Bifurcations of Critical Periods in Polynomial Systems of ODEs: An Algorithmic Approach

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$$\dot{u} = -v + P(u, v), \ \ \dot{v} = u + Q(u, v), \tag{1}$$

The singularity of system (1) at the origin is either a center or a focus. When it is a center we can consider the so-called period function $T(r)$, which gives the least period of the periodic solution passing through the point with coordinates $(u, v) = (r, 0)$ inside the period annulus.

A center is *isochronous* if all solutions in a neighborhood have the same period. For a center that is not isochronous any value $r > 0$ for which $T'(r) = 0$ is called a *critical period*.

The monotonicity properties were investigated by:
- L. P. Bonorino, E. H. M. Brietzke, J. P. Lukaszczyk,
C. A. Taschetto, (2005)
- C. Chicone (1987)
- S. N. Chow, J. A. Sanders (1986)
- S. N. Chow, D. Wang (1986)
- A. Cima, A. Gasull, F. Mañosas (2000)
- E. Freire, A. Gasull, A. Guillamon (2004)
- F. Mañosas, J. Villadelprat (2008) and others

The problem of critical period bifurcations was considered for the first time by Chicone and Jacobs (1989). The problem is to estimate the number of critical periods that can arise near the center under small perturbation of system (1) within some family that contains it. Chicone and Jacobs studied this problem for quadratic systems and some Hamiltonian systems. Bifurcations of critical periods for a linear center perturbed by homogeneous cubic polynomials were investigated by M. Grau and J. Villadelprat (2010), C. Rousseau, B. Toni (1997),V. R. & M. Han (2003). The problem has also been studied for reversible cubic systems by W. Zhang, X. Hou, Z. Zeng (2000), the reduced Kukles system by C. Rousseau, B. Toni(1997), for Liénard systems by L. Zou, X. Chen, W. Zhang (2008), generalized Loud systems (J. Villadelprat, 2012), reversible rigidly isochronous centers (X. Chen, V. R., W. Zhang, 2011; J. Zhou, N. Li, M. Han, 2013).

The period function via polar coordinates

$$\dot{u} = -v + P_n(u, v), \quad \dot{v} = u + Q_n(u, v), \tag{2}$$

P_n and Q_n are polynomials of degree at most n without constant and linear terms.
Polar coordinates: $u = r\cos\varphi$, $v = r\sin\varphi$.

$$\frac{dr}{d\varphi} = \frac{r^2 F\left(r, \cos\varphi, \sin\varphi\right)}{1 + rG\left(r, \cos\varphi, \sin\varphi\right)} = R\left(r, \varphi\right). \tag{3}$$

Expand $R\left(r, \varphi\right)$ in a convergent power series in r to obtain

$$\frac{dr}{d\varphi} = r^2 R_2\left(\varphi\right) + r^3 R_3\left(\varphi\right) + \cdots . \tag{4}$$

After some calculations:

$$T(r) = 2\pi \left(1 + \sum_{k=1}^{\infty} T_k r^k\right), \tag{5}$$

where T_k are polynomials of parameters of the system of ODEs. A center is isochronous iff $T_k = 0$ for $k \geq 1$.

Definition

For $T(t) \not\equiv 0$ any value $r > 0$ $(r < r^*)$ for which $T'(r) = 0$ is called a critical period.

The problem of critical period bifurcations:

Find an upper bound for the number of isolated zeros of $T'(r)$ near $r = 0$.

Bautin's approach to the multiplicity problem

The critical period bifurcations problem fits into the framework of the following bifurcational problem. Let $\mathscr{E} \subset \mathbb{R}^n$, $\mathscr{F} : \mathbb{R} \times \mathscr{E} \to \mathbb{R}$,

$$\mathscr{F}(z, \theta) = \sum_{j=0}^{\infty} f_j(\theta) z^j, \tag{6}$$

$f_j(\theta)$ is an analytic function and for any $\theta^* \in \mathscr{E}$ the series (6) is convergent near $(z, \theta) = (0, \theta^*)$. We wish to investigate, for any fixed value θ^*, the number of positive solutions of $\mathscr{F}(z, \theta^*) = 0$ in a neighborhood of $z = 0$ in $\mathbb{R}$ for θ near $\theta = \theta^*$ in $\mathscr{E}$.

Definition

Let $\theta^* \in \mathscr{E}$, $f_0(\theta^*) = 0$ and for $\theta \in \mathscr{E}$ let $n(\theta, \epsilon)$ denote the number of isolated zeros of $\mathscr{F}(z, \theta)$ in $0 < z < \epsilon$. The multiplicity of $\mathscr{F}(z, \theta)$ at θ^* with respect to the set $\mathscr{E}$ is m if $\exists \delta_0 > 0$ and $\epsilon_0 > 0$ such that for every pair of ϵ and δ, $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta_0$

$$\max\{n(\theta, \epsilon) : |\theta - \theta^*| < \epsilon\} = m.$$

If for

$$\mathscr{F}(z,\theta)=\sum_{j=0}^{\infty} f_j(\theta)z^j$$

there exists $s\in\mathbb{N}$ such that

$$f_0(\theta^*)=\cdots=f_{s-1}(\theta^*)=0 \text{ but } f_s(\theta^*)\neq 0$$

the multiplicity of θ^* is at most s (Weierstrass preparation theorem).

If $f_j(\theta^*)=0$ for all $j\in\mathbb{N}_0$ then an upper bound on the multiplicity can often be obtained by the following method of Bautin.

The idea is to rewrite $\mathscr{F}(z,\theta)$ in the form

$$\mathscr{F}(z,\theta) = f_{j_1}(\theta)z^{j_1}(1+\psi_1(z,\theta)) + \cdots + f_{j_s}(\theta)z^{j_s}(1+\psi_s(z,\theta)), \qquad (7)$$

where $j_u \in \mathbb{N}$ for $u = 1, \ldots, s$ and $j_1 < \cdots < j_s$, and where $\psi_j(z,\theta)$ and $f_k(\theta)$ are real analytic functions on N, for which $\psi_j(0,\theta^*) = 0$ for $j = 1, \ldots, s$. Then there exist numbers ϵ_1 and δ_1, $0 < \epsilon_1 \le \epsilon$ and $0 < \delta_1 \le \delta$, such that for each fixed $\theta \in \mathcal{S}$ satisfying $a|\theta - \theta^*| < \delta_1$, the equation

$$\mathscr{F}(z,\theta) = 0, \qquad (8)$$

regarded as an equation in z alone, has at most $s-1$ isolated solutions in the interval $0 < z < \epsilon_1$.

