Cyclicity of Nilpotent Centers

Douglas S. Shafer

April 21, 2015

Collaborator and Computer Algebra Tools

This research was done in collaboration with

Isaac A. García
Lleida, Spain

- Focus quantities computed using:
  MATHEMATICA 8.0—A General Purpose Computer Algebra System
- Computations with ideals done using:
  SINGULAR 3-1-6—A Computer Algebra System for Polynomial Computations

Polynomial Systems Contrasts

Non-degenerate

$$\begin{aligned} \dot{x} &= -y + R(x,y) \\ \dot{y} &= \quad x + S(x,y) \end{aligned}$$

- monodromic
- a center iff there is a suitable first integral
- the Lyapunov quantities are polynomials in the coefficients

Nilpotent

$$\begin{aligned} \dot{x} &= y + R(x,y) \\ \dot{y} &= \qquad S(x,y) \end{aligned}$$

- conditionally monodromic
- a center can exist without a formal or analytic first integral
- the Lyapunov quantities are conditionally polynomials in the coefficients

Andreev's Monodromy Theorem (1955, 1958)

$$\mathscr{X}: \quad \begin{aligned} \dot{x} &= y + R(x,y) \\ \dot{y} &= \qquad S(x,y) \end{aligned}$$

Let $y = F(X)$ be the unique solution of $y + R(x,y) = 0$ and

$$f(x) = S(x, F(x)) = ax^{\alpha} + \cdots$$
$$\varphi(x) = \operatorname{div} \mathscr{X}(x, F(x)) = bx^{\beta} + \cdots$$

The origin is monodromic if and only if

- $\alpha = 2n - 1$ is an odd integer
- $a < 0$
- $\varphi \equiv 0$ or

  $\beta \geqslant n$ or

  $\beta = n - 1$ and $b^2 + 4an < 0$.

Andreev Number

The *Andreev number* for the system

$$\mathscr{X}: \quad \begin{aligned} \dot{x} &= y + R(x,y) \\ \dot{y} &= \qquad S(x,y) \end{aligned}$$

with

$$\begin{aligned} f(x) &= S(x, F(x)) = ax^{2n-1} + \cdots, \quad a < 0 \\ \varphi(x) &= \operatorname{div} \mathscr{X}(x, F(x)) = bx^{\beta} + \cdots \end{aligned}$$

for which

$$\varphi \equiv 0 \quad \textit{or} \quad \beta \geqslant n \quad \textit{or} \quad \beta = n - 1 \text{ and } b^2 + 4an < 0$$

is the number n.

Standard Form

$$\begin{aligned} \dot{x} &= y + R(x,y) & f(x) &= S(x, F(x)) = ax^\alpha + \cdots \\ \dot{y} &= \qquad S(x,y) & \varphi(x) &= \operatorname{div} \mathscr{X}(x, F(x)) = bx^\beta + \cdots \end{aligned}$$

$$\downarrow \quad \begin{aligned} x &= u \\ y &= v + F(u) \end{aligned}$$

$$\begin{aligned} \dot{u} &= v + v\widetilde{R}(u,v) & f(u) &= au^\alpha + \cdots \\ \dot{v} &= f(u) + v\varphi(u) + v^2\widetilde{S}(u,v) & \varphi(u) &= bu^\beta + \cdots \end{aligned}$$

$$\downarrow \quad \begin{aligned} u &= \xi x \\ v &= -\xi y \end{aligned}$$

$$\begin{aligned} \dot{x} &= -y + y\widehat{R}(x,y) & \widehat{f}(x) &= -\xi^{-1} f(\xi x) = -a\xi^{\alpha-1}x^\alpha + \cdots \\ \dot{y} &= \widehat{f}(x) + y\widehat{\varphi}(x) + y^2\widehat{S}(x,y) & \widehat{\varphi}(x) &= \varphi(\xi x) = b\xi^\beta x^\beta + \cdots \end{aligned}$$

Lyaunov's Generalized Trigonometric Functions

For $n \in \mathbb{N}$ let

$$x = \operatorname{Cs}\theta \qquad y = \operatorname{Sn}\theta$$

denote the unique solution of

$$\begin{aligned} \tfrac{dx}{d\theta} &= -y & \qquad x(0) &= 1 \\ \tfrac{dy}{d\theta} &= x^{2n-1} & \qquad y(0) &= 0 \end{aligned}$$

