FINITE CYCLICITY OF SOME CENTER GRAPHICS

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WORK IN COLLABORATION

with

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The program for quadratic vector fields

To prove the existence of a finite bound $H(\mathcal{X})$ for any compact family $\mathcal{X}$ of vector fields on S^2, it is sufficient to prove that **any limit periodic set of $\mathcal{X}$ has a finite cyclicity inside the family.**

To apply this idea to the analytic family equivalent to the family of quadratic vector fields, Dumortier, Rousseau and myself have, in 1994, established a **list of all the possible limit periodic sets that have to be considered.**

Quadratic limit periodic sets

In this paper of 1994 we have presented the list of limit periodic sets, ordered by complexity, starting with the simpler ones :

Bounded hyperbolic graphics

and finishing with the more degenerate ones :

Graphics including non isolated singularities.

Any non trivial limit periodic set is a **graphic,** non degenerate or not.
The classification of possible graphics uses the type of singularities contained in the limit periodic set : they may be **hyperbolic, elementary, nilpotent, in family.**

Moreover the graphic may contains points or arcs at infinity (the more common case). It may be of **finite codimension** type or of **center** type. Finally it may contain non isolated singularities : in this case we have a **slow-fast unfolding.**

Present state of the program

Most of the cases have been now resolved.

The hyperbolic graphics were already solved in 1994, when they are of finite codimension and were solved by *Mourtada, Gavrilov, Zoladek* in the center cases. Graphics of finite codimension through a nilpopent point were systematically studied by *Rousseau and Zhu.* Some degenerate graphics were solved by *Fiedelers, Dumortier, Rousseau.*

Then its remains to solve the more degenerate cases, **essentially of center type.**

Nilpotent singular points of codimension 3

We will consider graphics through a nilpotent point at infinity :

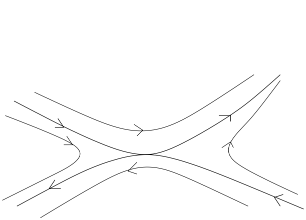


FIGURE: The different topological types of nilpotent points

A simplification : As the nilpotent singular point is located at infinity, the circle at infinity is invariant inside the quadratic family.

Graphics considered in this talk

They are **graphics containing a nilpotent singular point at infinity and of center type.** We have already some of them in a previous article : the graphics of pp-**type surrounding a center.** Now the graphic is of hh-**type surrounding a center and the nilpotent point is at infinity.**

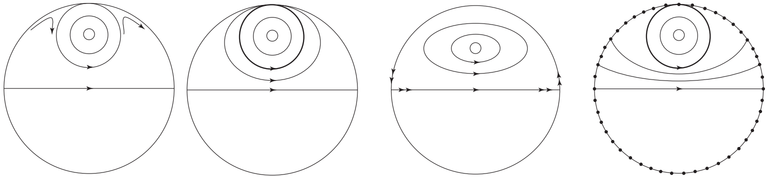


FIGURE: The graphics (I^1_{14}), (I^1_{6b}), (H^3_{13}) and (DI_{2b}).

Main results

(1) The boundary limit periodic sets in the graphics (I^1_{6b})*,* (H^3_{13}) *and* (DI_{2b}) *have a finite cyclicity.*

(2) The graphic (I^1_{14}) *has a finite cyclicity.*

A **boundary limit periodic set** is a secondary limit periodic set (obtained by blow up) containing an arc in the boundary of blow-up disk (see figure).

Normal form at a nilpotent point

A generic unfolding depending on a multi-parameter $\lambda = (\mu_1, \mu_2, \mu_3, \mu) \sim 0 \in \mathbb{R}^{k+3}$, has the form

$$\dot{x} = y + a(\lambda)x^2 + \mu_2$$

$$\dot{y} = \mu_1 + \mu_3 y + x^4 h_1(x, \varepsilon) + y(x + \eta x^2 + x^3 h_2(x, \lambda)) + y^2 Q$$

where $h_1(x, \lambda) = O(|\lambda|)$. Moreover, $h_1, h_2, Q = Q(x, y, \lambda)$ are C^∞ functions, and Q can be chosen of arbitrarily high order in λ. We have that $a(0) \neq \frac{1}{2}$.