The key point of the approach: *for $\mathscr{F}(z,\theta)$ find $\{f_{j_1}, \ldots, f_{j_s}\}$ such that* $\langle f_{j_1}, \ldots, f_{j_s}\rangle = \langle f_1, f_2, f_3, \ldots\rangle$.

$\mathscr{F} =$ Poincaré map $\to$ cyclicity; $\mathscr{F} = T'$ $\to$ critical periods

Definition

Given a Noetherian ring R and an ordered set $B = \{b_1, b_2, \ldots\} \subset R$, construct a basis M_I of the ideal $I = \langle b_1, b_2, \ldots \rangle$ as follows:

(a) initially set $M_I = \{b_p\}$, where b_p is the first non-zero element of B;

(b) sequentially check successive elements b_j, starting with $j = p + 1$, adjoining b_j to M_I if and only if $b_j \notin \langle M_I \rangle$.

The basis M_I is the *minimal basis* of the ideal I with respect to the ordered generating set B.

The cardinality of M_I is called the *Bautin depth* of I (Il'yashenko and Yakovenko).

$$\dot{u} = -v + \sum_{j+k=2}^{n} A_{jk} u^j v^k, \qquad \dot{v} = u + \sum_{j+k=2}^{n} B_{jk} u^j v^k.$$

$$T(r) = 2\pi \left(1 + \sum_{k=1}^{\infty} T_k(A, B) r^k \right) \tag{9}$$

$T_k(A, B)$ – polynomials.

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Find the variety of the ideal generated by the isochronicity quantities $\langle T_1, T_2, T_3 \ldots \rangle \subset \mathbb{R}[A, B]$. (This variety is called the isochronicity variety.)

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Find a (minimal) basis of the ideal generated by the isochronicity quantities $\langle T_1, T_2, T_3 \ldots \rangle \subset \mathbb{R}[A, B]$.

Connection between the variety of a polynomial ideal and its basis:

Strong Hilbert Nullstellensatz

Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ and let I be an ideal of $\mathbb{C}[x_1, \ldots, x_m]$. Then f vanishes on the variety of I if and only f belongs to the radical of I (for some positive integer ℓ $f^\ell \in I$).

Corollary

If polynomials $f_1, \ldots, f_s$ from an ideal I define the variety of I and the ideal I is a radical ideal (e.g. $I = \sqrt{I}$) then $I = \langle f_1, \ldots, f_s \rangle$.

Holds only over $\mathbb{C}$!

Complexification of system (2)

Introduce the complex variable $x = u + iv$, obtaining from (2)

$$\dot{x} = ix - \sum_{j+k=2}^{n} a_{jk} x^j \bar{x}^k, \tag{10}$$

which is family (2) in complex form. Adjoin to (10) its complex conjugate and consider $\bar{a}_{jk}$ as a new parameter b_{kj} and $\bar{x}$ as a distinct variable y, thereby obtaining the system

$$\begin{aligned} \dot{x} &= \quad ix - \sum_{j+k=2}^{n} a_{jk} x^j y^k \\ \dot{y} &= -iy + \sum_{j+k=2}^{n} b_{kj} x^k y^j \end{aligned} \tag{11}$$

on $\mathbb{C}^2$. This is the *complexification* of system (2), to which it is equivalent when $y = \bar{x}$ and $b_{kj} = \bar{a}_{jk}$.

Definition

System (11) has a *center* at the origin if it admits a formal first integral

$$\Psi(x,y) = xy + \sum_{j+k=3}^{\infty} \Psi_{jk} x^j y^k. \tag{12}$$

For (11) there exists a function Ψ of the form (12) such that

$$[ix+\widetilde{P}(x,y)]\Psi_x(x,y)+[-iy+\widetilde{Q}(x,y)]\Psi_y(x,y) = g_{11}(xy)^2+g_{22}(xy)^3+\cdots,$$

g_{kk} – the kth *focus quantity*.
The ideal

$$\mathscr{B} := \langle g_{kk} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a,b]$$

is called the *Bautin ideal* of system (11) and its variety $V_{\mathscr{C}} = \mathbf{V}(\mathscr{B})$ is the *center variety*.
$\mathscr{B}_K$ will denote the ideal generated by the first K focus quantities,

$$\mathscr{B}_K := \langle g_{kk} : k = 1,\ldots,K \rangle \subset \mathbb{C}[a,b].$$

(For any field $\mathbb{K}$ and ideal I in $\mathbb{K}[x_1,\ldots,x_n]$ we let $\mathbf{V}(I)$ denote the affine variety of I, the set of common zeros in $\mathbb{K}^n$ of elements of I.)

The period function via normal forms

By a change of the form

$$x = y_1 + \sum_{j+k\geq 2} h_1^{(j,k)} y_1^j y_2^k\,, \quad y = y_2 + \sum_{j+k\geq 2} h_2^{(j,k)} y_1^j y_2^k\,, \tag{13}$$

(11) is transformed to the normal form

$$\begin{aligned}
\dot{y}_1 &= y_1(i + \sum_{j=1}^{\infty} Y_1^{(j+1,j)}(y_1 y_2)^j) = y_1(i + Y_1(y_1 y_2)) \\
\dot{y}_2 &= y_2(-i + \sum_{j=1}^{\infty} Y_2^{(j,j+1)}(y_1 y_2)^j) = y_2(-i + Y_2(y_1 y_2)).
\end{aligned} \tag{14}$$

$Y_1^{(j+1,j)}$, $Y_2^{(j,j+1)}$ are elements of the polynomial ring $\mathbb{C}[a,b]$. They generate the ideal

$$\mathcal{Y} := \left\langle Y_1^{(j+1,j)},\ Y_2^{(j,j+1)} : j \in \mathbb{N} \right\rangle \subset \mathbb{C}[a,b]. \tag{15}$$

For any $K \in \mathbb{N}$ we set $\mathcal{Y}_K = \langle\, Y_1^{(j+1,j)},\ Y_2^{(j,j+1)} : j = 1, \ldots, K \,\rangle$.
The variety $V_{\mathscr{L}} := \mathbf{V}(\mathcal{Y})$ is called the *linearizability variety* of system (11).