$\operatorname{Cs}\theta$ and $\operatorname{Sn}\theta$ are periodic of least period

$$T_n = 2\sqrt{\frac{\pi}{n}}\frac{\Gamma(\frac{1}{2n})}{\Gamma(\frac{n+1}{2n})}$$

and satisfy

$$\operatorname{Cs}^{2n}\theta + n\operatorname{Sn}^2\theta = 1.$$

Generalized Polar Coordinates

For an analytic monodromic system with Andreev number n,

$$\dot{x} = -y + y\widehat{R}(x, y)$$
$$\dot{y} = \widehat{f}(x) + y\widehat{\varphi}(x) + y^2\widehat{S}(x, y)$$

define

$$x = r \operatorname{Cs}\theta, \quad y = r^n \operatorname{Sn}\theta$$

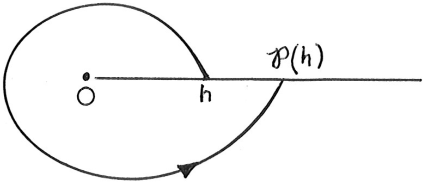
to obtain

$$\frac{dr}{d\theta} = \mathscr{F}[n](r, \theta)$$

for which $\mathscr{F}[n](r, \theta)$ is defined and analytic on a neighborhood of $r = 0$, is T_n-periodic, and satisfies $\mathscr{F}[n](0, \theta) \equiv 0$.

Generalized Lyapunov Quantities v_j

Let $\Psi(r;h)$ solve $\frac{dr}{d\theta} = \mathscr{F}[n](r,\theta)$, $\Psi(0;h) = h$.



$$\mathscr{P}(h) = \Psi(T_n;h) \qquad d(h) = \mathscr{P}(h) - h = \sum_{j \geqslant 1} v_j h^j$$

$$v_1 = \Psi_1(T_n) - 1, \quad v_j = \Psi_j(T_n), \ j \geqslant 2$$

Polynomial v_j

Given a monodromic polynomial family parametrized by the admissible coefficients, λ,

$$
\begin{aligned}
\dot{x} &= -y + y\,\widehat{R}(x,y) \\
\dot{y} &= \widehat{f}(x) + y\,\widehat{\varphi}(x) + y^2\,\widehat{S}(x,y)
\end{aligned}
$$

with

$$
\widehat{f}(x) = a_{2n-1}x^{2n-1} + \cdots\,, \quad \widehat{\varphi}(x) = b_\beta x^\beta + \cdots\,,
$$

the Poincaré-Lyapunov quantities v_i are polynomials in the parameters if and only if

1. a_{2n-1} is a fixed (positive) constant, not a parameter, which without loss of generality can be assumed to be 1; and
2. if $\widehat{\varphi}(x) \not\equiv 0$ and $\beta = n-1$ then b_β is a fixed constant, not a parameter.

Polynomial v_j: Proof

$$\begin{aligned}
\dot{x} &= -y + y\widehat{R}(x,y) \\
\dot{y} &= \widehat{f}(x) + y\widehat{\varphi}(x) + \overset{(y^2)}{\cdots} \\
\widehat{f} &= a_{2n-1}x^{2n-1} + \cdots \\
\widehat{\varphi} &= b_{n-1}x^{\beta} + \cdots
\end{aligned}
\quad \xleftarrow[v=-\xi y]{u=\xi x} \quad
\begin{aligned}
\dot{u} &= v + v\widetilde{R}(u,v) \\
\dot{u} &= f(u) + v\varphi(u) + \overset{(v^2)}{\cdots} \\
f &= ax^{2n-1} + \cdots \\
\varphi &= bx^{n-1} + \cdots
\end{aligned}$$

$$\frac{dr}{d\theta} = \frac{H_1 r + H_2 r^2 + \cdots}{J_0 + J_1 r + \cdots} = \frac{H_1}{J_0} r + \frac{H_2 J_0 - H_1 J_1}{J_0^2} r^2 + \cdots$$

where

- each H_i and J_i is a polynomial in λ, $\mathrm{Cs}\,\theta$, and $\mathrm{Sn}\,\theta$, and
- $$\begin{aligned} J_0 &= (\quad a_{2n-1}\ \mathrm{Cs}^{2n}\theta + n\,\mathrm{Sn}^2\theta) + b_{n-1}\ \mathrm{Cs}^n\theta Sn\theta \\ &= (-a\xi^{2n-2}\,\mathrm{Cs}^{2n}\theta + n\,\mathrm{Sn}^2\theta) + b\xi^{n-1}\,\mathrm{Cs}^n\theta Sn\theta \end{aligned}$$

