On periodic orbits in non-smooth differential equations with applications

Rafel Prohens

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1 Introduction

- A quick browse to some previous works
- Related works
- References on neuron models

2 Qualitative compatibility between PWL and smooth diff. eq.

- Maximal and faux canards in $\mathbb{R}^n$
- MMOs in PWL slow-fast dynamics in $\mathbb{R}^3$

3 Quantitative analysis in PWL diff. eq.

- Application: Estimation of the synaptic conductance in a McKean-model neuron
Introduction: A quick browse to some previous works

- [CGP-1999] *The center problem for discontinuous Liénard differential equation*
- [CGP-2000] *Center-focus and isochronous center problems for discontinuous differential equations*
- [CGP-2001] *Degenerate Hopf bifurcations in discontinuous planar systems*
- [PT-2013] *Canard trajectories in 3D piecewise linear systems*

\[
(\dot{x}, \dot{y}) = \begin{cases} 
(P^+(x, y), Q^+(x, y)) = X_n + X_m, \\
(P^-(x, y), Q^-(x, y)) = Y_p + Y_q,
\end{cases}
\]

where \(X_u\) and \(Y_u\) are homogeneous polynomial v.f. of degree \(u\)

**Theorem 1**

This system has one non simple singular limit cycle or, at most, four simple singular limit cycles. Moreover, at most two of these singular limit cycles surround the origin.

**Theorem 2**

All its periodic orbits surround the origin and it has, at most, one singular limit cycle (which is simple), and two regular limit cycles.

Tools: geometry of the set of points at which the derivative of the angular component of the vector field is zero; polar coordinates; the Schwarzian derivative of the Poincaré return map.

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[CGP-1999] The center problem for discontinuous Liénard differential equation

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(\dot{x}, \dot{y}) = \begin{cases} 
(-y + f^+(x), x), & \text{if } x \geq 0 \\
(-y + f^-(x), x), & \text{if } x \leq 0,
\end{cases}
\]  

(1)

and

\[
(\dot{x}, \dot{y}) = \begin{cases} 
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\end{cases}
\]  

(2)

where \( f^\pm(x) = \sum_{i \geq 2} a_i^\pm x^i \).

We derive the general expression of the Lyapunov constants.
Theorem 1

For (1), the $n$-th Lyapunov constant, $n \geq 2$ is

$$V_n = \begin{cases} 
\frac{2n!!}{(n+1)!!} (a_n^+ - a_n^-) & \text{if } n \text{ is even}, \\
\frac{\pi n!!}{(n+1)!!} (a_n^+ + a_n^-) & \text{if } n \text{ is odd},
\end{cases}$$

Theorem 2

For (2), the $n$-th Lyapunov constant, $n \geq 2$ is: $V_2 = 0$ and

$$V_n = \begin{cases} 
\sum_{j=1}^{n/2-1} C_{n/2,j} a_{2j+1}^+ (a_{n-2j}^+ + a_{n-2j}^-) & \text{if } n \text{ is even}, \\
\frac{\pi n!!}{(n+1)!!} (a_n^+ + a_n^-) & \text{if } n \text{ is odd},
\end{cases}$$

for some $C_{n/2,j}$ constant numbers.

Tools: the Lyapunov constants for nonsmooth differential equations are quasi-homogeneous polynomials in the variables given by the coefficients of the differential equation.
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Tools: the Lyapunov constants for nonsmooth differential equations are quasi-homogeneous polynomials in the variables given by the coefficients of the differential equation.
[CGP-2000] Center-focus and isochronous center problems for discontinuous differential equations

\[
\dot{z} = \begin{cases} 
F_1(z, \bar{z}), & \text{if } \text{Im}(z) \geq 0 \\
F_2(z, \bar{z}), & \text{if } \text{Im}(z) \leq 0,
\end{cases} 
\tag{1}
\]

where \( z = x + i y = \text{Re}(z) + i \text{Im}(z) \in \mathbb{C} \)

\[
\dot{z} = F_j(z, \bar{z}), \quad \text{where} \quad j = 1, 2. \tag{1.j}
\]

We relate the order of degeneracy of the critical point with the orders of the two components.
Definition

The origin is a \((m, k)\)-monodromic point if

\[ \Pi(\rho) - \rho = O(\rho^m) \quad \text{and} \quad T(\rho) - 2\pi = O(\rho^k). \]

\((m, *)\) means that the origin of (1) is a \((m, k)\)-monodromic point for some \(k\).

Theorem

If (1) has a \((m, k)\)-monodromic point and (1.1) has a \((n, \ast)\)-monodromic point, then (1.2) has \((\tilde{m}, \ast)\)-monodromic point with \(\tilde{m} \geq \min(m, n)\).