Writing

$$G = Y_1 + Y_2, \quad H = Y_1 - Y_2, \tag{16}$$

the origin is a center for

$$\begin{aligned} \dot{x} &= \quad ix - \sum_{j+k=2}^{n} a_{jk} x^j y^k \\ \dot{y} &= -iy + \sum_{j+k=2}^{n} b_{kj} x^k y^j \end{aligned}$$

if and only if $G \equiv 0$, in which case the distinguished normalizing transformation converges. We define $\widetilde{H}$ by

$$\widetilde{H}(w) = -\tfrac{1}{2} iH(w).$$

If system (11) is the complexification of a real system we recover the real system by replacing every occurrence of y_2 by $\bar{y}_1$ in each equation of the normal form. Setting $y_1 = re^{i\varphi}$ we obtain from them

$$\dot{r} = \frac{1}{2r}(\dot{y}_1\bar{y}_1 + y_1\dot{\bar{y}}_1) = 0, \qquad \dot{\varphi} = \frac{i}{2r^2}(y_1\dot{\bar{y}}_1 - \dot{y}_1\bar{y}_1) = 1 + \widetilde{H}(r^2)\,. \quad (17)$$

Integrating the expression for $\dot{\varphi}$ in (17) yields

$$T(r) = \frac{2\pi}{1 + \widetilde{H}(r^2)} = 2\pi\left(1 + \sum_{k=1}^{\infty} p_{2k} r^{2k}\right) \quad (18)$$

for some coefficients p_{2k}. The coefficients p_{2k} are polynomials in the parameters $(a, b) \in E(a, b) = \mathbb{C}^{2\ell}$.

Although the isochronicity quantities p_{2k} lose their geometric significance when (11) does not correspond to the complexification of any real system (10), they still exist, hence so does the function

$$T(r, a, b) = 2\pi\left(1 + \sum_{k=1}^{\infty} p_{2k}(a, b) r^{2k}\right), \quad (19)$$

which coincides with the period function (18) when $b = \bar{a}$.

Crucial feature of our approach: *We study complex function* (19).

Since values of the isochronicity quantity p_{2k} are of interest only on the center variety we are primarily interested in the equivalence class $[p_{2k}]$ of p_{2k} in the coordinate ring $\mathbb{C}[V_{\mathscr{C}}]$ of the center variety, which can be viewed as the set of equivalence classes of polynomials in $\mathbb{C}[a,b]$ by which f and g are equivalent if for every $(a^*, b^*) \in V_{\mathscr{C}}$, $f(a^*, b^*) = g(a^*, b^*)$. In fact $[f] = [g]$ iff $f \equiv g \mod \mathbf{I}(V_{\mathscr{C}})$, setting up an isomorphism between $\mathbb{C}[V_{\mathscr{C}}]$ and $\mathbb{C}[a,b]/\mathbf{I}(V_{\mathscr{C}})$.

We denote

$$P = \langle p_{2k} : k \in \mathbb{N} \rangle \subset \mathbb{C}[a,b] \qquad \text{and} \qquad \widetilde{P} = \langle [p_{2k}] : k \in \mathbb{N} \rangle \subset \mathbb{C}[V_{\mathscr{C}}]$$

and for $K \in \mathbb{N}$

$$P_K = \langle p_2, \ldots, p_{2K} \rangle \qquad \text{and} \qquad \widetilde{P}_K = \langle [p_2], \ldots, [p_{2K}] \rangle ,$$

It is known (R. & Shafer, 2009) that

$$V_{V_{\mathscr{C}}}(\widetilde{P}) = \mathbf{V}(P) \cap V_{\mathscr{C}} \quad \text{and} \quad V_{V_{\mathscr{C}}}(\widetilde{P}_K) = \mathbf{V}(P_K) \cap V_{\mathscr{C}}. \tag{20}$$

$$\mathbf{V}(P) \cap V_{\mathscr{C}} = \mathbf{V}(\mathcal{Y}) \cap V_{\mathscr{C}} \quad \text{and} \quad \mathbf{V}(P_K) \cap V_{\mathscr{C}} = \mathbf{V}(\mathcal{Y}_K) \cap V_{\mathscr{C}} \forall K \in \mathbb{N}. \tag{21}$$

The case of radical isochronicity ideal

To obtain an upper bound on the number of critical periods we show that

$$T'(r, a, \bar{a}) = \sum_{k=1}^{\infty} 2k p_{2k}(a, \bar{a}) r^{2k-1}$$

can be expressed for $(a, \bar{a}) \in V_{\mathscr{C}}$ as

$$T'(r, a, \bar{a}) = 2p_2(a, \bar{a}) r^{j_1}(1+\psi_1(r, (a, \bar{a})) + \cdots + 2s p_{2s}(a, \bar{a}) r^{j_s}(1+\psi_s(r, (a, \bar{a})).$$

Equivalently, it is sufficient to show that the period function

$$\mathscr{T}(r, (a, \bar{a})) = T(r, a, \bar{a}) - 2\pi = \sum_{k=1}^{\infty} p_{2k}(a, \bar{a}) r^{2k}, \qquad (22)$$

can be expressed for $(a, \bar{a}) \in \mathcal{S} = V_{\mathscr{C}}$ as

$$\mathscr{T}(r, (a, \bar{a})) = p_2(a, \bar{a}) r^{j_1}(1+\psi_1(r, (a, \bar{a})) + \cdots + p_{2s}(a, \bar{a}) r^{j_s}(1+\psi_s(r, (a, \bar{a}))$$

Theorem 1

Suppose that for the complexification (11) of the family (10):

(a) $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$,

(b) the Bautin depth (i.e., the cardinality of the minimal basis) of $\widetilde{P}_K$ in $\mathbb{C}[V_{\mathscr{C}}]$ is m, and

(c) $\widetilde{P}_K$ is a radical ideal in $\mathbb{C}[V_{\mathscr{C}}]$.