Bautin Ideal

Analytic	Polynomial
$\dot{x} = y + R(x,y,\lambda)$ $\dot{y} = \quad S(x,y,\lambda)$	$\dot{x} = y + yR(x,y)$ $\dot{y} = \quad S(x,y)$ parametrized by admissible coefficients

$$d(h) = \mathscr{P}(h) - h = \sum_{j\geqslant 1} v_j(\lambda) h^j$$

$\mathscr{B} = \langle v_1(\lambda), v_2(\lambda), \cdots \rangle \in \mathscr{G}_{\lambda^*}$	$\mathscr{B} = \langle v_1(\lambda), v_2(\lambda), \cdots \rangle \in \mathbb{R}[\lambda]$

Minimal Basis

The *minimal basis* of a finitely generated ideal I with respect to an *ordered* basis $B = \{f_1, f_2, f_3, \ldots\}$ is the basis M_I defined by the following procedure:

(a) initially set $M_I = \{f_p\}$, where f_p is the first non-zero element of B;

(b) sequentially check successive elements f_j, starting with $j = p + 1$, adjoining f_j to M_I if and only if $f_j \notin \langle M_I \rangle$, the ideal generated by M_I.

Example

For $I = \langle x^3, x^2, x \rangle$ in $\mathbb{R}[x]$, $M_I = \{x, x^2, x^3\}$

Small Zeros of Analytic Functions (Bautin)

Technical Lemma

If $\{f_{j_1}, \ldots, f_{j_s}\}$ is the minimal basis for the ideal $\langle f_j : j \in \mathbb{N} \rangle$ with generators ordered by the indices, then the analytic function

$$Z(h, \lambda) = \sum f_j(\lambda) h^j$$

can be validly expressed as

$$Z(h, \lambda) = f_{j_1}(\lambda)[1 + \psi_1(h, \lambda)]h^{j_1} + \cdots + f_{j_s}(\lambda)[1 + \psi_s(h, \lambda)]h^{j_s}.$$

Zeros Theorem

Suppose $\psi_j(0, \lambda^*) = 0$ for all j. Then there exist δ and ϵ such that for each λ satisfying $|\lambda - \lambda^*| < \delta$ the equation $Z(h, \lambda) = 0$ has at most $s - 1$ isolated solutions in the interval $0 < h < \epsilon$.

Cyclicity Bound Theorem

$$d(h) = \mathscr{P}(h) - h = \sum_{j \geqslant 1} v_j h^j$$

If the minimal basis of the Bautin ideal $\mathscr{B} = \langle v_1, v_2, \cdots \rangle$ is

$$M_{\mathscr{B}} = \{v_{j_1}, v_{j_2}, \cdots, v_{j_s}\}$$

then the cyclicty of a center at the origin of any member of the family is at most $s - 1$.

Problems:

1. The generalized Lyapunov quantities are difficult to compute.
2. Even if we know a collection $\{v_{j_1}, \ldots, v_{v_s}\}$ whose vanishing at λ^* implies that all v_j vanish, we do not necessarily know a basis of $\mathscr{B}$.

Odd Degree Homogeneous Nonlinearities

Henceforth restrict to

$$\begin{aligned}\dot{x} &= y + P_{2m+1}(x,y)\\ \dot{y} &= \qquad Q_{2m+1}(x,y)\end{aligned}$$

Andreev's Monodromy Theorem:

$$(0,0) \text{ is monodromic iff } Q_{2m+1}(1,0) < 0$$

Make the change

$$\begin{aligned}x &= u - p(-q)^{-1/2}v\\ y &= (-q)^{1/2}v\\ t &= (-q)^{-1/2}\tau\end{aligned}$$

where

$$p = P_{2m+1}(1,0) \text{ and } q = Q_{2m+1}(1,0)).$$

Odd Degree Homogeneous Nonlinearities in Standard Form

$$\begin{aligned}\dot{x} &= y + yP(x,y) \\ \dot{y} &= \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1}\end{aligned}$$

for which

$$f(x) = -x^{2m+1}$$

$$\varphi(x) = \begin{cases} bx^{2m} & \text{if } b \neq 0 \\ 0 & \text{if } b = 0 \end{cases}$$

Andreev number $n = m + 1$

$v_j(\lambda) \in \mathbb{R}[\lambda]$, λ the admissible coefficients

Focus Quantities (Amel'kin, Lukashevich, Sadovskii, 1982)

For

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array}$$

there exists a formal series

$$W(x,y) = (m+1)y^2 + \sum_{k\geqslant 1} W_{2(km+1)}(x,y),$$

$$W_j \text{ homogeneous of degree } j$$

such that

- $\mathscr{X}W = x^{2(2m+1)}\sum_{k\geqslant 1} g_k x^{2km} = \sum_{k\geqslant 1} g_k x^{K(k)}$
- $g_k \in \mathbb{R}[\lambda]$
- $(0,0)$ is a center for system λ^* iff $g_k(\lambda^*) = 0$ for all k