Blowing up

First, we make the change of parameters

$$(\mu_1, \mu_2, \mu_3) = (\nu^3 \bar{\mu}_1, \nu^2 \bar{\mu}_2, \nu \bar{\mu}_3).$$

Next, we perform the blow-up transformation

$$(x, y, \nu) = (r\bar{x}, r^2\bar{y}, r\rho),$$

with $r > 0$ and $(\bar{x}, \bar{y}, \rho) \in S^2$.
The blown-up vector field $\bar{X}_A$ is defined on a 3-dimensional space with boundary the **critical locus** $\{r\rho = 0\}$. It is tangent to the **foliation** defined by $\{\nu = r\rho = \mathrm{Cst}\}$. It depends on the parameter $A = (a - a_0, \bar{M}, \mu) \sim 0$, where $\bar{M} = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$.

The critical locus

The critical locus $\{r\rho = 0\}$ contains two strata :

- $\mathbb{S}^1 \times \mathbb{R}^+$ is the blown-up space for X_0 (for $\lambda = 0$) ;
- $D_{\bar{\mu}} = \{\bar{x}^2 + \bar{y}^2 + \rho^2 = 1 \mid \rho \geq 0\}$, for any $\bar{\mu} \in \mathbb{S}^2$.

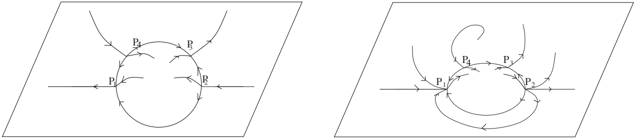
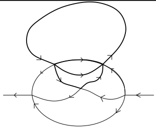
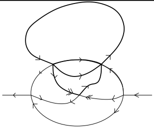
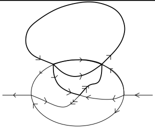
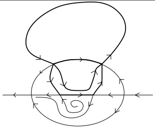


FIGURE: Saddle and Elliptic cases

Secondary limit periodic sets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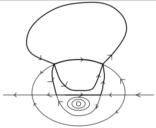	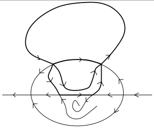	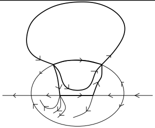	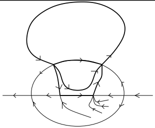
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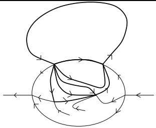
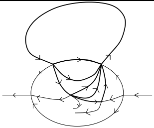
			
Sxhh9	Sxhh10		

Strategy to obtain the finite cyclicity

The initial graphic Γ is replaced by a whole family of secondary limit periodic sets in the blown-up space. For instance, in the case $Sxhh1$ this family begins with the boundary limit periodic set and finishes with a limit periodic set through a saddle on the critical locus.
The graphic Γ has a finite cyclicity if any secondary limit periodic set has a finite cyclicity.
We prove that **the boundary limit periodic set has always a finite cyclicity.** We finally prove that **any secondary limit periodic set of $\Gamma = (I^1_{14})$ has a finite cyclicity.**

We begin with the boundary limit set of (I^1_{14}),(I_{6b}),(DI_{2b}).

The boundary limit periodic set

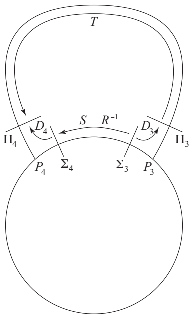


FIGURE: The boundary graphic through P_3 and P_4 and the four sections Σ_i and Π_i, $i = 3, 4$, in the normalizing coordinates.

We have to study the zeros of the displacement map :

$$V = D_4 \circ S - T \circ D_3.$$

The 3-dimensional saddles P_i

Near the points P_i, the vector field $\bar{X}_A$ is equivalent to :

$$\begin{cases} \dot{u} = u, \\ \dot{v} = -v, \\ \dot{y} = -\sigma y + O(|(u, v, y)|^2), \end{cases}$$

where the eigenvalue σ depends on the parameter b. We write $\sigma_0 = \sigma(b_0)$. We verify that $\nu = uv$ is a first integral.