Tools: the use of the relation between the extensions of the half return maps to a whole neighbourhood of zero.
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[CGP-2001] *Degenerate Hopf bifurcations in discontinuous planar systems*

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(\dot{x}, \dot{y}) = \begin{cases} 
(X^+(x, y), Y^+(x, y)), & \text{if } y \geq 0 \\
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where \(X^\pm, Y^\pm\) are real analytical functions. The role of foci points is inherited by four types of singular points, pseudo-focus.

Obtain the general expressions for the first three Lyapunov constants.
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• [PT-2013] *Canard trajectories in 3D piecewise linear systems*

Singularly perturbed 3–dimensional piecewise linear differential systems

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\begin{align*}
\dot{u}_1 &= \varepsilon(a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1), \\
\dot{u}_2 &= \varepsilon(a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2), \\
\dot{v} &= u_1 + |v|,
\end{align*}
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where \(0 < \varepsilon \ll 1\).

Fenichel’s geometric theory allows us to analyze the dynamics of

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\dot{u} &= \frac{du}{dt} = \varepsilon g(u, v, \varepsilon), \\
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\end{align*}
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where \((u, v) \in \mathbb{R}^s \times \mathbb{R}^q\) when \(f\) and \(g\) are sufficiently smooth functions. The coordinates of \(u\) are called *slow variables*, while the coordinates of \(v\) are called *fast variables*. 
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This analysis follows by combining the behaviour of the singular orbits, corresponding to the limiting problems, \( \varepsilon = 0 \)

layer: \[ \dot{u} = 0, \quad \dot{v} = f(u, v, 0), \]

reduced: \[ u' = g(u, v, 0), \quad 0 = f(u, v, 0), \quad u \in \mathbb{R}^s \]

where \( u' = d/d\tau, \tau = \varepsilon t. \) Critical manifold

\[ S = \{ (u, v) \in \mathbb{R}^{s+q} \mid f(u, v, 0) = 0 \}. \]

We call normally hyperbolic the singular points \((u_0, v_0) \in S\) for which the eigenvalues of the Jacobian matrix \( D_v f(u_0, v_0) \) have nonzero real part.
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where $' = d/d\tau$, $\tau = \varepsilon t$. Critical manifold

\[ \mathcal{S} = \{(u, v) \in \mathbb{R}^{s+q} \mid f(u, v, 0) = 0\}. \]

We call normally hyperbolic the singular points $(u_0, v_0) \in \mathcal{S}$ for which the eigenvalues of the Jacobian matrix $D_vf(u_0, v_0)$ have nonzero real part.
Consider $S_0 \subset S$ a compact set such that every point in $S_0$ is a normally hyperbolic singular point.

\[ S_0 \Rightarrow S_\varepsilon \text{ locally invariant slow manifold and} \]

The restriction of the flow of the perturbed system to the slow manifold $S_\varepsilon$ is a small smooth perturbation of the flow of the reduced problem.

Orbits of the perturbed system are composed by:

- slow segments are close to the flow of the reduced problem
- fast segments are close to the flow of the layer problem
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Related to the loss of normal hyperbolicity is the appearance of relaxation oscillation and canard orbits.
Canard Orbits in 2-D

Canard orbits cross from the attracting manifold to the repelling manifold.
Question

What remains of previous dynamical behaviour when smoothness is no longer present?

In [PRSZ2011]$^1$ the authors prove the existence of canard cycles in singly perturbed piecewise differential systems with $s = 2$ and $q = 1$.

This fact suggests that canards are not exclusively a differential phenomenon, but rather a geometric one.

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For next singularly perturbed 3–dimensional piecewise linear differential system

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\begin{align*}
\dot{u}_1 &= \varepsilon (a_{11} u_1 + a_{12} u_2 + a_{13} v + b_1), \\
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\dot{v} &= u_1 + |v|,
\end{align*}
\]

where \(0 < \varepsilon \ll 1\),

- we present results similar to those obtained by the Geometric Singular Perturbation Theory.
- we obtain the global expression of the slow manifold \(S_\varepsilon\).
- we characterize the existence of canard orbits in such systems.
- we provide numerical arguments for the existence of a canard cycle.
Representation of the canard cycle $\gamma_{P_c}$, slow manifolds $\tilde{S}_-^\epsilon \cup \tilde{S}_+^\epsilon$ and the border planes $\{v = \eta\}, \{v = 0\}$ and $\{v = -\eta\}$, which separate the regions where the system is linear. We highlighted the points of intersection of $\gamma_{P_c}$ with the border planes.
Canard dynamics has been investigated in planar PWL slow-fast systems, from the 1990s.