Then at most $m-1$ critical periods bifurcate from centers of family (10).

Proof: We have

$$V_{V_{\mathscr{C}}}(\widetilde{P}) \overset{(20)}{=} \mathbf{V}(P) \cap V_{\mathscr{C}} \overset{(21)}{=} V_{\mathscr{L}} \overset{(a)}{=} \mathbf{V}(P_K) \cap V_{\mathscr{C}} \overset{(20)}{=} V_{V_{\mathscr{C}}}(\widetilde{P}_K)$$

so that by the Strong Hilbert Nullstellensatz for coordinate rings for the first and third equalities and hypothesis (c) for the last,

$$\widetilde{P} \subset \sqrt{\widetilde{P}} = I_{V_{\mathscr{C}}}(V_{V_{\mathscr{C}}}(\widetilde{P})) = I_{V_{\mathscr{C}}}(V_{V_{\mathscr{C}}}(\widetilde{P}_K)) = \sqrt{\widetilde{P}_K} = \widetilde{P}_K.$$

Thus $\widetilde{P} \subset \widetilde{P}_K$, hence $\widetilde{P} = \widetilde{P}_K$.

If $\{[p_{2k_1}], \ldots, [p_{2k_m}]\}$ is the minimal basis of $\widetilde{P}_K$ then for any $[p_{2k}] \in \widetilde{P}$ there exist $w_j \in \mathbb{C}[a, b]$ such that

$$[p_{2k}] = [w_1][p_{2k_1}] + \cdots + [w_m][p_{2k_m}] = [w_1 p_{2k_1} + \cdots + w_m p_{2k_m}]$$

in $\mathbb{C}[V_{\mathscr{C}}]$. The isomorphism $\mathbb{C}[V_{\mathscr{C}}] \cong \mathbb{C}[a, b]/\mathbf{I}(V_{\mathscr{C}})$ and the Strong Hilbert Nullstellensatz (recalling that $V_{\mathscr{C}} = \mathbf{V}(\mathscr{B})$) then give

$$p_{2k} - (w_1 p_{2k_1} + \cdots + w_m p_{2k_m}) \in \mathbf{I}(V_{\mathscr{C}}) = \sqrt{\mathscr{B}}.$$

Thus if $\sqrt{\mathscr{B}} = \langle g_1, \ldots, g_s \rangle$ then

$$p_{2k} = w_1 p_{2k_1} + \cdots + w_m p_{2k_m} + w_{m+1} g_1 + \cdots + w_{m+s} g_s. \quad (23)$$

Thus,

$$\mathscr{T}(r,(a,\bar{a})) = T(r) - 2\pi = \sum_{j=1}^{m}(1+\psi_j(r,a,\bar{a}))p_{2k_j}(a,\bar{a})r^{2k_j} + \sum_{j=1}^{s} W_j(r,a,\bar{a})g_j(a,\bar{a}), \tag{24}$$

where the W_j are analytic functions and the ψ_j are real analytic functions in the original real parameters (A,B) of real system of which (10) is the complex form. For $(A,B) \in V_{\mathscr{C}}^{\mathbb{R}}$ (i.e., such that $(a(A,B),\bar{a}(A,B)) \in V_{\mathscr{C}}$), the polynomials g_j evaluate to zero, so the conclusion follows.

Example

$$\dot{x} = i(x + ax^2 + bx^3 + cx^2\bar{x} + dx\bar{x}^2), \tag{25}$$

Associated complex system

$$\begin{aligned} \dot{x} &= \quad ix(1 - a_{10}x - a_{20}x^2 - a_{11}xy - a_{02}y^2), \\ \dot{y} &= -iy(1 - b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2). \end{aligned} \tag{26}$$

The center problem for (26) was solved by Ferčec and Mahdi (2013). They established that $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_4) \Longrightarrow \sqrt{\mathscr{B}} = \sqrt{\mathscr{B}_4}$. Computing the normal form of the system (26) we obtain

$$\begin{aligned} p_2 =& \tfrac{1}{2} a_{11} + \tfrac{1}{2} b_{11}, \\ p_4 =& a_{02} a_{10}^2 + \tfrac{1}{2} a_{02} a_{20} + b_{01}^2 b_{20} + a_{02} b_{20} + \tfrac{1}{2} b_{02} b_{20}, \\ p_6 =& \tfrac{3}{2} a_{02} b_{11} a_{10}^2 + \tfrac{1}{2} b_{02} b_{11} a_{10}^2 + 4 a_{02} b_{01} b_{20} a_{10} - \tfrac{1}{2} a_{20} b_{01}^2 b_{11} \\ & - \tfrac{7}{2} b_{01}^2 b_{11} b_{20} - a_{02} b_{11} b_{20} - b_{02} b_{11} b_{20}, \end{aligned}$$

where p_4 has been reduced modulo p_2 and p_6 has been reduced modulo $\langle p_2, p_4 \rangle$.