The Focus Quantities and the Generalized Lyapunov Quantities

Generalized Lyapunov quantities v_j:

control stability and cyclicity

recursively computed via integrations

Focus quantities g_j:

pick out centers

recursively computed via algebra

Theorem

Let $I_k = \langle g_1, g_2, \ldots, g_k \rangle$. There exist positive constants w_k that are independent of λ such that

- $v_1 = \cdots = v_m = 0$ and $v_{m+1} = w_1 g_1$
- for $k \in \mathbb{N}$,
  - $v_{(2k-1)m+j} \in I_k$ for $j = 2, \ldots, 2m$
  - $v_{(2k+1)m+1} - w_{k+1} g_{k+1} \in I_k$

The Lyapunov and Focus Quantities

$$\begin{array}{cccc}
v_1 = 0 & v_{m+2} \in \langle g_1 \rangle & v_{3m+2} \in \langle g_1, g_2 \rangle & \\
\vdots & \vdots & \vdots & \cdots \\
v_m = 0 & v_{3m} \in \langle g_1 \rangle & v_{5m} \in \langle g_1, g_2 \rangle & \\
v_{m+1} = w_1 g_1 & v_{3m+1} - w_2 g_2 \in \langle g_1 \rangle & v_{5m+1} - w_3 g_3 \in \langle g_1, g_2 \rangle &
\end{array}$$

The Lyapunov and Focus Quantities: Sketch of the Proof

Truncate the formal series for W at sufficiently large $N = 2(\kappa m + 1)$.

Relate ΔW and Δh in one turn about the origin.

$$
\begin{aligned}
&\Delta W(h;\lambda) \\
&\quad = \int_0^{\tau(h)} \frac{d}{dt}\Big[W(x(t;h,\lambda), y(t;h,\lambda)\Big] dt \\
&\quad = \int_0^{\tau(h)} \sum_{k=1}^{\kappa} g_k(\lambda) x^{K(k)}(t;h,\lambda) dt \\
&\quad = \int_0^{T_{m+1}} \sum_{k=1}^{\kappa} g_k(\lambda) x^{K(k)}(t(\theta);h,\lambda) h^{-m}\Big[1 + \sum_{j\geqslant 1} u_j(\theta;\lambda) h^j\Big] d\theta
\end{aligned}
$$

The Lyapunov and Focus Quantities: Sketch of the Proof

Apply

$$x(\theta(t); h, \lambda) = r(\theta; h, \lambda)\,\mathrm{Cs}\,\theta = \Big[\sum_{i\geqslant 1} \Psi_i(\theta;\lambda)h^i\Big]\,\mathrm{Cs}\,\theta = \Big[h + \cdots\Big]\,\mathrm{Cs}\,\theta.$$

$$\begin{aligned}
&\Delta W(h;\lambda)\\
&= h^{-m}\sum_{k=1}^{\kappa}\left[\int_0^{T_{m+1}} x^{K(k)}(t(\theta); h, \lambda)\Big[1 + \sum_{j\geqslant 1} u_j(\theta;\lambda)h^j\Big]\,d\theta\right] g_k(\lambda)\\
&= h^{-m}\sum_{k=1}^{\kappa}\left[\int_0^{T} \mathrm{Cs}^{K(k)}\,\theta\Big[h^{K(k)} + \sum_{j\geqslant 2}\widehat{u}_j(\theta;\lambda)h^{K(k)+j}\Big]\,d\theta\right] g_k(\lambda)\\
&= h^{-m}\sum_{k=1}^{\kappa}\Big[w_k h^{K(k)} + g_{k,1}(\lambda)h^{K(k)+1} + g_{k,2}(\lambda)h^{K(k)+2} + \cdots\Big] g_k(\lambda)
\end{aligned}$$

$w_k = \int_0^T \mathrm{Cs}^{K(k)}\,\theta\,d\theta > 0$, independent of λ

The Lyapunov and Focus Quantities: Sketch of the Proof

$$\Delta W(h;\lambda)$$
$$= h^{-m}\sum_{k=1}^{\kappa}\Big[w_k h^{K(k)} + g_{k,1}(\lambda)h^{K(k)+1} + g_{k,2}(\lambda)h^{K(k)+2} + \cdots\Big]g_k(\lambda)$$

and

$$\Delta W = \zeta(h+\Delta h) - \zeta(h)$$

for the invertible function

$$\zeta(h) = W(h,0) = h^{2m+1} + \cdots$$

Apply Taylor's Theorem to the inverse to obtain

$$\Delta h = \frac{1}{h^{2m+1}}[c_0 + \cdots]\Delta W - \frac{1}{\widetilde{h}^{4m+3}}[d_0 + \cdots]\Delta W^2$$