After a brief mention of the “loss” of canards in PWL systems with two corners in


The first study of a PWL van der Pol system from the perspective of canards (McKean ODE model)

Related works

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Up to very recently


The four piece of the critical manifold

The cubic critical manifold is replaced by a PWL caricature consisting of three straight line segments. The corners play the role of the fold points and cycles resembling canards and evolving around these corners were identified by simulation.


reasons for the equivalent of canards with head can arise only in systems with one more piece in between the two corners.

The main idea to obtain true canard cycles in a planar PWL systems consists in approximating the critical manifold near a fold by a three-piece PWL function.
A large class of neuron models based on the approximation that the membrane of the neuron behaves like a circuit. The voltage equation is obtained by applying Kirchoff’s law.

After the model by (HH model):

The first reduction to a planar system (FHN model):


where, the vector field of the HH model was approximated by a polynomial system through the crucial observation that the voltage nullcline is roughly cubic shaped.

Hence, the FHN model appears as a modified van der Pol system.
The FHN model was investigated from the slow-fast perspective and in


further simplified by approximating the cubic voltage nullcline by a PWL function.
Since the late 1990s several papers have shown that the canard phenomenon can be reproduced with piecewise-linear (PWL) dynamical systems in two and three dimensions, exhibiting an slow-fast dynamics.

Smooth slow-fast dynamical systems models in neuroscience displaying canard-induced MMOs.

Goal
We aim to explore the gap between PWL and smooth slow-fast dynamical systems by analysing canonical PWL systems that display folded singularities.

**Slow-Fast Piecewise Linear System (PWLS)**

\[
\begin{align*}
\dot{u} &= \frac{du}{dt} = \varepsilon (Au + av + b), \\
\dot{v} &= \frac{dv}{dt} = u_1 + |v|.
\end{align*}
\]

This paper is mainly concerned with maximal canard orbits occurring in \(n\)-dimensional piecewise linear slow-fast systems.

More precisely, conditions for the existence of maximal canard orbits and/or faux canard orbits are established.

We show that these maximal canards perturb from singular orbits (singular canards) whose order of contact with the fold manifold is greater than or equal to two.
**Slow-Fast Piecewise Linear System (PWLS)**

\[ \dot{u} = \frac{du}{dt} = \varepsilon(Au + av + b), \quad \dot{v} = \frac{dv}{dt} = u_1 + |v|. \]

- \( u \in \mathbb{R}^s \) slow variable
- \( v \in \mathbb{R} \) fast variable
- \( 0 < \varepsilon \ll 1 \) ratio of time scales
- \( n = s + 1 \) system dimension

- Rather general: \( f(u, v, \varepsilon) = d^T u + |v| \), with \( d \neq 0 \), can be transformed into our system \( (u \rightarrow (d^T u, u_2, \ldots, u_n)^T) \).

- \( A = (a_{ij})_{1 \leq i, j \leq s} \) \( s \times s \) real matrix
- \( a = (a_1, a_2, \ldots, a_s)^T \) vector in \( \mathbb{R}^s \)
- \( b = (b_1, b_2, \ldots, b_s)^T \) vector in \( \mathbb{R}^s \)

---

**Rafel Prohens**

On periodic orbits in non-smooth differential equations with applications
Slow-Fast Piecewise Linear System (PWLS)

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\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \varepsilon (A\mathbf{u} + a\mathbf{v} + b), \quad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = u_1 + |v|.
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\(\mathbf{u} \in \mathbb{R}^s\) slow variable

\(\mathbf{v} \in \mathbb{R}\) fast variable

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- Continuous and nonlinear system (but, piecewise linear).
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- 2 regimes: \(\{v \leq 0\}\) and \(\{v \geq 0\}\) and 1 common boundary \(\{v = 0\}\).
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- Continuous and nonlinear system (but, piecewise linear).
- 2 regimes: \( \{v \leq 0\} \) and \( \{v \geq 0\} \) and 1 common boundary \( \{v = 0\} \).
Unperturbed Dynamics:

Associated to

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\begin{align*}
\dot{u} &= \varepsilon (Au + av + b), \\
\dot{v} &= u_1 + |v|,
\end{align*}
\]

we have the:

- fast subsystem (layer problem)
- slow subsystem (reduced problem)
- critical manifold, where the slow subsystem is defined, \( S \)
- fold manifold, \( \mathcal{F} \), when normal hyperbolicity fails, (points where \( S \) folds)
Unperturbed Dynamics:

- **Fast subsystem**
  \[
  \begin{cases}
  \dot{u} = 0, \\
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  \[S = \{(u, v) \in \mathbb{R}^n : u_1 + |v| = 0\}\]
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- \[ S = S^+ \cup F \cup S^- \]
  \[ S^+ = \{u_1 + v = 0; \ v > 0\} \]
  \[ S^- = \{u_1 - v = 0; \ v < 0\} \]
  \[ F = \{u_1 = 0, \ v = 0\} \]

where \( S^+ \) and \( S^- \) are normally hyperbolic and \( F \) is the fold manifold.
Unperturbed Dynamics:

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where \(S^+\) and \(S^-\) are normally hyperbolic and \(F\) is the fold manifold.
Unperturbed Dynamics:

- Slow subsystem associated to

\[
\begin{align*}
\dot{u} &= Au + av + b, \\
\dot{v} &= u_1 + |v|,
\end{align*}
\]
Unperturbed Dynamics:

- **Slow subsystem**

\[
\begin{align*}
    \dot{u} &= Au + av + b, \\
    0 &= u_1 + |v|, 
\end{align*}
\]

The slow subsystem is a linear differential equation defined on the critical manifold $S$, but it is not defined on $F$. To overcome this problem, we consider the Filippov’s convention.
Singularity Perturbed System

**Slow-Fast Piecewise Linear System (PWLS)**

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\dot{u} = \frac{du}{dt} = \varepsilon (Au + av + b), \quad \dot{v} = \frac{dv}{dt} = u_1 + |v|.
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\[S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-\]

The manifold \(S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-\) is a Fenichel’s manifold.
Sporadically Perturbed System

Slow-Fast Piecewise Linear System (PWLS)

\[ \dot{u} = \frac{du}{dt} = \varepsilon (Au + av + b), \quad \dot{v} = \frac{dv}{dt} = u_1 + |v|. \]

The manifold \( S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^- \) is a Fenichel's manifold.
Theorem 1.

The manifold \( S_\varepsilon = S^+_\varepsilon \cup S^-_\varepsilon \) is a Fenichel’s manifold.

- \( S^+_\varepsilon = \left\{ (u, v) \in \mathbb{R}^n : v \geq 0, \; -e_1^T(\varepsilon A - \lambda^+_n I)^{-1}u + v = \frac{\varepsilon}{\lambda^+_n}e_1^T(\varepsilon A - \lambda^+_n I)^{-1}b \right\} \).

- \( S^-_\varepsilon = \left\{ (u, v) \in \mathbb{R}^n : v \leq 0, \; -e_1^T(\varepsilon A - \lambda^-_n I)^{-1}u + v = \frac{\varepsilon}{\lambda^-_n}e_1^T(\varepsilon A - \lambda^-_n I)^{-1}b \right\} \).

For \( \varepsilon > 0 \) and sufficiently small, \( S_\varepsilon \) satisfies:

a) \( S_\varepsilon \) is locally invariant manifold.
Theorem 1.

The manifold $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$ is a Fenichel’s manifold.

For $\varepsilon > 0$ and sufficiently small, $S_\varepsilon$ satisfies:

a) $S_\varepsilon$ is locally invariant manifold.

b) The flow on $S_\varepsilon$ is a regular perturbation of the flow on $S$. 
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The manifold $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$ is a Fenichel’s manifold.

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c) $S_\varepsilon^+$ and $S_\varepsilon^-$ are the repelling and attracting branch, respectively.
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c) \( S_\varepsilon^+ \) and \( S_\varepsilon^- \) are the repelling and the attracting branch, respectively.

d) Given a compact subset \( \hat{S} \) of the critical manifold \( S \), \( \exists \hat{S}_\varepsilon \) compact subsets of the slow manifold \( S_\varepsilon \) (diffeomorphic to \( \hat{S} \)) such that \( d_H(\hat{S}_\varepsilon, \hat{S}) = O(\varepsilon) \), \( (d_H := \) Hausdorff distance).
The manifold $S_\varepsilon = S_\varepsilon^+ \cup S_\varepsilon^-$ is a Fenichel’s manifold.

For $\varepsilon > 0$ and sufficiently small, $S_\varepsilon$ satisfies:

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d) Given a compact subset $\hat{S}$ of the critical manifold $S$, $\exists \hat{S}_\varepsilon$ compact subsets of the slow manifold $S_\varepsilon$ (diffeomorphic to $\hat{S}$) such that $d_H(\hat{S}_\varepsilon, \hat{S}) = O(\varepsilon)$, ($d_H :=$ Hausdorff distance).

e) $S_\varepsilon$ is a regular perturbation of $S$. 

Rafel Prohens

On periodic orbits in non-smooth differential equations with applications
Theorem 1.

The manifold $S_\varepsilon = S^+_\varepsilon \cup S^-_\varepsilon$ is a Fenichel’s manifold.