Lemma (Giné & R. (2009)) For system (26),

$$V_{\mathscr{L}} = \mathbf{V}(\mathcal{Y}_3) = V_{\mathscr{C}} \cap \mathbf{V}(P_3). \tag{27}$$

Theorem

At most two critical periods bifurcate from centers of system (25)*.*

Proof. By Lemma hypothesis (a) in Theorem 1 holds with $K = 3$. $[p_2]$, $[p_4]$, and $[p_6]$ form the minimal basis of $\widetilde{P}_3$, so that hypothesis (b) in Theorem 1 holds with $m = 3$. $P_3 + \sqrt{\mathscr{B}_4} (\simeq \widetilde{P}_3) = \cap_{j=1}^{4} J_j$ (using Singular), where J_j are the prime ideals

$$\begin{aligned}
J_1 &= \langle b_{02}, b_{11}, a_{02}, a_{11} \rangle, \\
J_2 &= \langle b_{11}, a_{02} + b_{02}, a_{11}, a_{20} + b_{20} \rangle, \\
J_3 &= \langle b_{11}, b_{20}, a_{02}, a_{11} \rangle, \\
J_4 &= \langle b_{11}, b_{20}, a_{11}, a_{20} \rangle.
\end{aligned}$$

An intersection of prime ideals is radical, hence the ideal $\widetilde{P}_3$ is a radical ideal in $\mathbb{C}[\mathbf{V}(\mathscr{B}_4)]$, establishing hypothesis (c). Thus by Theorem 1 at most two critical periods bifurcate from any center of system (25).

The case of non-radical isochronicity ideal

Always exists a finite value of K such that condition (a) of Theorem 1 holds, and the Bautin depth of the corresponding ideal $\widetilde{P}_K$ must have a finite value $m \leq K$.
But it is possible that $\widetilde{P}_K$ fails to be radical. This is the case even for the quadratic system.

Proposition 2

Suppose $I = \langle h_1, \dots, h_r \rangle$, A, and B are ideals in $\mathbb{C}[x_1, \dots, x_n]$, A radical, such that $I = A \cap B$. Let

$$W = \mathbf{V}(I) = \mathbf{V}(A) \cup \mathbf{V}(B).$$

Then for any $f \in \mathbf{I}(W)$ and any $x^* \in \mathbb{C}^n \setminus \mathbf{V}(B)$ there exist a neighborhood U^* of x^* in $\mathbb{C}^n$ and rational functions $f_1, \dots, f_r$ on U^* such that

$$f = f_1 h_1 + \cdots + f_r h_r \quad \text{on} \quad U^*.$$

Theorem 2

Suppose that for the complexification (11) of the family (10):

(a) $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$,

(b) the Bautin depth (i.e., the cardinality of the minimal basis) of $\widetilde{P}_K$ in $\mathbb{C}[V_{\mathscr{C}}]$ is m, and

(c) a primary decomposition of $P_K + \mathscr{B}$ ($\simeq \widetilde{P}_K$) can be written $R \cap N$, where R is the intersection of the ideals in the decomposition that are prime and N is the intersection of the remaining ideals in the decomposition.

Then for any system of family (10) corresponding to $(a^*, \bar{a}^*) \in V_{\mathscr{C}} \setminus \mathbf{V}(N)$, at most $m-1$ critical periods bifurcate from a center at the origin.

We illustrate the theorem by applying it to the family

$$\dot{x} = i(x + ax^3 + bx^2\bar{x} + cx\bar{x}^2 + d\bar{x}^3), \tag{28}$$

$$\begin{aligned} \dot{x} &= \quad i(x - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2 - a_{-13}y^3) \\ \dot{y} &= -i(y - b_{3,-1}x^3 - b_{20}x^2y - b_{11}xy^2 - b_{02}y^3). \end{aligned} \tag{29}$$

Lemma 1 (Sadovskii, 1974)

The center variety $V_{\mathscr{C}}$ of family (29) is $\mathbf{V}(\mathscr{B}_5)$.

Lemma 2 (Christopher & Rousseau, 2001)

For family (29)

$$V_{\mathscr{L}} = \mathbf{V}(\mathcal{Y}_4) = V_{\mathscr{C}} \cap \mathbf{V}(P_4).$$

Theorem

At most three critical periods bifurcate from any nonlinear center at the origin of system (28).

Proof. By Lemma 2 hypothesis (a) in Theorem 2 holds with $K = 4$.
A direct computation shows that the Bautin depth is $m = 4$.
It is well-known that $\mathscr{B} = \mathscr{B}_5$.
There is a primary decomposition of $P_4 + \mathscr{B}_5$ of the form $\cap_{s=1}^{16} J_s$ in which the last nine components are not prime (we computed with `primdecGTZ` in SINGULAR. Thus hypothesis (c) in Theorem 1 fails.

To proceed we compute the associated prime ideals $I_s = \sqrt{J_s}$ of the last nine components of the primary decomposition of $P_4 + \mathscr{B}_5$; they are

$$
\begin{aligned}
I_8 &= \langle b_{20}, b_{3,-1}, 3a_{02} + b_{02}, a_{20}, b_{11}, a_{11} \rangle, \\
I_9 &= \langle b_{20}, b_{3,-1}, a_{02} + b_{02}, a_{20}, b_{11}, a_{11} \rangle, \\
I_{10} &= \langle b_{02}, a_{-13}, a_{02}, a_{20} + 3b_{20}, b_{11}, a_{11} \rangle, \\
I_{11} &= \langle b_{02}, a_{-13}, a_{02}, a_{20} + b_{20}, b_{11}, a_{11} \rangle, \\
I_{12} &= \langle b_{02}, b_{20}, a_{-13}, a_{02}, b_{11}, a_{11} \rangle, \\
I_{13} &= \langle b_{20}, b_{3,-1}, a_{02}, a_{20}, b_{11}, a_{11} \rangle, \\
I_{14} &= \langle b_{02}, b_{20}, b_{3,-1}, a_{02}, a_{20}, b_{11}, a_{11} \rangle, \\
I_{15} &= \langle b_{02}, b_{20}, a_{-13}, a_{02}, a_{20}, b_{11}, a_{11} \rangle, \\
I_{16} &= \langle b_{02}, b_{20}, b_{3,-1}, a_{-13}, a_{02}, a_{20}, b_{11}, a_{11} \rangle.
\end{aligned}
$$

Their intersection is the ideal

$$N = \langle b_{11}, a_{11}, b_{20}b_{02}, b_{3,-1}b_{02}, a_{20}b_{02}, a_{-13}b_{20}, a_{02}b_{20},$$
$$a_{20}a_{-13}, a_{20}a_{02}, a_{20}^2 b_{20} + 4a_{20}b_{20}^2 + 3b_{20}^3, 3a_{02}^3 + 4a_{02}^2 b_{02} + a_{02}b_{02}^2\rangle,$$

which can be computed by using, for example, the command `intersect` of SINGULAR.

