On periodic orbits in non-smooth differential equations with applications

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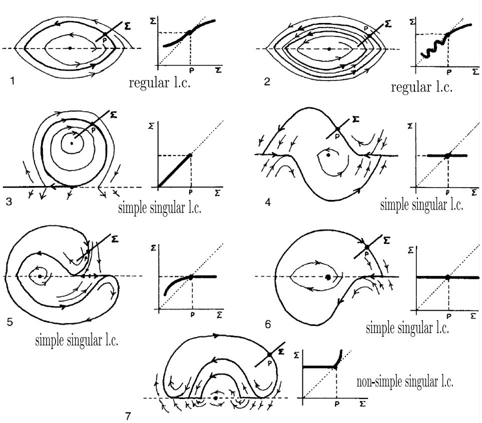
Introduction: A quick browse to some previous works

- [CGP-1996] *Limit Cycles for Non Smooth Differential Equations via Schwarzian Derivative*
- [CGP-1999] *The center problem for discontinuous Liénard differential equation*
- [CGP-2000] *Center-focus and isochronous center problems for discontinuous differential equations*
- [CGP-2001] *Degenerate Hopf bifurcations in discontinuous planar systems*
- [PT-2013] *Canard trajectories in $3D$ piecewise linear systems*

- [CGP-1996] *Limit Cycles for Non Smooth Differential Equations via Schwarzian Derivative*

$$(\dot{x}, \dot{y}) = \begin{cases} (P^+(x,y), Q^+(x,y)) = X_n + X_m, \\ (P^-(x,y), Q^-(x,y)) = Y_p + Y_q, \end{cases}$$

where X_u and Y_u are homogeneous polynomial v.f. of degree u



A. F. Filippov, “Differential Equations with Discontinuous Righthand Sides,” Kluwer, Dordrecht, 1988.

Theorem 1

This system has one non simple singular limit cycle or, at most, four simple singular limit cycles. Moreover, at most two of these singular limit cycles surround the origin.

Theorem 2

All its periodic orbits surround the origin and it has, at most, one singular limit cycle (which is simple), and two regular limit cycles.

Tools: geometry of the set of points at which the derivative of the angular component of the vector field is zero; polar coordinates; the Schwarzian derivative of the Poincaré return map.

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- [CGP-1999] *The center problem for discontinuous Liénard differential equation*

$$(\dot{x},\dot{y}) = \begin{cases} (-y+f^{+}(x),x), & \text{if} \quad x \geq 0 \\ (-y+f^{-}(x),x), & \text{if} \quad x \leq 0, \end{cases} \tag{1}$$

and

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where $f^{\pm}(x) = \sum_{i\geq 2} a_i^{\pm} x^i$.

We derive the general expression of the Lyapunov constants.

Theorem 1

For (1), the n-th Lyapunov constant, $n \geq 2$ is

$$V_n = \begin{cases} \frac{2n!!}{(n+1)!!}(a_n^+ - a_n^-) & \text{if} \quad n \quad \text{is even,} \\ \frac{\pi n!!}{(n+1)!!}(a_n^+ + a_n^-) & \text{if} \quad n \quad \text{is odd,} \end{cases}$$

Theorem 2

For (2), the n-th Lyapunov constant, $n \geq 2$ is: $V_2 = 0$ and

$$V_n = \begin{cases} \sum_{j=1}^{n/2-1} C_{n/2,j}\, a_{2j+1}^+ (a_{n-2j}^+ + a_{n-2j}^-) & \text{if} \quad n \quad \text{is even,} \\ \frac{\pi n!!}{(n+1)!!}(a_n^+ + a_n^-) & \text{if} \quad n \quad \text{is odd,} \end{cases}$$

for some $C_{n/2,j}$ constant numbers.