Normal form at the saddles P_i

To compute the Dulac maps we use a normal coordinate Y where the last line of the above differential equation takes the form :

- If $\sigma_0 \notin \mathbb{Q}$: $\dot{Y} = -(\sigma + \varphi_A(\nu))Y$;
- If $\sigma_0 = \frac{p}{q}$:
  $$\dot{Y} = -(\sigma + \varphi_A(\nu))Y + \Phi_A(\nu, u^p Y^q)Y + v^p \eta_A(\nu)$$
  .

$y \to Y$ can be taken of class $\mathcal{C}^k$ for any k.
$\varphi_A, \Phi_A, \eta_A$: polynomials of degree $K(k) >> k$ with smooth coefficient in A.
$\Phi_A(\nu, 0) \equiv 0$ and $\eta_A \equiv 0$ if $\sigma_0 \notin \mathbb{N}$.

Expression of the Dulac maps D_i

We introduce $\bar\sigma = \bar\sigma(\sigma,\nu) = \sigma + \varphi_A(\nu)$ and $\alpha = \alpha(\sigma,\nu) = \bar\sigma - \sigma_0$.
We consider the Dulac map from the section $\{\bar Y = Y_0\}$, parametrized by (r,ρ) to a section $\{r = r_0\}$ parameterized by $(\bar Y,\nu)$. It has the form $(r,\rho) \mapsto (D_A(r,\rho),\nu)$, with its $\bar Y$-component, $(D_A(r,\rho)$, given by :

1. If $\sigma_0 \notin \mathbb{Q}$:
$$D_A(r,\rho) = \left(\frac{r}{r_0}\right)^{\bar\sigma} Y_0.$$
2. If $\sigma_0 = \frac{p}{q} \in \mathbb{Q}$ with $(p,q) = 1$ when $\sigma_0 \notin \mathbb{N}$:
$$D_A(r,\rho) = \eta_A(\nu)\rho^p\left(\frac{r}{r_0}\right)^{\bar\sigma}\omega\left(\frac{r}{r_0},\alpha\right)+\left(\frac{r}{r_0}\right)^{\bar\sigma}\Big(Y_0+\phi_A(r,\rho)\Big),$$
with $\eta_A \equiv 0$ when $\sigma_0 \notin \mathbb{N}$.

The function family ϕ_A is of order $O(r^{p+q\alpha}\omega^{q+1}\left(\frac{r}{r_0},\alpha\right)|\ln r|)$ and, for any integer $l \geq 2$, is of class $\mathcal{C}^{l-2}$ in $(r^{1/l}, r^{1/l}\omega\left(\frac{r}{r_0},\alpha\right), \rho, \mu, \sigma)$.

Compensators

For $\xi > 0$:

$$\omega(\xi,\alpha)=\omega_\alpha(\xi)=\begin{cases}\frac{\xi^{-\alpha}-1}{\alpha}, & \alpha\neq 0,\\ -\ln\xi, & \alpha=0.\end{cases}$$

$$\Omega(\xi,\alpha,\beta)=\Omega_{\alpha,\beta}(\xi)=\begin{cases}\frac{\omega(\xi,\alpha)-\omega(\xi,\beta)}{\alpha-\beta}, & \alpha\neq\beta,\\ \frac{1}{2}(\ln\xi)^2, & \alpha=\beta,\end{cases}$$

$\omega = O(\xi^{-|\alpha|}|\ln\xi|)$, $\Omega = O(\xi^{-\mathrm{Max}\{|\alpha|,|\beta|\}}\ln^2\xi)$,
$\omega\to+\infty$, $\Omega\to+\infty$ for $(\xi,\alpha)\to(0,0)$ or
$(\xi,\alpha,\beta)\to(0,0,0)$

$\mathcal{C}^k$ function on monomials

The functions $f(r,\rho)$ are $\mathcal{C}^k$ **function on monomials for arbitrary large** k :

$$f(r,\rho) = \tilde{f}(M_1,\dots,M_l),$$

with $\tilde{f}$ a $\mathcal{C}^k$-function and the M_i are monomials in $r^a, \rho^b, r^a\omega_\gamma^c,\ r^a\Omega_{\gamma_1,\gamma_2}^d, \omega_\gamma^{-e}$, where $a,b,c,d,e > 0$.