We now limit our consideration to the space $E(a)$ of parameters of system (28), that is, we set $b = \bar{a}$. Then $\mathbf{V}(N) \cap E(a) = \{0\}$, which corresponds to the linear system in (28). By Theorem 2 we conclude that at most three critical periods bifurcate from any nonlinear center at the origin of system (28). $\square$

The theorem was also proved by Rousseau and Toni (1997) using another approach.

A Further Extension of the Approach

For the studied families we obtained as good a result as possible from Theorem 2, since only nonlinear centers were excluded in the conclusion. In this section we show that this is not always the case, and describe a method that can sometimes improve the result given by a direct application of Theorem 2.

We first apply Theorem 2 to the family

$$\dot{x} = i(x + a\bar{x}^2 + bx^3 + cx^2\bar{x} + dx\bar{x}^2). \tag{30}$$

whose complexification is

$$\begin{aligned} \dot{x} &= i(x - a_{-12}y^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2) \\ \dot{y} &= -i(y - b_{20}x^2y - b_{11}xy^2 - b_{02}y^3 - b_{2,-1}x^2). \end{aligned} \tag{31}$$

Lemma 3 (B. Ferčec, V. Levandovskyy, V. R. & D.S. Shafer, 2013)

For family (31), $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_7)$, $\quad V_{\mathscr{L}} = \mathbf{V}(\mathcal{Y}_4) = \mathbf{V}(P_4) \cap V_{\mathscr{C}}$.

By Lemma 3 $K = 4$, hence $m \leq 4$. Since $\mathbf{V}(\mathscr{B}) = \mathbf{V}(\mathscr{B}_7)$, hence $\sqrt{\mathscr{B}} = \sqrt{\mathscr{B}_7}$, we compute a primary decomposition of the ideal $P_4 + \sqrt{\mathscr{B}_7}$. Prime ideals corresponding to non-prime components are

$$
\begin{aligned}
I_1 &= \langle b_{11}, b_{02}, a_{02}, a_{-12}, a_{11} \rangle \\
I_2 &= \langle b_{11}, b_{20}, b_{2,-1}, a_{20}, a_{11} \rangle \\
I_3 &= \langle b_{11}, b_{20} b_{2,-1}, a_{02} + b_{02}, a_{20}, a_{11} \rangle \\
I_4 &= \langle b_{11}, b_{02}, b_{20}, a_{02}, a_{-12}, a_{11} \rangle \\
I_5 &= \langle b_{11}, b_{02}, a_{02}, a_{20} + b_{20}, a_{-12}, a_{11} \rangle \\
I_6 &= \langle b_{11}, b_{20}, b_{2,-1}, a_{02}, a_{20}, a_{11} \rangle \\
I_7 &= \langle b_{11}, b_{20}, b_{2,-1}, a_{02}, a_{-12}, a_{11} \rangle \\
I_8 &= \langle b_{11}, b_{02}, b_{20}, b_{2,-1}, a_{02}, a_{20}, a_{11} \rangle \\
I_9 &= \langle b_{11}, b_{02}, b_{20}, a_{02}, a_{20}, a_{-12}, a_{11} \rangle \\
I_{10} &= \langle b_{11}, b_{02}, b_{20}, b_{2,-1}, a_{02}, a_{20}, a_{-12}, a_{11} \rangle .
\end{aligned}
$$

The intersection of the ideals above is the ideal

$$N = \langle b_{11}, a_{11}, b_{20}b_{02}, b_{02}b_{2,-1}, a_{02}b_{20}, a_{-12}b_{20}, a_{02}b_{2,-1},$$

$$a_{-12}b_{2,-1}, a_{20}a_{02}\rangle.$$

The intersection of the variety $\mathbf{V}(N)$ with the parameter space $E(a)$ (that is, when the condition $b = \bar{a}$ is imposed) is the variety $\mathbf{V}(a_{02}, a_{-12}, a_{11})$. Thus $E(a) \setminus \mathbf{V}(N)$ is the set $|a_{11}| + |a_{02}| + |a_{-12}| \neq 0$. Comparing (30) and (31), by Theorem 2 we conclude that at most three critical periods bifurcate from isochronous centers of systems (30) for which $|a| + |c| + |d| \neq 0$.

Using a certain structure that is inherent in g_{kk} and p_{2k} we can sometimes extends the reach of Theorem 2.

$$p_2 = 4/3a_{-12}b_{2,-1} + a_{11} + b_{11}$$
$$p_4 = 2/3a_{11}a_{-12}b_{2,-1} + 2/3b_{11}a_{-12}b_{2,-1} + a_{20}a_{02} + 2a_{02}b_{20} + b_{20}b_{02}$$
$$p_6 = -20/81a_{11}a_{-12}^2b_{2,-1}^2 - 20/81b_{11}a_{-12}^2b_{2,-1}^2 + 8/9a_{11}^2a_{-12}b_{2,-1} - 16/27a_{11}b$$

$$c_1 = a_{-12}\,b_{2,-1},\ c_2 = a_{20}\,b_{02},\ c_3 = a_{02}\,b_{20},\ c_4 = b_{20}\,b_{02},$$
$$c_5 = a_{02}^3\,b_{2,-1}^2,\ c_6 = a_{02}^2\,b_{2,-1}^2\,b_{02},\ c_7 = a_{02}\,b_{2,-1}^2\,b_{02}^2,\ c_8 = b_{2,-1}^2\,b_{02}^3,$$
$$c_9 = a_{20}\,a_{02},\ c_{10} = a_{-12}^2\,b_{20}^3,\ c_{11} = a_{-12}^2\,a_{20}\,b_{20}^2,\ c_{12} = a_{-12}^2\,a_{20}^2\,b_{20},$$
$$c_{13} = a_{-12}^2\,a_{20}^3, c_{14} = a_{11},\ c_{15} = b_{11}$$

$$p_2^c = 4/3c_1 + c_{14} + c_{15}$$
$$p_4^c = 2/3c_1c_{14} + 2/3c_1c_{15} + 2c_3 + c_4 + c_9$$
$$p_6^c = -20/81c_1^2c_{14} + 8/9c_1c_{14}^2 - 20/81c_1^2c_{15} - 16/27c_1c_{14}c_{15} + 8/9c_1c_{15}^2 + 59/1$$

For $\langle p_2^c, p_4^c, p_6^c, p_8^c\rangle$ "non-radical component" is $c = 0$.