Tools: the Lyapunov constants for nonsmooth differential equations are quasi-homogeneous polynomials in the variables given by the coefficients of the differential equation

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$$\dot{z} = \begin{cases} F_1(z,\bar{z}), & \text{if} \quad \mathrm{Im}(z) \geq 0 \\ F_2(z,\bar{z}), & \text{if} \quad \mathrm{Im}(z) \leq 0, \end{cases} \qquad (1)$$

where $z = x + iy = \mathrm{Re}(z) + i\mathrm{Im}(z) \in \mathbb{C}$

$$\dot{z} = F_j(z,\bar{z}), \qquad \text{where} \qquad j = 1,2. \qquad (1.j)$$

We relate the order of degeneracy of the critical point with the orders of the two components.

Definition

The origin is a (m,k)-monodromic point if

$$\Pi(\rho)-\rho=O(\rho^m) \qquad \text{and} \qquad T(\rho)-2\pi=O(\rho^k).$$

$(m,*)$ means that the origin of (1) is a (m,k)-monodromic point for some k.

Theorem

If (1) has a (m,k)-monodromic point and (1.1) has a $(n,*)$-monodromic point, then (1.2) has $(\widetilde{m},*)$-monodromic point with $\widetilde{m}\geq\min(m,n)$.

Tools: the use of the relation between the extensions of the half return maps to a whole neighbourhood of zero

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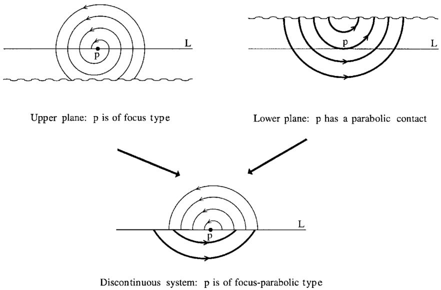
where $X^{\pm}, Y^{\pm}$ are real analytical functions. The role of foci points is inherited by four types of singular points, pseudo-focus.



Obtain the general expressions for the first three Lyapunov constants.

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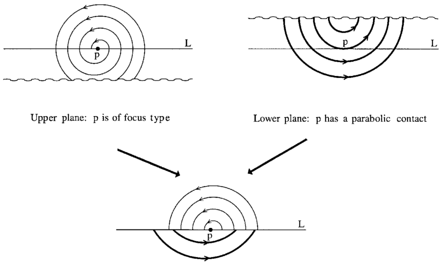

Upper plane: p is of focus type

Lower plane: p has a parabolic contact

Discontinuous system: p is of focus-parabolic type

Obtain the general expressions for the first three Lyapunov constants.

- [PT-2013] *Canard trajectories in $3D$ piecewise linear systems*

Singularly perturbed 3–dimensional piecewise linear differential systems

$$\begin{cases} \dot{u}_1 = \varepsilon(a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1), \\ \dot{u}_2 = \varepsilon(a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2), \\ \dot{v} = u_1 + |v|, \end{cases}$$

where $0 < \varepsilon \ll 1$.

Fenichel's geometric theory allows us to analyze the dynamics of

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \varepsilon g(\mathbf{u}, \mathbf{v}, \varepsilon), \quad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = f(\mathbf{u}, \mathbf{v}, \varepsilon),$$

where $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^s \times \mathbb{R}^q$ when f and g are sufficiently smooth functions. The coordinates of $\mathbf{u}$ are called *slow variables*, while the coordinates of $\mathbf{v}$ are called *fast variables*.

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This analysis follows by combining the behaviour of the singular orbits, corresponding to the limiting problems, $\varepsilon = 0$

layer:	$\dot{\mathbf{u}} = \mathbf{0},$	$\dot{\mathbf{v}} = f(\mathbf{u}, \mathbf{v}, 0),$	
reduced:	$\mathbf{u}' = g(\mathbf{u}, \mathbf{v}, 0),$	$\mathbf{0} = f(\mathbf{u}, \mathbf{v}, 0),$	$\mathbf{u} \in \mathbb{R}^s$

where $' = d/d\tau$, $\tau = \varepsilon t$. Critical manifold

$$\mathcal{S} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{s+q} \mid f(\mathbf{u}, \mathbf{v}, 0) = \mathbf{0}\}.$$

We call normally hyperbolic the singular points $(\mathbf{u}_0, \mathbf{v}_0) \in \mathcal{S}$ for which the eigenvalues of the Jacobian matrix $D_{\mathbf{v}} f(\mathbf{u}_0, \mathbf{v}_0)$ have nonzero real part.