for some $\widetilde{h} = O(h)$

The Lyapunov and Focus Quantities: Sketch of the Proof

$$\begin{aligned}\Delta h = {} & w_1 g_1 h^{m+1} + g_1 [\widetilde{g}_{1,1} h^{m+2} + \widetilde{g}_{1,2} h^{m+3} + \cdots] \\ & + w_2 g_2 h^{3m+1} + g_2 [\widetilde{g}_{2,1} h^{3m+2} + \widetilde{g}_{2,2} h^{3m+3} + \cdots] \\ & + w_3 g_3 h^{5m+1} + g_3 [\widetilde{g}_{3,1} h^{5m+2} + \widetilde{g}_{3,2} h^{5m+3} + \cdots] \\ & + \cdots \\ & + w_\kappa g_\kappa) h^{(2\kappa-1)m+1} \\ & \qquad + g_\kappa [\widetilde{g}_{\kappa,1} h^{(2\kappa-1)m+2} + \widetilde{g}_{\kappa,2} h^{(2\kappa-1)m+3} + \cdots].\end{aligned}$$

and

$$\Delta h = d(h; \lambda) = v_1 h + v_2 h^2 + v_3 h^3 + \cdots$$

The Bautin Ideal and Its Minimal Bases

$$\begin{array}{rclrclrcl} v_1 &=& 0 & v_{m+2} &\in& \langle g_1 \rangle & v_{3m+2} &\in& \langle g_1, g_2 \rangle \\ &\vdots& && \vdots & && \vdots & \\ v_m &=& 0 & v_{3m} &\in& \langle g_1 \rangle & v_{5m} &\in& \langle g_1, g_2 \rangle \\ v_{m+1} &=& w_1 g_1 & v_{3m+1} - w_2 g_2 &\in& \langle g_1 \rangle & v_{5m+1} - w_3 g_3 &\in& \langle g_1, g_2 \rangle \end{array} \quad \cdots$$

Thus

$$\mathscr{B} \stackrel{\text{def}}{=} \langle v_k : k \in \mathbb{N} \rangle = \langle v_{(2k-1)m+1} : k \in \mathbb{N} \rangle = \langle g_k : k \in \mathbb{N} \rangle$$

and the minimal bases

$$\{v_{k_1}, \ldots, v_{k_r}\} \text{ and } \{g_{j_1}, \ldots, g_{j_s}\}$$

satisfy

$$r = s \text{ and } k_p = (2j_p - 1)m + 1.$$

Summary: The Bautin Ideal and Its Minimal Bases

For

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array}$$

and the formal series

$$W(x,y) = (m+1)y^2 + \sum_{k \geqslant 1} W_{2(km+1)}(x,y)$$

such that $\mathscr{X}W = x^{2(2m+1)} \sum_{k \geqslant 1} g_k x^{2km} = \sum_{k \geqslant 1} g_k x^{K(k)}$

the Bautin ideal is

$$\mathscr{B} \stackrel{\text{def}}{=} \langle v_k : k \in \mathbb{N} \rangle = \langle v_{(2k-1)m+1} : k \in \mathbb{N} \rangle = \langle g_k : k \in \mathbb{N} \rangle$$

and the minimal bases

$$\{v_{k_1}, \ldots, v_{k_r}\} \text{ and } \{g_{j_1}, \ldots, g_{j_s}\}$$

satisfy

$$r = s \text{ and } k_p = (2j_p - 1)m + 1.$$

The Minimal Bases and Cyclicity of Centers

For

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array}$$

if the minimal bases of the Bautin ideal $\mathscr{B}$ are

$$\{v_{k_1}, \ldots, v_{k_r}\} \text{ and } \{g_{j_1}, \ldots, g_{j_s}\}$$

then the cyclicity of a center at the origin of member of the family is at most $r - 1 = s - 1$.

Notation: Ideals and Affine Varieties

Let $\mathbb{F}$ be a field.

