
$$L_7 = -\frac{10}{11 \cdot 13 \cdot 31} \cdot \frac{(14!)^2}{29!}\left(-\frac{62}{5}c_1 + 4c_2 - c_3 + \frac{8}{35}c_4\right),$$
$$L_9 = -\frac{9}{13 \cdot 17 \cdot 37} \cdot \frac{(17!)^2}{35!}\left(-\frac{37}{3}c_1 + 4c_2 - c_3 + \frac{28}{123}c_4\right).$$

Proof. This is obtained by straightforward computation of L_k using the identities $\widetilde{T}_6 = \widetilde{T}_3^2 = \widetilde{T}_2^2 + 4\widetilde{T}_2^3$, $d\widetilde{T}_6 = 2\widetilde{T}_3 d\widetilde{T}_3 = 2(\widetilde{T}_2 + 6\widetilde{T}_2^2)d\widetilde{T}_2$ and $\int_0^1 \widetilde{T}_2^n(x)\,dx = (-1)^n \cdot (n!)^2/(2n+1)!$. □

Now we come back to system (6.6). Let us start with the special case where $\alpha_2 = 0$.

PROPOSITION 6.3. *Let $P = \widetilde{T}_6$, $Q = S_1(\widetilde{T}_2) + \alpha_1\widetilde{T}_3$, with $S_1(T) = \sum_{i=1}^4 c_i T^i$. If the first three equations (6.1) of Proposition 6.1 are satisfied, then either $Q = S_1(\widetilde{T}_2)$ or $Q = c_2\widetilde{T}_6 + \alpha_1\widetilde{T}_3$. In each of these cases Q has either $\widetilde{T}_2$ or $\widetilde{T}_3$ as a right composition factor.*

Proof. Substitution to the equations (6.6) gives the following system of equations on the coefficients $\alpha_1, c_1, c_2, c_3, c_4$:

$$\begin{aligned} \alpha_1\big(-13c_1 + 4c_2 - c_3 + \tfrac{4}{17}c_4\big) &= 0,\\ \alpha_1\big(-\tfrac{38}{3}c_1 + 4c_2 - c_3 + \tfrac{16}{69}c_4\big) &= 0,\\ \alpha_1\big(-\tfrac{25}{2}c_1 + 4c_2 - c_3 + \tfrac{20}{87}c_4\big) &= 0. \end{aligned} \tag{6.7}$$

The result follows immediately from this system. □

Let us consider now the remaining case, where $\alpha_2 \neq 0$.

PROPOSITION 6.4. *Let $P = \widetilde{T}_6$, $Q = S_1(\widetilde{T}_2) + \alpha_1\widetilde{T}_3 + \alpha_2\widetilde{T}_3^3$, with $S_1(T) = \sum_{i=1}^4 c_i T^i$ and $\alpha_2 \neq 0$. If all the four equations (6.1) of Proposition 6.1 are satisfied, then $Q = c_2\widetilde{T}_6 + \alpha_1\widetilde{T}_3 + \alpha_2\widetilde{T}_3^3$ and hence Q has $\widetilde{T}_3$ as a right composition factor.*

Proof. Substitution to the equations (6.6) gives a system of equations on the coefficients $\alpha_1, \alpha_2, c_1, c_2, c_3, c_4$, which, putting $K := \alpha_1/\alpha_2$, can be brought to the following form:

$$\begin{aligned} (-4199K - 19)c_1 + \big(323K + \tfrac{3}{2}\big)(4c_2 - c_3) + \big(76K + \tfrac{8}{23}\big)c_4 &= 0,\\ (-874K - 5)c_1 + \big(69K + \tfrac{2}{5}\big)(4c_2 - c_3) + \big(16K + \tfrac{8}{87}\big)c_4 &= 0,\\ (-40\,600K - 252)c_1 + \big(3248K + \tfrac{630}{31}\big)(4c_2 - c_3) + \big(\tfrac{2240}{3}K + \tfrac{144}{31}\big)c_4 &= 0,\\ (-7750K - 49)c_1 + \big(625K + \tfrac{147}{37}\big)(4c_2 - c_3) + \big(\tfrac{1000}{7}K + \tfrac{1372}{1517}\big)c_4 &= 0. \end{aligned} \tag{6.8}$$

System (6.8) contains four equations with respect to four variables K, c_1, c_4 and $t = 4c_2 - c_3$. It presents a system of four linear equations with respect to c_1, c_4, t, but K enters the coefficients. We shall show that for each K this system has only the trivial solution $c_1 = c_4 = t = 0$. Indeed, existence of a non-trivial solution would imply simultaneous vanishing of, for example, the determinants

$$\Delta_1(K) = \tfrac{21\,280}{3}K^3 + \tfrac{2736}{31}K^2 + \tfrac{1368}{4495}K + \tfrac{24}{103\,385}$$

and

$$\Delta_2(K) = 76\,000K^3 + \tfrac{101\,998\,240}{104\,673}K^2 + \tfrac{3\,934\,112}{1\,081\,621}K + \tfrac{3528}{1\,081\,621},$$

formed by the coefficients of system (6.3) in the rows 1, 2, 3 and 1, 3, 4, respectively. However, the resultant $\mathrm{res}(\Delta_1(K), \Delta_2(K))$ of the polynomials $\Delta_1(K)$, $\Delta_2(K)$ in K is approximately equal to 21.514 474 38, so it is non-zero and hence these polynomials do not have common roots. (The final calculation of $\Delta_1(K)$, $\Delta_2(K)$ and of their resultant has been performed with the help of the 'MATLAB' system).

We conclude that for any α_1, α_2 with $\alpha_2 \neq 0$, and for $K = \alpha_1/\alpha_2$, system (6.3) implies $c_1 = c_4 = 0$, $c_3 = 4c_2$. Consequently, any polynomial Q satisfying this system has a form

$$Q = c_2\widetilde{T}_2^2 + 4c_2\widetilde{T}_2^3 + \alpha_1\widetilde{T}_3 + \alpha_2\widetilde{T}_3^3 = c_2\widetilde{T}_6 + \alpha_1\widetilde{T}_3 + \alpha_2\widetilde{T}_3^3.$$

In particular, Q has $\widetilde{T}_3$ as its right composition factor. This completes the proof of Theorem 6.4: vanishing of the moments and of the initial four Melnikov coefficients implies composition for P, Q up to degree nine. □

Finally, we consider center sets in the subspaces $V = U_{\mathcal{R}}$, as defined in §4.3.

THEOREM 6.6. *Let a subset $\mathcal{R} = \{r_1, r_2, \ldots\}$ of prime numbers be fixed. Put $V = U(\mathcal{R})$, as defined in §4.3 above. Then, for any $a \neq b$ and for each fixed polynomial $Q \in U_1(\mathcal{R})$, the center set $\mathrm{CS}_{V,Q}$ of the Abel equation (1.1) inside the space V consists of a composition set with possibly a finite number of additional points.*

Proof. This is a direct consequence of Corollary 4.5 and Theorem 6.1. □

The results of this section cover all the results of Theorems 1.2–1.6 stated in the Introduction.

The methods developed in this paper work not only in the setting of the center equations at infinity. They can be applied also to the study of the local structure of the affine center set, extending the approach of [**7**]. Here we use the 'second degree' Nakayama lemma in order to conclude that the center set (locally near the origin) coincides with the composition set, defined by the moments and the second Melnikov function. We plan to present these results separately.

Our recent paper [**9**] applies the results of the present paper on definite polynomials to the parametric versions of the center-focus problem for the polynomial Abel equation.

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