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Consider $\mathcal{S}_0 \subset \mathcal{S}$ a compact set such that every point in $\mathcal{S}_0$ is a normally hyperbolic singular point.

$\mathcal{S}_0 \Rightarrow \mathcal{S}_\varepsilon$ locally invariant slow manifold and

The restriction of the flow of the perturbed system to the slow manifold $\mathcal{S}_\varepsilon$ is a small smooth perturbation of the flow of the reduced problem.

Orbits of the perturbed system are composed by:

- slow segments are close to the flow of the reduced problem
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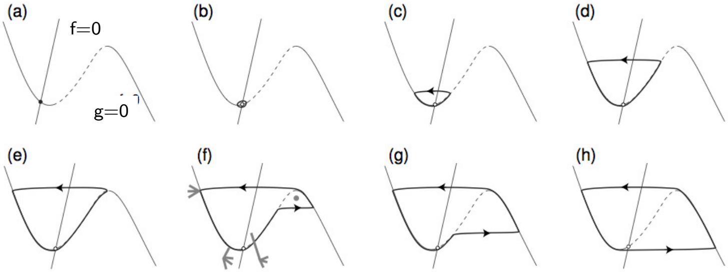
Canard Orbits in 2-D



[Izhikevich, Springer(2007)]

Canard orbits cross from the attracting manifold to the repelling manifold.

Question

What remains of previous dynamical behaviour when smoothness is no longer present?

In [PRSZ2011][1] the authors prove the existence of canard cycles in singularly perturbed piecewise differential systems with $s = 2$ and $q = 1$.

This fact suggests that canards are not exclusively a differential phenomenon, but rather a geometric one.

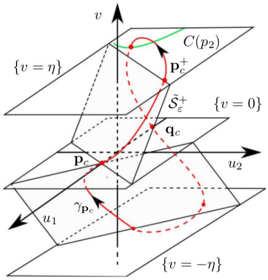
[1] A. Pokrovskii, D. Rachinskii, V. Sobolev and A. Zhezherun, *Topological degree in analysis of canard-type trajectories in 3-D systems*, Applicable Analysis: An International Journal, 90 (2011), 1123–1139.

For next singularly perturbed 3–dimensional piecewise linear differential system

$$\begin{cases} \dot{u}_1 = \varepsilon(a_{11}u_1 + a_{12}u_2 + a_{13}v + b_1), \\ \dot{u}_2 = \varepsilon(a_{21}u_1 + a_{22}u_2 + a_{23}v + b_2), \\ \dot{v} = u_1 + |v|, \end{cases}$$

where $0 < \varepsilon \ll 1$,

- we present results similar to those obtained by the Geometric Singular Perturbation Theory.
- we obtain the global expression of the slow manifold $\mathcal{S}_\varepsilon$.
- we characterize the existence of canard orbits in such systems.
- we provide numerical arguments for the existence of a canard cycle.



Representation of the canard cycle $\gamma_{\mathbf{p}_c}$, slow manifolds $\tilde{\mathcal{S}}_\varepsilon^- \cup \tilde{\mathcal{S}}_\varepsilon^+$ and the border planes $\{v=\eta\}, \{v=0\}$ and $\{v=-\eta\}$, which separate the regions where the system is linear. We highlighted the points of intersection of $\gamma_{\mathbf{p}_c}$ with the border planes.

Related works

Canard dynamics has been investigated in planar PWL slow-fast systems, from the 1990s.

After a brief mention of the "loss" of canards in PWL systems with two corners in

1991 M. Itoh and R. Tomiyasu, *Canards and irregular oscillations in a nonlinear circuit*, in Circuits and Systems, 1991., IEEE International Sympoisum on, IEEE, 1991, pp. 850–853.