For $\varepsilon > 0$ and sufficiently small, $S_\varepsilon$ satisfies:

a) $S_\varepsilon$ is locally invariant manifold.
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e) $S_\varepsilon$ is a regular perturbation of $S$. 
Maximal Canard Orbits

A point $p_\varepsilon$ in $S^+_{\varepsilon} \cap S^-_{\varepsilon}$, it is said to be a maximal canard (resp. faux canard) point if the orbit, $\gamma_{p_\varepsilon}$, through $p_\varepsilon$ is a maximal canard (resp. faux canard) orbit.

$S_{\varepsilon} = S^+_{\varepsilon} \cup S^-_{\varepsilon}$

- Maximal canard orbits cross from $S^-_{\varepsilon}$ to $S^+_{\varepsilon}$.
- To locate them we study:
  - Behaviour of the flow on $\mathcal{F}$,
  - order of contact of $p_\varepsilon \in S^+_{\varepsilon} \cap S^-_{\varepsilon}$.
Existence of Maximal Canard Orbits.

Study of $S^+_{\epsilon} \cap S^-_{\epsilon}$

**Theorem 2.2**

a) If $a_{1j} \neq 0$ for some $j \in \{2, \ldots, s\}$, then $\dim(S^+_{\epsilon} \cap S^-_{\epsilon}) = n - 3$

a.1) If $u_1^* > 0$, ∃ maximal canard through $p_\epsilon$ and order of contact 1;

a.2) If $u_1^* < 0$, ∃ faux canard through $p_\epsilon$ and order of contact 1;

a.3) If $u_1^* = 0$, order of contact greater than or equal to 2.

b) If $a_{1j} = 0$ for all $j \in \{2, \ldots, s\}$ and $b_1 = 0$, then

$\dim(S^+_{\epsilon} \cap S^-_{\epsilon}) = n - 2$ and neither maximal nor faux canard orbits exist.
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Source of Maximal Canard Orbits

Theorem 3.

a) Each point $p_\varepsilon$ in $S_\varepsilon^+ \cap S_\varepsilon^-$ lies in the unfolding of a contact point of order greater than or equal to 2 of the slow subsystem with the fold hyperplane $\mathcal{F}$.

b) If $n = 3$, then the maximal canard point (or faux canard point) of order 1 lies in the unfolding of the two-fold visible-visible (or invisible-invisible) point of the slow subsystem.
Source of Maximal Canard Orbits

Representation of a 2-dimensional reduced flow. Upper panels: unperturbed case surrounding the invisible two-fold $p^*_0$. Bottom panels: perturbed flow where the black point $p^*_\varepsilon$ stands for the faux canard point, while the white points $p^+$ and $p^-$ are the breaking points of $p^*_0$. These white points are invisible two-fold singularities for $S^+_{\varepsilon}$ and $S^-_{\varepsilon}$. 
Conclusions


### Slow-Fast Piecewise Linear System (PWLS)

\[
\begin{align*}
\dot{u} &= \frac{d u}{d t} = \varepsilon (A u + a v + b), \\
\dot{v} &= \frac{d v}{d t} = u_1 + |v|.
\end{align*}
\]

- An explicit expression for the slow manifold have been derived
- This expression allows to find maximal canard orbits

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- Introduce a theory for slow-fast dynamics by using PWL systems,
- and then deriving simplified models that are meaningful for neuroscience applications.

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Idea: reproduce canard-induced MMO behaviour in three-dimensional PWL slow-fast systems and investigate the equivalent of maximal canards (primary, secondary) and folded nodes.

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Introduction

Qualitative compatibility between PWL and smooth diff. eq.

Maximal and faux canards in $\mathbb{R}^3$

MMOs in PWL slow-fast dynamics in $\mathbb{R}^3$

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Goals:

This paper analyses PWL systems displaying folded singularities, primary and secondary canards, with a similar control of the maximal winding number as in the smooth case.
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Strategies to construct canard type dynamics 3D PWL

1) To construct transient canard trajectories in three-dimensional systems, building up on the knowledge from the planar case, we proceed as follows.

From the planar case, the simplest way to consider three-dimensional models is to put a slow drift on the parameter that displays the canard (or quasi-canard).

For instance, for systems in Liénard form\(^2\)

\[
\varepsilon \dot{x} = y - f(x), \quad \dot{y} = a - x. \tag{1}
\]

We will simply add a trivial slow dynamics on the parameter displaying the explosion in the planar system. Consider the slow drift

\[
\dot{a} = c, \quad c \in \mathbb{R}. \tag{2}
\]

---

2) To approximate a quadratic fold of a smooth slow-fast system, we distinguishing between two-piece local systems and three-piece local systems given by $f$.