The *affine variety* determined by $f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_p]$:

$$V = \mathbf{V}(f_1, \ldots, f_s) := \{x : f_j(x) = 0 \text{ for } j = 1, \ldots, s\} \subset \mathbb{F}^p.$$

For $I = \langle f_1, \ldots, f_s \rangle$, we also write

$$V = \mathbf{V}(I)$$

The *ideal detemined by a variety* V:

$$\mathbf{I}(V) := \{f : f(a) = 0 \text{ for all } a \in V\} \subset \mathbb{F}[x_1, \ldots, x_p]$$

The Center Variety

For

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array}$$

and the formal series

$$W(x,y) = (m+1)y^2 + \sum_{k \geqslant 1} W_{2(km+1)}(x,y)$$

such that $\quad \mathscr{X}W = x^{2(2m+1)} \sum_{k \geqslant 1} g_k x^{2km} = \sum_{k \geqslant 1} g_k x^{K(k)}$

the parameter values corresponding to a center is the variety

$$V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}) = \mathbf{V}(v_k : k \in \mathbb{N}) = \mathbf{V}(g_k : k \in \mathbb{N}) \subset \mathbb{R}^{4m+2}.$$

The Computational Challenge

Knowing the solution of the center problem for the family

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array}$$

means, at the least, knowing something like

$$V_{\mathscr{C}} \overset{\text{def}}{=} \mathbf{V}(v_1, v_2, \dots) = \mathbf{V}(v_{j_1}, \dots, v_{j_r})$$

or

$$V_{\mathscr{C}} = \mathbf{V}(g_1, g_2, \dots) = \mathbf{V}(g_{k_1}, \dots, g_{k_s}).$$

We need the minimal basis of $\mathscr{B}$ but it is possible that

$$\langle g_{k_1}, \dots, g_{k_s} \rangle \subsetneq \langle g_1, g_2, \dots \rangle$$

Finding the Minimal Basis of $\mathscr{B}$

Knowing

$$V_{\mathscr{C}} \stackrel{\text{def}}{=} \mathbf{V}(g_1, g_2, \dots) = \mathbf{V}(g_{k_1}, \dots, g_{k_s}) \qquad (1)$$

for the family

$$\mathscr{X}: \begin{array}{l} \dot{x} = y + yP(x,y) \\ \dot{y} = \quad -x^{2m+1} + bx^{2m}y + \cdots + cy^{2m+1} \end{array} \qquad (2)$$

- view (2) as a family on $\mathbb{C}^2$ with coefficients in $\mathbb{C}$
- W and the focus quantities g_k exist as before
- prove by non-geometric methods that

$$g_{k_1}(\lambda^*) = \cdots = g_{k_s}(\lambda^*) = 0 \text{ yields } W \text{ such that } \mathscr{X}W = 0$$

- obtaining (1) in $\mathbb{C}^{2(2m+1)}$
- and use the Strong Nullstellensatz to finish when $\langle g_{k_1}, \dots, g_{k_s} \rangle$ is radical.

Homogeneous Cubic Nonlinearities

$$\begin{aligned} \dot{x} &= y + Ax^2y + Bxy^2 + Cy^3 \\ \dot{y} &= \quad -x^3 + Px^2y + Kxy^2 + Ly^3 \end{aligned} \tag{3}$$

Andreev, 1953:
System (3) has a center at the origin if and only if

$$h_1 = P \qquad h_2 = B + 3L \qquad h_3 = (A + K)L$$

all vanish.

Theorem
A sharp global upper bound for the cyclicity of centers at the origin for systems in family (3) is two.

Homogeneous Cubic Nonlinearities: The Focus Quantities

$$\begin{aligned} \dot{x} &= y + Ax^2y + Bxy^2 + Cy^3 \\ \dot{y} &= \quad -x^3 + Px^2y + Kxy^2 + Ly^3 \end{aligned}$$

Focus quantities:

$g_1 = P$

$g_2 = 3B + 9L - 3AP - 4KP$

$g_3 = 10\text{-term cubic}$

Reduced Focus quantities:

$g_1 = h_1 = P$

$\widetilde{g}_2 = h_2 = B + 3L$

$\widetilde{g}_3 = h_3 = (A + K)L$

$$V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}) = \mathbf{V}(h_1, h_2, h_3) = \mathbf{V}(g_1, g_2, g_3) \overset{\text{nota.}}{=} \mathbf{V}(\mathscr{B}_3)$$

Finding the Center Variety in the Complex Setting

On $\mathbb{R}^2$: $V_{\mathscr{C}} = \mathbf{V}(P,\ B + 3L,\ (A+K)L)$.

Using SINGULAR compute the primary decomposition

$$\langle h_1, h_2, h_3 \rangle = J_1 \cap J_2 = \langle P, A+K, B+3L \rangle \cap \langle P, B, L \rangle$$

- $\lambda^* \in \mathbf{V}(J_1)$ implies the system is Hamiltonian with Hamiltonian W of the desired form
- $\lambda^* \in \mathbf{V}(J_2)$ implies existence of invariance under $(x, y, t) \to (-x, y, -t)$
  - this suggests: there is a first integral containing no odd power of x, which can be proved by induction; or
  - quote a theorem of Chavarriga, Giacomini, Giné, and Llibre (2003) to this effect

On $\mathbb{C}^2$: $V_{\mathscr{C}} = \mathbf{V}(P,\ B + 3L,\ (A+K)L)$.