The first study of a PWL van der Pol system from the perspective of canards (McKean ODE model)

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Up to very recently

2011 D. J. Simpson and R. Kuske, *Mixed-mode oscillations in a stochastic, piecewise-linear system*, Physica D, 240 (2011), pp. 1189–1198.

2012 H. G. Rotstein, S. Coombes, and A. M. Gheorge, *Canard-like explosion of limit cycles in two-dimensional piecewise-linear models of FitzHugh-Nagumo type*, SIAM Journal on Applied Dynamical Systems, 11 (2012), pp. 135–180.

2013 M. Desroches, E. Freire, S. J. Hogan, E. Ponce, P. Thota, *Canards in piecewise-linear systems: explosions and super-explosions*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science 469 (2013).

2014 S. Fernández-García, M. Desroches, M. Krupa, and A. E. Teruel, *Canards in planar piecewise linear systems with three zones*. Preprint.

The four piece of the critical manifold

The cubic critical manifold is replaced by a PWL caricature consisting of three straight line segments. The corners play the role of the fold points and cycles resembling canards and evolving around these corners were identified by simulation.

1997 N. Arima, H. Okazaki, H. Nakano, *A generation mechanism of canards in a piecewise linear system*, IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences 80 (1997) 447–453.

reasons for the equivalent of canards with head can arise only in systems with one more piece in between the two corners.

The main idea to obtain true canard cycles in a planar PWL systems consists in approximating the critical manifold near a fold by a three-piece PWL function.

A large class of neuron models based on the approximation that the membrane of the neuron behaves like a circuit. The voltage equation is obtained by applying Kirchoff's law.

After the model by (HH model):

1952 A. L. Hodgkin and A. F. Huxley, *A quantitative description of membrane current and its application to conduction and excitation in nerve*, The Journal of physiology, 117 (1952), p. 500.

The first reduction to a planar system (FHN model):

1961 R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical Journal, 1 (1961), pp. 445–466.

1962 J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proceedings of the IRE, 50 (1962), pp. 206–2070.

where, the vector field of the HH model was approximated by a polynomial system through the crucial observation that the voltage nullcline is roughly cubic shaped.

Hence, the FHN model appears as a modified van der Pol system.

The FHN model was investigated from the slow-fast perspective and in

1970 H. P. McKean Jr, Nagumo's equation, Advances in mathematics, 4 (1970), pp. 209–223,

further simplified by approximating the cubic voltage nullcline by a PWL function.

PWL slow-fast systems

Since the late 1990s several papers have shown that the canard phenomenon can be reproduced with piecewise-linear (PWL) dynamical systems in two and three dimensions, exhibiting an slow-fast dynamics.

Smooth slow-fast dynamical systems models in neuroscience displaying canard-induced MMOs.

Goal

We aim to explore the gap between PWL and smooth slow-fast dynamical systems by analysing canonical PWL systems that display folded singularities.

R.P., A. E. Teruel and C. Vich, *Slow-fast n-dimensional piecewise linear differential systems*. Preprint 2015.

Slow-Fast Piecewise Linear System (PWLS)

$$\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt} = \varepsilon(A\mathbf{u} + \mathbf{a}v + \mathbf{b}), \qquad \dot{\mathbf{v}} = \frac{d\mathbf{v}}{dt} = u_1 + |v|.$$

This paper is mainly concerned with maximal canard orbits occurring in n-dimensional piecewise linear slow-fast systems.

More precisely, conditions for the existence of maximal canard orbits and/or faux canard orbits are established.

We show that these maximal canards perturb from singular orbits (singular canards) whose order of contact with the fold manifold is greater than or equal to two.

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$\mathbf{u} \in \mathbb{R}^s$ slow variable
$\mathbf{v} \in \mathbb{R}$ fast variable

$0 < \varepsilon \ll 1$ ratio of time scales
$n = s + 1$ system dimension

$A = (a_{ij})_{1 \le i,j \le s}$ $s \times s$ real matrix
$\mathbf{a} = (a_1, a_2, \dots, a_s)^T$ vector in $\mathbb{R}^s$
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- Rather general: $f(\mathbf{u}, v, \varepsilon) = \mathbf{d}^T\mathbf{u} + |v|$, with $\mathbf{d} \neq \mathbf{0}$, can be transformed into our system $(\mathbf{u} \rightarrow (\mathbf{d}^T\mathbf{u}, u_2, \dots, u_n)^T)$.