Hence, (1)+(2),

\[
\begin{align*}
\varepsilon \dot{x} &= y - f(x), \\
\dot{y} &= a - x, \\
\dot{a} &= c.
\end{align*}
\]

That is, we will consider,

\[
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\end{align*}
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where $0 < \varepsilon \ll 1$, 

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where \( 0 < \varepsilon \ll 1 \),
Generating mechanism: quasi-canard explosion

\[ f(x) = x + \frac{1}{2}(1 + k)(|x - 1| - |x + 1|) \]

Transient MMO in a three-dimensional version of the two-piece local system (1). Parameter values for this transient MMO trajectory are: \( \varepsilon = 0.1, \ k = 0.5, \ c = -0.001. \)
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Transient MMO in a three-dimensional version of the two-piece local system (1). Parameter values for this transient MMO trajectory are: $\varepsilon = 0.1$, $k = 0.5$, $c = -0.001$. 
Generating mechanism: quasi-canard explosion

1) an explosive behaviour in the growth of small oscillations
2) no repelling slow manifold

Hence, one can create transient MMO dynamics but not of true canard type.
Generating mechanism: quasi-canard explosion

1) an explosive behaviour in the growth of small oscillations
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Hence, one can create transient MMO dynamics but not of true canard type.
Generating mechanism: canard explosion

$$f(x) = F_\delta(x) = \begin{cases} 
-x + (\beta + 1)\delta & \text{if } x \geq \delta, \\
\beta x & \text{if } |x| \leq \delta, \\
x - (\beta - 1)\delta & \text{if } x_0 < x < -\delta, \\
-x + 2x_0 - (\beta - 1)\delta & \text{if } x \leq x_0.
\end{cases}$$

Canard-induced MMOs in transient dynamics exactly as in the smooth case
Generating mechanism: canard explosion

Observe:
1) the four-piece PWL critical manifold
2) a dynamic canard explosion, and hence, folded node type dynamics

The parameter values of the critical manifold are the same as in [AON97], and the speed of the drift is $c = -0.01$.

Therefore, this could be -in PWL systems- the correct framework to find where the equivalent of the folded node is.
Three-piece local system,

\[
\begin{align*}
\varepsilon \dot{x} &= -y + f(x) \\
\dot{y} &= p_1 x + p_2 z \\
\dot{z} &= p_3, \quad \text{where} \quad f = f_\delta.
\end{align*}
\]

\[
f_\delta(x) = \begin{cases} 
0 & \text{if } |x| \leq \delta, \\
|x| - \delta & \text{if } |x| \geq \delta.
\end{cases}
\]
Dynamics near the PWL equivalent of folded singularities

Case $p_1 > 0$: (a) folded saddle, (b) folded node.

In the central zone, $H(x, y, z) = \varepsilon p_1(p_1 x + p_2 z)^2 + (p_1 y - \varepsilon p_2 p_3)^2$, is a first integral. It is either a hyperbola ($p_1 < 0$) or a cylinder ($p_1 > 0$), with axis $x = -\frac{p_2}{p_1} z$, $y = \frac{\varepsilon p_2 p_3}{p_1}$. If $p_1 < 0$ no rotation can happen in this region.
If $p_1 > 0$, the eigenvalues are $\pm i \sqrt{\varepsilon p_1}$, therefore trajectories do rotate in this region.

The line segment organises the dynamics of the full system by acting as an axis of rotation for trajectories that display Small-Amplitude Oscillations (SAOs) in the central zone, which corresponds to the so-called weak canard in the smooth case.

It can be proved that the associated maximal winding number $\mu$ is obtained as

$$\mu = \frac{\delta}{\pi \sqrt{\varepsilon}} \frac{p_1 \sqrt{p_1}}{|p_2 p_3|}$$

Note that $\mu$ is reminiscent of the eigenvalue ratio at a folded singularity in the smooth setting.
In the smooth case, this maximal winding number is independent of $\varepsilon$.

Thus, in order to reproduce quantitatively the behaviour observed in the smooth context, we choose

$$\delta = \pi \sqrt{\varepsilon},$$

and hence, the maximal winding number is

$$\mu = \frac{p_1 \sqrt{p_1}}{|p_2 p_3|}$$

This choice gives a complete match (qualitative and quantitative) with the behaviour of smooth slow-fast systems near folded singularities. That is,
the $\varepsilon$-dependence of $\delta$ given by

$$\delta = \pi \sqrt{\varepsilon},$$

forces the central zone collapse to a single corner-line in the singular limit $\varepsilon = 0$, that is, the three-piece local system for $\varepsilon > 0$ converges, in the singular limit, to a two-piece local system. Hence,

- one can see the central zone, needed to obtain canard dynamics, as a blow-up of the corner-line that exists in the singular limit.
- the size of this blow-up, $O(\sqrt{\varepsilon})$, matches that of the smooth case.