We will have to consider the derivation $L_{\mathcal{X}}f$ with $\mathcal{X} = r\frac{\partial}{\partial r} - \rho\frac{\partial}{\partial \rho}$.

This derivation has good properties. For instance, if f **is a** $\mathcal{C}^k$ **function on monomials, then** $L_{\mathcal{X}}f$ **is also a** $\mathcal{C}^{k-1}$ **function on monomials.**

The functions depend smoothly on parameters $(\bar{\mu}_3, \mu, \dots)$.

Expression of the difference map V

1. **Case** $\sigma_0 \notin \mathbb{N}$.

$$V = a_0(1+\cdots) + a_1 r^{\bar{\sigma}_3}(1+\cdots) + a_2 r^{\bar{\sigma}_3}\rho(1+\cdots),$$

2. **Case** $\sigma_0 = p \neq 1$.

$$V = a_0(1+\cdots)+a_1 r^{\bar{\sigma}_3}(1+\cdots)+a_2 r^{\bar{\sigma}_3}\rho(1+\cdots)+a_3 r^{\bar{\sigma}_3}\rho^p\omega.$$

3. **Case** $\sigma_0 = 1$.

$$V = a_0(1+\cdots)+a_1 r^{\bar{\sigma}_3}(1+\cdots)+a_2 r^{\bar{\sigma}_3}\rho(1+\cdots) + a_3\rho\omega(1+\cdots)$$

a_i : functions of the parameter and $\omega = \omega(\frac{r}{r_0}, \bar{\sigma}_3 - 1)$.
$+\cdots$: $\mathcal{C}^k$ on monomials, for large k.

Obtention of leading monomials

Transition S along the boundary arc :

$$S(r,\rho) = (rF(r,\rho), \rho F^{-1}(r,\rho).$$

To obtain the leading monomial $r^{\bar{\sigma}_3}\rho$ behind the coefficient a_2, we need to prove that :

$$F(r,\rho) = 1 + *\bar{\mu}_3\rho(1+O(M)) + O(r)O(M_C),$$

where $*$ **is a strictly positive constant** and $M_C = (\bar{\mu}_3, \mu_4, \mu_5)$ and $M = (M_C, \bar{\mu}_2, b_0 - 1)$.
This comes from the properties of quadratic vector fields.

Starting with the blow-up system, restricted to $\{r = 0 :\}$

$$\dot{\rho} = -\rho(\bar{x} - \bar{\mu}_3\rho) = P(\rho, \bar{x}),$$

$$\dot{\bar{x}} = 2 + (1 - 2B)\bar{x}^2 - 2\bar{\mu}_2\rho^2 - \bar{\mu}_3\bar{x}\rho = Q(\rho, \bar{x}),$$

we have that

$$S''(0) = 2\left(f_4'(0)S'(0)\left(\frac{P_\rho'}{Q}\right)(0, f_4(0)) - f_3'(0)\left(\frac{P_\rho'}{Q}\right)(0, f_3(0))\right)$$

$$+\int_{f_3(0)}^{f_4(0)}\left(\frac{P_{\rho\rho}''}{Q}(0,\bar{x}) - 2\frac{P_\rho' Q_\rho'}{Q^2}(0,\bar{x})\right)\exp\left(\int_{f_3(0)}^{\bar{x}}\left(\frac{P_\rho'}{Q}\right)(0,x)dx\right)d\bar{x}$$

where the f_i depend on the choice of sections.

For instance, the last term is given by :

$$I_3 = 4\bar{\mu}_3(2+(1-2B)x_0^2)^{\frac{1}{2(1-2B)}} \int_{x_0}^{-x_0} (1-B\bar{x}^2)(2+(1-2B)\bar{x}^2)^{\frac{8B-5}{2(1-2B)}} d\bar{x}.$$

We have to prove that $\frac{1}{\bar{\mu}_3}S''(0) \neq 0$. It is rather involved. For instance, in the case $B_0 \neq \frac{3}{4}$, we have to use an expression of I_3 through the Gauss hypergeometric function.

Finite cyclicity for the boundary l.p.s.