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Unperturbed Dynamics:

Associated to

$$\begin{cases} \dot{\mathbf{u}} = \varepsilon(A\mathbf{u} + \mathbf{a}v + \mathbf{b}), \\ \dot{v} = u_1 + |v|, \end{cases}$$

we have the:

- fast subsystem (layer problem)
- slow subsystem (reduced problem)
- critical manifold, where the slow subsystem is defined, $\mathcal{S}$
- fold manifold, $\mathcal{F}$, when normal hyperbolicity fails, (points where $\mathcal{S}$ folds)

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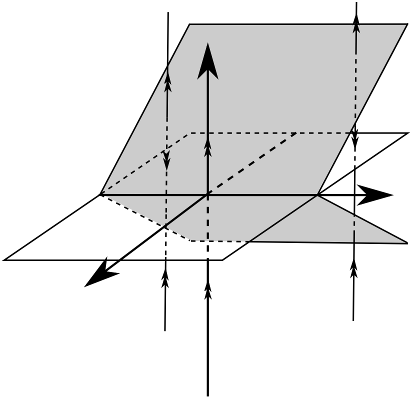
- $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{F} \cup \mathcal{S}^-$

$$\mathcal{S}^+ = \{u_1 + v = 0; \ v > 0\}$$
$$\mathcal{S}^- = \{u_1 - v = 0; \ v < 0\}$$
$$\mathcal{F} = \{u_1 = 0, \ v = 0\}$$

where $\mathcal{S}^+$ and $\mathcal{S}^-$ are normally hyperbolic and $\mathcal{F}$ is the fold manifold



Unperturbed Dynamics:

- Fast subsystem

$$\begin{cases} \dot{\mathbf{u}} = \mathbf{0}, \\ \dot{v} = u_1 + |v|, \end{cases}$$

- Critical manifold

$$\mathcal{S} = \{(\mathbf{u}, v) \in \mathbb{R}^n : \ u_1 + |v| = 0\}$$

- $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{F} \cup \mathcal{S}^-$

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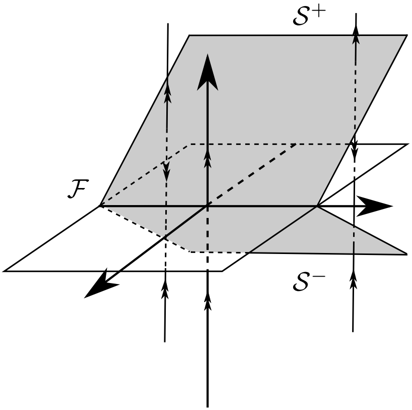


Unperturbed Dynamics:

- Slow subsystem associated to

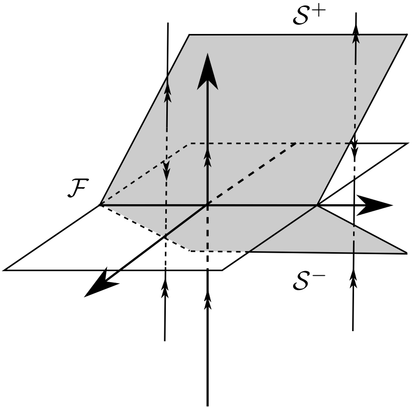
$$\begin{cases} \mathbf{u}' = A\mathbf{u} + \mathbf{a}v + \mathbf{b}, \\ \varepsilon v' = u_1 + |v|, \end{cases}$$

Unperturbed Dynamics:

- Slow subsystem

$$\begin{cases} \mathbf{u}' = A\mathbf{u} + \mathbf{a}v + \mathbf{b}, \\ 0 = u_1 + |v|, \end{cases}$$

The slow subsystem is a linear differential equation defined on the critical manifold $\mathcal{S}$, but it is not defined on $\mathcal{F}$. To overcome this problem, we consider the Filippov's convention.