Recall: when blow-up is performed near non-hyperbolic points in smooth slow-fast systems, it can be proven that the region of hyperbolicity, where canards are shown to exist, is extended in the blown-up locus by a size of $O(\sqrt{\varepsilon})$. 
Maximal canards and weak canards

But, are there maximal canards? i.e. explicit solutions passing from the attracting slow manifold to the repelling one.

It can be shown that, yes.

In particular it can be shown that, indeed, there is a unique maximal canard, that passes from one side to the other without completing a full rotation; by definition, this special solution is the primary canard or strong canard.

There are, also, maximal canards completing full $k$ rotations, for some values of $k$, named secondary canards.
Proposition

Consider $p_3 > 0$, $\delta = \pi \sqrt{\varepsilon}$ and $\varepsilon$ small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

a) If $p_1 > 0$ and $p_2 < 0$, for every integer $k$ with $0 \leq k \leq \lfloor \mu \rfloor$, there exists a maximal canard $\gamma_k$ intersecting the switching plane $\{x = -\delta\}$ at $p_k = (-\delta, y_k, z_k)$ where

$$
y_k = - \left( \left( k + \frac{1}{2} \right) \frac{p_2 p_3}{\sqrt{p_1}} + p_1 \right) \pi \varepsilon^{\frac{3}{2}} - p_2 p_3 \varepsilon^2 + O(\varepsilon^{\frac{5}{2}}),
$$

(3)

$$
z_k = - \left( k + \frac{1}{2} \right) \frac{p_3}{\sqrt{p_1}} \pi \sqrt{\varepsilon} + O(\varepsilon).
$$

Moreover, $\gamma_k$ turns $k$ times around the weak canard $\gamma_w$, therefore $\gamma_0$ is the strong canard.
Consider $p_3 > 0$, $\delta = \pi \sqrt{\varepsilon}$ and $\varepsilon$ small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

b) If $p_1 > 0$ and $p_2 > 0$, there exists a unique maximal canard $\gamma_0$ intersecting the switching plane at $p_0 = (-\delta, y_0, z_0)$ where the coordinates $y_0$ and $z_0$ satisfy equation (3) with $k = 0$. Since, $\gamma_0$ turns less than one time around the faux canard $\gamma_f$, therefore $\gamma_0$ is the strong canard.

c) If $p_1 < 0$, there are no maximal canards.

Finally, we show on an example how to construct a (linear) global return and obtain PWL MMOs.
A PWL example with MMOs

A global return near a PWL folded node, so that we can create canard-induced MMOs.

First, adding a fourth zone to allow for LAOs

\[
\tilde{f}_\delta(x) = \begin{cases} 
-x - \delta & \text{if } x \leq -\delta, \\
0 & \text{if } |x| \leq \delta, \\
x - \delta & \text{if } \delta < x < x_0, \\
-x + 2x_0 - \delta & \text{if } x \geq x_0.
\end{cases}
\]

Then, add linear terms to the \(z\) equation in order to obtain a global return mechanism.

\[
\begin{align*}
\varepsilon \dot{x} &= -y + \tilde{f}_\delta(x) \\
\dot{y} &= p_1 x + p_2 z \\
\dot{z} &= p_3 + \alpha_1 (x - \kappa) + \alpha_2 (y - \zeta) + \alpha_3 (z - \xi).
\end{align*}
\]
Periodic PWL MMO $\Gamma$ near a folded node. Panels (a1) and (a2) show a phase-space representation of $\Gamma$ together with the 4-piece PWL critical manifold $C^0$; panel (a2) is a zoom of panel (a1) near the central flat zone, highlighting the SAOs.
Panel (b1) shows the time profile of $\Gamma$ for the fast variable $x$. Panel (b2) shows a similar MMO obtained by imposing conditions so that $\Gamma$ has SAOs with a constant amplitude.

MMO in this model have SAOs with increasing amplitude as the trajectory travels through the central zone. This is simply due to the fact that the eigenvalues in the central zone have non-zero real part because of the new terms in the $z$-equation.
Application: Estimation of the synaptic conductance in a McKean-model neuron


To understand the flow of information in the brain, estimating the synaptic conductances impinging on a single neuron, directly from its membrane potential, is one of the open problems.