We have to prove that V has a finite multiplicity along the curves $r\rho = \nu$. To this end we use the **procedure of "division-derivation"** with the derivation $L_{\mathcal{X}}$, where $\mathcal{X} = r\frac{\partial}{\partial r} - \rho\frac{\partial}{\partial \rho}$.
To control the recurrence in the procedure we use the following result :

If $M = r^a \rho^b \omega_\alpha^c$ *is a non resonant monomial, i.e. such that* $a \neq b$ *then*

$$L_{\mathcal{X}}\Big[M(1+o(1))\Big] = (a-b+\alpha c)M(1+o(1))$$

where terms $o(1)$ are $\mathcal{C}^k$-functions on monomials.

1. **Case** $\sigma_0 \notin \mathbb{N}$. Up to the relation $r\rho = \nu$, the sequence of leading monomials reduces to $\{1, r^{\sigma_0}, r^{\sigma_0 - 1}\}$ and is non degenerate. The procedure works directly. The cyclicity is less than 2.
2. **Case** $\sigma_0 = p \neq 1$. The sequence of leading monomials reduces to $\{1, r^p, r^{p-1}, \omega\}$. After one preliminary step, the sequence of leading monomials reduces to $\{r^{p+\alpha}, r^{p-1+\alpha}, r^{\alpha}\}$ and we can apply the procedure. The cyclicity is less than 3.
3. **Case** $\sigma_0 = 1$. We begin with the sequence $\{1, r^{\alpha}, r^{1+\alpha}, r^{\alpha}\omega\}$. In order to apply the procedure we have to take into account that the remainders are of order $O(r^{\delta})$ for a $\delta > 0$ After one preliminary step we reduce to the sequence $\{1, \rho\}$. The cyclicity is less than 2.

The difficulty in the two last cases is that we encounter resonant leading monomials in the recurrence.

Boundary l.p.s. of (H_{13}^3)

The regular transition T is replaced by a transition near two saddle points at infinity. If τ is the hyperbolicity ratio at these points, we obtain that

$$V = a_0(1+\cdots)+a_1 r^{\bar{\sigma}_2+\tau}(1+\cdots)+a_2 r^{\bar{\sigma}_2+\tau}\rho(1+\cdots)$$

We compute that $\bar{\sigma}_2+\tau \neq 1$. Then we can apply the procedure of division-derivation to obtain that the cyclicity is less than 2.

Finite cyclicity of the graphic (I_{14}^1)

Recall that depending on the parameter we may have different families of secondary l.p.s : $Sxhh1, Sxhh2, \dots, Sxhh10$.

In each case, to obtain the finite cyclicity of (I_{14}^1), we have to prove that every secondary l.p.s has itself a finite cyclicity.

Almost every case was treated in the previous work by Rousseau and Zhu.

It remained to treat the boundary l.p.s (studied above) and the intermediate and lower l.p.s. of $Sxhh1$ and $Sxhh5$.

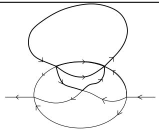
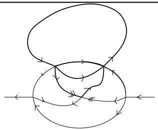
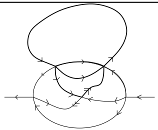
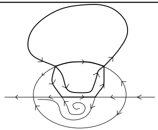
For the intermediate l.p.s. it is sufficient to use that $\frac{\partial}{\partial \bar{\mu}_3} S$ is non linear. For the lower l.p.s. we have a difference map $V(x)$ of one variable x :

$$V(x) = a_0(1 + \cdots) + a_1 x(1 + \cdots) + a_2 x\omega(1 + \cdots),$$

where $\{x = 0\}$ corresponds to the l.p.s. This gives a cyclicity less than 2 for the lower l.p.s.

This method does not give an explicit cyclicity for (I_{14}^1).

Secondary limit periodic sets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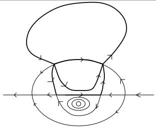	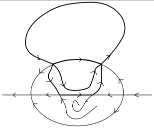	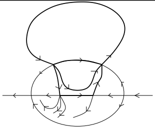	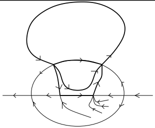
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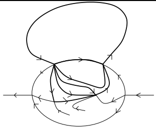	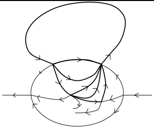		
Sxhh9	Sxhh10		

Thanks you for your attention !