New coordinates

$$\begin{aligned} \dot{x} &= \quad ix(1 - a_{p_1 q_1} x^{p_1} y^{q_1} - a_{p_2 q_2} x^{p_2} y^{q_2} - \cdots - a_{p_\ell q_\ell} x^{p_\ell} y^{q_\ell}) \\ \dot{y} &= -iy(1 - b_{q_1 p_1} x^{q_1} y^{p_1} - b_{q_2 p_2} x^{q_2} y^{p_2} - \cdots - b_{q_\ell p_\ell} x^{q_\ell} y^{p_\ell}), \end{aligned} \tag{32}$$

where $p_i \geq -1$. Define a mapping $L : \mathbb{N}_0^{2\ell} \to \mathbb{Z}^2$ by

$$L(\nu) = \nu_1 \binom{p_1}{q_1} + \cdots \nu_\ell \binom{p_\ell}{q_\ell} + \nu_{\ell+1} \binom{q_\ell}{p_\ell} + \cdots + \nu_{2\ell} \binom{q_1}{p_1}. \tag{33}$$

$$\mathscr{M} = \{\nu = (\nu_1, \ldots, \nu_{2\ell}) :\in \mathbb{N}_0^{2\ell} : L(\nu) = \binom{s}{s} \text{ for some } s \in \mathbb{N}_0\}. \tag{34}$$

$$\nu \longleftrightarrow a_{p_1 q_1}^{\nu_1} \ldots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell, p_\ell}^{\nu_{\ell+1}} \ldots b_{q_1 p_1}^{\nu_{2\ell}},$$

$\mathbb{C}[\mathcal{M}]$ – polynomial subalgebra generated by $\mathcal{M}$.

$$
\begin{aligned}
\dot{x} &= \quad i(x - a_{-12}y^2 - a_{20}x^3 - a_{11}x^2y - a_{02}xy^2) \\
\dot{y} &= -i(y - b_{20}x^2y - b_{11}xy^2 - b_{02}y^3 - b_{2,-1}x^2).
\end{aligned}
$$

$$
\nu_1\begin{pmatrix}-1\\2\end{pmatrix}+\nu_2\begin{pmatrix}2\\0\end{pmatrix}+\nu_3\begin{pmatrix}1\\1\end{pmatrix}+\nu_4\begin{pmatrix}0\\2\end{pmatrix}+\nu_5\begin{pmatrix}2\\0\end{pmatrix}+\nu_6\begin{pmatrix}1\\1\end{pmatrix}+\nu_7\begin{pmatrix}0\\2\end{pmatrix}+\nu_8\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}s\\s\end{pmatrix}.
$$

$(1,0,0,0,0,0,0,1) \mapsto a_{-12}\, b_{2,-1} = c_1,$

$(0,1,0,1,0,0,0,0) \mapsto a_{20}a_{02} = c_9$

Proposition 2 (Yirong Liu & Jibin Li (1990); R. (1991); R. & Han (2003))

For family (32) $g_{kk},\ p_{2k} \in \mathbb{C}[\mathcal{M}]$.

Generating set of $\mathbb{C}[\mathcal{M}] \longleftrightarrow$ Hilbert basis of $\mathcal{M}$.

An algorithm for computing generators of $\mathbb{C}[\mathcal{M}]$ (V.R. 2008)

Let

$$H = \langle 1 - \alpha w, a_{p_k q_k} - t_k, b_{q_k p_k} - \alpha^{p_k - q_k} t_k \mid k = 1, \dots, \ell \rangle.$$

- Compute a Groebner basis G_H for H with respect to any elimination order with $\{w, \alpha, t_k\} > \{a_{p_k q_k}, b_{q_k p_k} \mid k = 1, \dots, \ell\}$;
- The set $B = G_H \cap k[a, b]$ is a set of binomials;
- Monomials of B along with monomials $a_{ks} b_{sk}$ generate $\mathbb{C}[\mathcal{M}]$; exponents of the monomials form the Hilbert basis of $\mathcal{M}$.

For system (11) we look for the reduced Gröbner basis of the ideal

$$\langle 1 - w\alpha^5, a_{-12} - t_1, \alpha^3 b_{2,-1} - t_1, a_{20} - t_2, b_{02} - \alpha^2 t_2,$$

$$a_{11} - t_3, b_{11} - t_3, a_{02} - t_4, \alpha^2 b_{20} - t_4\rangle$$

with respect to a monomial order eliminating variables $\{w, \alpha, t_1, t_2, t_3, t_4\}$.

$$
\begin{aligned}
&a_{-12}^2 a_{20}^2 b_{20} - a_{02} b_{02}^2 b_{2,-1}^2, && a_{-12}^2 a_{20} b_{20}^2 - a_{02}^2 b_{02} b_{2,-1}^2,\\
&a_{02} a_{20} - b_{02} b_{20}, && a_{-12}^2 b_{20}^3 - a_{02}^3 b_{2,-1}^2,\\
&a_{11} - b_{11}, && a_{-12}^2 a_{20}^3 - b_{02}^3 b_{2,-1}^2.
\end{aligned}
$$

The generators of $\mathbb{C}[\mathscr{M}]$ are the monomials that appear in any of these binomials, along with the monomials

$$a_{20} b_{02},\ a_{-12} b_{2,-1},\ a_{02} b_{20}.$$

Rewrite polynomials $p_{2k}(a, b)$ as polynomials $p_{2k}(c)$

$$F : \mathbb{C}^{2\ell} \to \mathbb{C}^M : (a, b) \mapsto (c_1, \dots, c_M) = (h_1(a, b), \dots, h_M(a, b)) \quad (35)$$