In this work, we aim at giving a first proof of concept to address the estimation of synaptic conductances when the neuron is spiking.
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Estimation of the synaptic conductance in a McKean-model neuron

Simplified model of neuronal activity, namely a piecewise linear version of the Fitzhugh-Nagumo model, the McKean model

\[
C \frac{dv}{dt} = f(v) - w - w_0 + I - I_{\text{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0,
\]

where \( f \) is a 3-zone piecewise linear function,

\[
f(v) = \begin{cases} 
-v & v < a/2, \\
v - a & a/2 \leq v \leq (1 + a)/2, \\
1 - v & v > (1 + a)/2.
\end{cases}
\]

- variables: membrane potential, \( v \), the fast variable and \( w \) the slow component,
- parameters: membrane capacitance, \( C \), \( 0 < C < 0.1 \); total current that the neuron is receiving from non-synaptic inputs, \( I \); \( v_0 \), \( w_0 \), \( \gamma \) and \( a \) conductance properties and combinations of membrane reversal potentials.
\[ C \frac{dv}{dt} = f(v) - w - w_0 + I - I_{\text{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0, \]

- we consider the synaptic current $I_{\text{syn}} = g_{\text{syn}}(v - v_{\text{syn}})$ apart from the total one
- $g_{\text{syn}}$ stands for the synaptic conductance and is considered to be constant

Therefore, $I_{\text{syn}}$ can be understood as a representation of the mean field of the synaptic inputs.

---

3 Synaptic current is the movement of charge through the postsynaptic membrane due to synaptic transmission. The post-synaptic membrane is the membrane of the nerve after the synapse.
Existence and uniqueness of the periodic orbit


If

\[ g_{syn} > 1 - \frac{1}{\gamma}, \quad I_1 < I < I_2 \quad \text{and} \quad |g_{syn} + C\gamma| < 1, \]

Th.1 gives that there exists a limit cycle which is unique and stable.
⇒ At a first step, we infer steady synaptic conductances from the cell’s oscillatory activity.

The idea is to get $g_{syn}$ as follows:

- Once we get the analytical expression of the period of oscillation $T(g_{syn})$, and
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T(g_{syn}) = \tilde{T} \quad \text{(inverse problem)}
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In practice...
to solve

\[ T(g_{syn}) = \tilde{T}, \quad \text{(inverse problem)} \]

we approximate \( T(g_{syn}) \) and \( \tilde{T} \) by

- \( T_a \) an analytical approximation of \( T(g_{syn}) \)
- \( \tilde{T}_a \) a numerical approximation of the period of oscillation of \( \tilde{T} \).
to solve

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and then we solve

\[ T_a(g_{syn}) = \tilde{T}_a \rightarrow g_{syn,a} \]

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instead.
Steady synaptic conductances estimation

\[ T_{\text{lateral}} = \frac{1}{2\lambda_s} \ln \left( \left| \frac{\gamma(I - I_i)}{\gamma(I - I_i) - K} \right| \right) \]

\[ T_{\text{central}} = \frac{1}{2\lambda_q} \ln \left( \left| \frac{\gamma(I - I_i) + K_{1,i}}{\gamma(I - I_i) + K_{2,i}} \right| \right) , \quad i = 1, 2. \]

\( T_{\text{lateral}} \) stands for \( T_L \) when \( i = 1 \) and for \( T_R \) when \( i = 2 \). \( T_{\text{central}} \) stands for \( T_{c,up} \) when \( i = 1 \) and for \( T_{c,down} \) when \( i = 2 \). \( K, K_{1,i} \) and \( K_{2,i} \) are functions depending on the system parameters and they have a non-linear dependence with \( g_{\text{syn}} \).
Relative error when we estimate the synaptic conductance. Different traces correspond to different values of $g_{syn}$

(A) versus the applied current, $I$, for $C = 10^{-4}$.

(B) versus the membrane capacitance, $C$, for $I = I_1 + 10^{-3}$

$a = 0.25$, $v_0 = 0$, $\gamma = 0.5$, $w_0 = 0$, $v_{syn} = 0.25 + a/2$. 
Variable synaptic conductances estimation

We want to estimate $g_{syn}$ when the neuron is regularly spiking.

Idea:

1. Solve McKean system using the RK78 method.
2. Once we have $v(t)$, we find the different peaks of $v(t)$ and we compute the differences in time to obtain the sequence of periods $\{T_1, \ldots, T_k\}$.
3. For each $T_k$ we get $g_{syn}^k$ by using the steady synaptic conductance estimation for $T(g_{syn}^k, C, I) = T_k$.
4. We interpolate to obtain $g_{syn}(t)$.
Computational network that models layer 4Cα of primary visual cortex (McLaughlin et al (2000) and Tao et al (2004)).
Estimation: Panel A shows the real and the estimated conductances vs time. The estimation fits the synaptic conductance with a small shift which is larger as $C$ increase. $C = 0.001$.

B and C: scatter plot of the real vs the estimated
Panel B: after interpolation; Panel C: only with estimated values.

Parameters: $a = 0.25$, $v_0 = 0$, $\gamma = 0.5$, $w_0 = 0$, $v_{syn} = 0.25 + a/2$, $C = 0.001 \mu F/cm^2$, $I = 0.625 \mu A/cm^2$, $g_{syn}(t_0) = 0.6278$. 
We still are working in these problems...

Thanks for your attention