Singularly Perturbed System

Slow-Fast Piecewise Linear System (PWLS)

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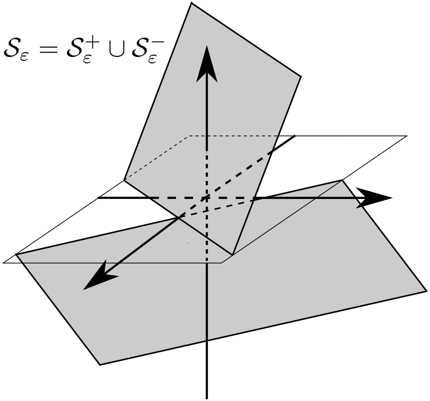

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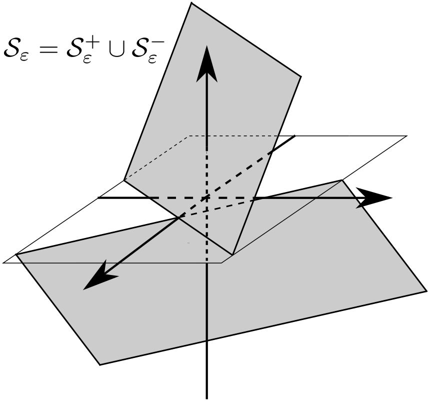

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- $$\mathcal{S}_\varepsilon^+ = \Big\{ (\mathbf{u}, v) \in \mathbb{R}^n : \ v \geq 0, \\ -\mathbf{e}_1^T(\varepsilon A - \lambda_n^+ I)^{-1}\mathbf{u} + v = \frac{\varepsilon}{\lambda_n^+}\mathbf{e}_1^T(\varepsilon A - \lambda_n^+ I)^{-1}\mathbf{b} \Big\}.$$
- $$\mathcal{S}_\varepsilon^- = \Big\{ (\mathbf{u}, v) \in \mathbb{R}^n : \ v \leq 0, \\ -\mathbf{e}_1^T(\varepsilon A - \lambda_n^- I)^{-1}\mathbf{u} + v = \frac{\varepsilon}{\lambda_n^-}\mathbf{e}_1^T(\varepsilon A - \lambda_n^- I)^{-1}\mathbf{b} \Big\}.$$

For $\varepsilon > 0$ and sufficiently small, $\mathcal{S}_\varepsilon$ satisfies:

a) $\mathcal{S}_\varepsilon$ is locally invariant manifold.

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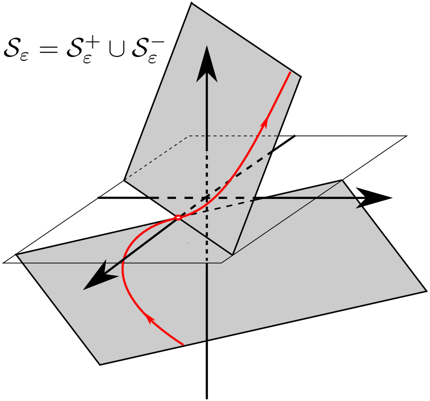
Maximal Canard Orbits

A point $\mathbf{p}_\varepsilon$ in $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$, it is said to be a maximal canard (resp. faux canard) point if the orbit, $\gamma_{\mathbf{p}_\varepsilon}$, through $\mathbf{p}_\varepsilon$ is a maximal canard (resp. faux canard) orbit.



- Maximal canard orbits cross from $\mathcal{S}_\varepsilon^-$ to $\mathcal{S}_\varepsilon^+$.
- To locate them we study:
  - Behaviour of the flow on $\mathcal{F}$,
  - order of contact of $\mathbf{p}_\varepsilon \in \mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$.

Existence of Maximal Canard Orbits.