The Minimal Basis of $\mathscr{B}$ and an Upper Bound

A computation yields $\sqrt{\langle g_1, g_2, g_3 \rangle} = \langle g_1, g_2, g_3 \rangle$.

Using the Strong Nullstellensatz (valid over $\mathbb{C}$):

$$\mathscr{B} \subset \sqrt{\mathscr{B}} = \mathbf{I}(\mathbf{V}(\mathscr{B})) = \mathbf{I}(\mathbf{V}(\mathscr{B}_3)) = \sqrt{\mathscr{B}_3} = \mathscr{B}_3 \subset \mathscr{B}$$

hence

$$M_{\mathscr{B}} = \{g_1, g_2, g_3\}$$

so by the Cyclicity Bound Theorem the cyclicity of any center is at most two.

The Global Upper Bound Is Sharp I

$$M_{\mathscr{B}} = \{g_1, g_2, g_3\} \text{ implies } M_{\mathscr{B}} = \{v_2, v_4, v_6\}$$

hence

$$d(h,\lambda) = v_2(\lambda)[1+\psi_1(h,\lambda)]h^2 + v_4(\lambda)[1+\psi_2(h,\lambda)]h^4 + v_6(\lambda)[1+\psi_3(h,\lambda)]h^6.$$

hence

$$d(h,\lambda) = g_1(\lambda)[1+\widetilde{\psi}_1(h,\lambda)]h^2 + g_2(\lambda)[1+\widetilde{\psi}_2(h,\lambda)]h^4 + g_3(\lambda)[1+\widetilde{\psi}_3(h,\lambda)]h^6.$$

The Global Upper Bound Is Sharp II

$$d(h,\lambda) = g_1(\lambda)[1+\widetilde{\psi}_1(h,\lambda)]h^2 + g_2(\lambda)[1+\widetilde{\psi}_2(h,\lambda)]h^4 + g_3(\lambda)[1+\widetilde{\psi}_3(h,\lambda)]h^6.$$

$$g_1 = P$$
$$g_2 = 3B + 9L - 3AP - 4KP$$
$$g_3 = -60AB - 66BK - 120AL - 138KL + 30A^2P - 45CP + 61AKP + 23K^2P + 25BP^2 + 50LP^2$$

Independently adjust the g_j to produce two small cycles.

Remark. Romanovski (1986) and Andreev, Sadovskii, Tsikalyuk (2003): two cycles can be made to bifurcate from a third order focus.

Homogeneous Quintic Nonlinearities

$$\begin{aligned}\dot{x} &= y + Ax^4y + Bx^3y^2 + Cx^2y^3 + Dxy^4 + Ey^5 \\ \dot{y} &= \quad -x^5 + Qx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5.\end{aligned} \tag{4}$$

Sadovskii, 1968:
System (4) has a center at the origin if and only if

- either B, D, Q, L, and N all vanish
- or Q, $2A + K$, $B + L$, $C + 2M$, and $D + 5N$ all vanish.

That is, the center variety is

$$V_{\mathscr{C}} = \mathbf{V}(B, D, L, N, Q) \cup \mathbf{V}(Q, 2A + K, B + L, C + 2M, D + 5N)$$

Finding the Center Variety in the Complex Setting

On $\mathbb{R}^2$: $V_{\mathscr{C}} = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) = \mathbf{V}(J_1 \cap J_2)$, where

$$J_1 = \langle B, D, Q, L, N \rangle$$
$$J_2 = \langle Q, 2A + K, B + L, C + 2M, D + 5N \rangle$$

- $\lambda^* \in \mathbf{V}(J_1)$ implies existence of invariance under $(x, y, t) \to (-x, y, -t)$
- $\lambda^* \in \mathbf{V}(J_2)$ implies the system is Hamiltonian with Hamiltonian W of the desired form

On $\mathbb{C}^2$: $V_{\mathscr{C}} = \mathbf{V}(J_1) \cup \mathbf{V}(J_2)$.