Induces the homomorphism (letting $c = (c_1, \dots, c_M)$)

$$F^\sharp : \mathbb{C}[c] \to \mathbb{C}[a, b] : \sum d_{(\alpha)} c_1^{\alpha_1} \cdots c_M^{\alpha_M} \mapsto \sum d_{(\alpha)} h_1^{\alpha_1}(a, b) \cdots h_M^{\alpha_M}(a, b),\ d_{(\alpha)} \in \quad (36)$$

In our case $M = 15,\ h_1 = a_{-12}\, b_{2,-1},\ h_2 = a_{20}\, b_{02}, \dots$.
$J = \langle h_k(a, b) - c_k : k = 1, \dots, M \rangle$.
In our case

$$\begin{aligned} J = \langle & c_1 - a_{-12}\, b_{2,-1},\ c_2 - a_{20}\, b_{02},\ c_3 - a_{02}\, b_{20},\ c_4 - b_{20}\, b_{02}, \\ & c_5 - a_{02}^3\, b_{2,-1}^2,\ c_6 - a_{02}^2\, b_{2,-1}^2\, b_{02},\ c_7 - a_{02}\, b_{2,-1}^2\, b_{02}^2,\ c_8 - b_{2,-1}^2\, b_{02}^3, \\ & c_9 - a_{20}\, a_{02},\ c_{10} - a_{-12}^2\, b_{20}^3,\ c_{11} - a_{-12}^2\, a_{20}\, b_{20}^2,\ c_{12} - a_{-12}^2\, a_{20}^2\, b_{20}, \\ & c_{13} - a_{-12}^2\, a_{20}^3, c_{14} - a_{11},\ c_{15} - b_{11} \rangle. \end{aligned}$$

$p_{2k}^c(c)$ is the remainder of division of $p_{2k}(a, b)$ by the Groebner basis of J using the lexicographic order.

Let $W \subset \mathbb{C}^M$ denote the image of $\mathbb{C}^{2\ell}$ under F, $\overline{W} \subset \mathbb{C}^M$ its Zariski closure.
Let $R = \ker(F^\sharp) \subset \mathbb{C}[c]$. Since $\mathbb{C}[a, b]$ is an integral domain, R is a prime ideal. By a classical fact $\overline{W} = \mathbf{V}(R)$.
Main idea:
- Let $J = \langle h_j(a, b) - c_j : 1 \leq j \leq M \rangle \subset \mathbb{C}[a, b, c]$. Then $R = \ker(F^\sharp) = J \cap \mathbb{C}[c]$. Choose any monomial order $\prec$ on $\mathbb{C}[a, b, c]$ eliminating $\{a_{pq}, b_{qp}\}$ and let $J_G \subset \mathbb{C}[a, b, c]$ denote a Gröbner basis of J with respect to $\prec$. Then $R_G := J_G \cap \mathbb{C}[c]$ is a Gröbner basis of R.
- Compute the preimages $p^c_{2k}(c)$ as remainders of the division of $p_{2k}(a, b)$ by J_G and try to apply one of Theorems 1 or 2 to $\langle p^c_2, p^c_4 p^c_6, \ldots \rangle \subset \mathbb{C}[\bar{W}] \simeq \mathbb{C}[c]/R$.
Difficulties:

- images of varieties under F are not necessary varieties;
- preimages $p^c_{2k}(c)$ are not uniquely defined.

Theorem 3

For the complexification (11) of the family (10):

i. let $\{\mu_1, \dots, \mu_M\}$ be a Hilbert basis of $\mathscr{M}$ of (34), and let h_j be corresponding monomials in $\mathbb{C}[a, b]$;

ii. let F and $F^\sharp$ be the mappings defined by (35) and (36);

iii. let $J = \langle h_j(a, b) - c_j : 1 \le j \le M \rangle \subset \mathbb{C}[a, b, c]$ and $R = J \cap \mathbb{C}[c]$;

iv. let $J_G \subset \mathbb{C}[a, b, c]$ be a Gröbner basis of J after elimination a_{pq}, b_{qp};

v. for any $f \in \text{Image}(F^\sharp) \subset \mathbb{C}[a, b]$ let $f^c \in \mathbb{C}[c]$ denote a representative of $(F^\sharp)^{-1}(f)$;

vi. for any set $I = \{f_1, \dots, f_s\} \subset \mathbb{C}[a, b]$ such that $f_j \in \text{Image}(F^\sharp)$ for all j let $I^c = \langle f_1^c, \dots, f_s^c \rangle$ in $\mathbb{C}[c]/R$.

(a) $V_{\mathscr{C}} = \mathbf{V}(\mathscr{B}_L)$ and $V_{\mathscr{L}} = \mathbf{V}(P_K) \cap V_{\mathscr{C}}$,

(b) the Bautin depth of $\widetilde{P}_K$ in $\mathbb{C}[V_{\mathscr{C}}]$ is m,

(c) $(\mathbb{C}[a, b, c]P_K + \mathbb{C}[a, b, c]\mathscr{B}_L + J) \cap \mathbb{C}[c] = P_K^c + \mathscr{B}_L^c + R$, and

(d) a primary decomposition of $P_K^c + \mathscr{B}_L^c + R \subset \mathbb{C}[c]$ is $A \cap B$, where A is the intersection of the ideals in the decomposition that are prime and B is the intersection of the remaining non-prime ideals.

Then for any system of family (10) corresponding to $(a^*, \bar{a}^*) \in V_{\mathscr{C}} \setminus F^{-1}(\mathbf{V}(B))$, at most $m - 1$ critical periods bifurcate from a center at the origin.

By Theorem 2 at most three critical periods bifurcate from centers of systems (30) for which $|a|+|c|+|d| \neq 0$. Using Theorem 3 we have

Theorem

At most three critical periods bifurcate from nonlinear centers at the origin of system

$$\dot{x} = i(x + a\bar{x}^2 + bx^3 + cx^2\bar{x} + dx\bar{x}^2).$$

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