Homogeneous Quintic Nonlinearities: The Focus Quantities

$$\begin{aligned}\dot{x} &= y + Ax^4y + Bx^3y^2 + Cx^2y^3 + Dxy^4 + Ey^5 \\ \dot{y} &= \quad -x^5 + Qx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5.\end{aligned}$$

Focus quantities:

$g_1 = Q$

$g_2 = 10B + 10L - 10AQ - 7KQ$

$g_3 = $ 13-term cubic

$\vdots$

Reduced Focus quantities:

$g_1 = Q$

$\widetilde{g}_2 = B + L$

$\widetilde{g}_3 = 3D + 4AL + 2KL + 15N$

$\vdots$

Using SINGULAR compute the prime decomposition to obtain

$$\sqrt{\mathscr{B}_6} = J_1 \cap J_2$$

hence

$$V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2) = \mathbf{V}(J_1 \cap J_2) = \mathbf{V}(\mathscr{B}_6)$$

Houston, we have a problem

$$V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}) = \mathbf{V}(\sqrt{\mathscr{B}_6}) = \mathbf{V}(\mathscr{B}_6)$$

but a computation shows that

$$\sqrt{\mathscr{B}_6} \supsetneq \mathscr{B}_6$$

- we cannot conclude that $\mathscr{B} = \mathscr{B}_6$:

$$\mathscr{B} \subset \sqrt{\mathscr{B}} = \mathbf{I}(\mathbf{V}(\mathscr{B})) = \mathbf{I}(\mathbf{V}(\mathscr{B}_6)) = \sqrt{\mathscr{B}_6} \supsetneq \mathscr{B}_6$$

- we do not know that the obvious minimal basis $\{g_1, \ldots, g_6\}$ of $\mathscr{B}_6$ is even a basis of $\mathscr{B}$

A Second Cyclicity Bound Theorem

Suppose

- $\{g_{j_1}, \dots, g_{j_s}\}$ is the minimal basis of the ideal $I = \langle g_{j_1}, \dots, g_{j_s} \rangle$ that it generates
- $V_{\mathscr{C}} = \mathbf{V}(I)$
- $I = R \cap N = \text{(primes)} \cap \text{(primaries)}$

Then for the system corresponding to any $\lambda^* \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$, the cyclicity of the center at the origin is at most $s - 1$.

(An adaptation to this setting of a result of Ferčec, Levandovskyy, Romanovski, Shafer, 2015/6.)

Quintics: An Upper Bound On a Subset of $V_{\mathscr{C}}$

Let R_3 denote the prime ideal

$$R_3 = \langle B,\ D,\ Q,\ L,\ N,\ 2ACK + CK^2 - 4A^2M + K^2M + C^2 + 4CM + 4M^2 \rangle.$$

Then for any system in the quintic family corresponding to a parameter value λ lying in $V_{\mathscr{C}} \setminus \mathbf{V}(R_3)$ the cyclicity of the center at the origin is at most five.

Proof.

$$\mathscr{B}_6 = (J_1 \cap J_2) \cap (J_3 \cap J_4) = (\text{primes}) \cap (\text{primaries})$$

$$\sqrt{J_3} = R_3 \subset R_4 = \sqrt{J_4}$$

$$\mathbf{V}(N) = \mathbf{V}(\sqrt{N}) = \mathbf{V}(\sqrt{J_3} \cap \sqrt{J_4}) = \mathbf{V}(R_3 \cap R_4) = \mathbf{V}(R_3)$$

Restatement and Global Sharpness Result

The cyclicity of a center at $(0, 0)$ of any element of the family

$$\begin{aligned} \dot{x} &= y + Ax^4y + Bx^3y^2 + Cx^2y^3 + Dxy^4 + Ey^5 \\ \dot{y} &= \quad -x^5 + Qx^4y + Kx^3y^2 + Lx^2y^3 + Mxy^4 + Ny^5, \end{aligned}$$

except those of the form

$$\begin{aligned} \dot{x} &= y + Ax^4y + Cx^2y^3 + Ey^5 \\ \dot{y} &= \quad -x^5 + Kx^3y^2 + Mxy^4 \end{aligned}$$

satisfying $2ACK + CK^2 - 4A^2M + K^2M + C^2 + 4CM + 4M^2 = 0$, is at most five.

In each irreducible component $\mathbf{V}(J_1)$ and $\mathbf{V}(J_2)$ of $V_{\mathscr{C}}$ there are points from which five limit cycles can be made to bifurcate.

In Closing

- for quintics:
  - $g_j \in \mathscr{B}_6$ for $j \leqslant 11$ making it likely that $\mathscr{B} = \mathscr{B}_6$ so we conjecture that a global upper bound on cyclicity of quintic centers is five
  - by imposing a relation among coefficients the ideal $\mathscr{B}_6$ can become radical in the polynomial ring in the remaining coefficients
  - in particular, this is so if any one of B, D, Q, L, or N is fixed, and the cyclicity is bounded above by five
- in general:
  - the ideas and methods described in this lecture have been further extended to larger families by Isaac García.