Study of $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$

Theorem 2.2

a) If $a_{1j} \neq 0$ for some $j \in \{2, \dots, s\}$, then $\dim(\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-) = n - 3$
   - a.1) If $u_1^* > 0$, $\exists$ maximal canard through $\mathbf{p}_\varepsilon$ and order of contact 1;
   - a.2) If $u_1^* < 0$, $\exists$ faux canard through $\mathbf{p}_\varepsilon$ and order of contact 1;
   - a.3) If $u_1^* = 0$, order of contact greater than or equal to 2.

b) If $a_{1j} = 0$ for all $j \in \{2, \dots, s\}$ and $b_1 = 0$, then $\dim(\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-) = n - 2$ and neither maximal nor faux canard orbits exist.

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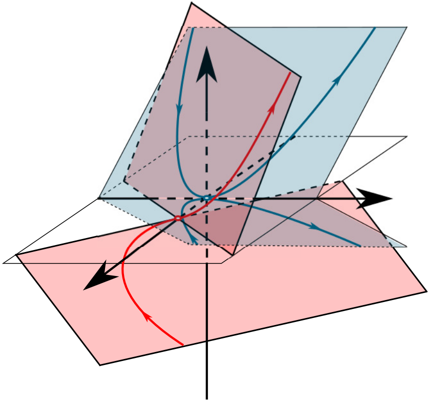
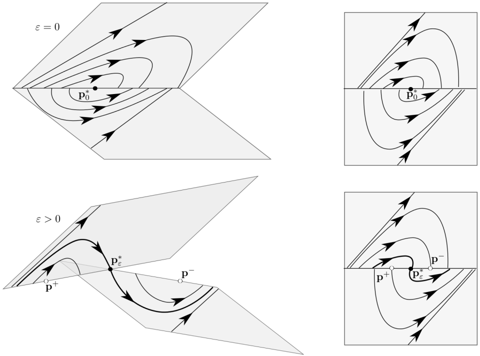


Theorem 3.

a) Each point $\mathbf{p}_\varepsilon$ in $\mathcal{S}_\varepsilon^+ \cap \mathcal{S}_\varepsilon^-$ lies in the unfolding of a contact point of order greater than or equal to 2 of the slow subsystem with the fold hyperplane $\mathcal{F}$.

b) If $n = 3$, then the maximal canard point (or faux canard point) of order 1 lies in the unfolding of the two-fold visible-visible (or invisible-invisible) point of the slow subsystem.

Source of Maximal Canard Orbits



Representation of a 2-dimensional reduced flow. Upper panels: unperturbed case surrounding the invisible two-fold $\mathbf{p}_0^*$. Bottom panels: perturbed flow where the black point $\mathbf{p}_\varepsilon^*$ stands for the faux canard point, while the white points $\mathbf{p}^+$ and $\mathbf{p}^-$ are the breaking points of $\mathbf{p}_0^*$. These white points are invisible two-fold singularities for $\mathcal{S}_\varepsilon^+$ and $\mathcal{S}_\varepsilon^-$.

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Strategies to construct canard type dynamics 3D PWL

1) To construct transient canard trajectories in three-dimensional systems, building up on the knowledge from the planar case, we proceed as follows.

From the planar case, the simplest way to consider three-dimensional models is to put a slow drift on the parameter that displays the canard (or quasi-canard).

For instance, for systems in Liénard form[2]

$$\varepsilon \dot{x} = y - f(x), \qquad \dot{y} = a - x. \tag{1}$$

We will simply add a trivial slow dynamics on the parameter displaying the explosion in the planar system. Consider the slow drift

$$\dot{a} = c, \quad c \in \mathbb{R}. \tag{2}$$

[2] M. Wechselberger, *Existence and bifurcation of canards in* $\mathbb{R}^3$ *in the case of a folded node*, SIAM Journal on Applied Dynamical Systems, 4 (2005), pp. 101–139.

Strategies to construct canard type dynamics 3D PWL

$2)$ To approximate a quadratic fold of a smooth slow-fast system, we distinguishing between two-piece local systems and three-piece local systems given by f.

Hence, (1)+(2),

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That is, we will consider,

$$\varepsilon \dot{x} = -y + f(x), \qquad \dot{y} = p_1 x + p_2 z, \qquad \dot{z} = p_3.$$

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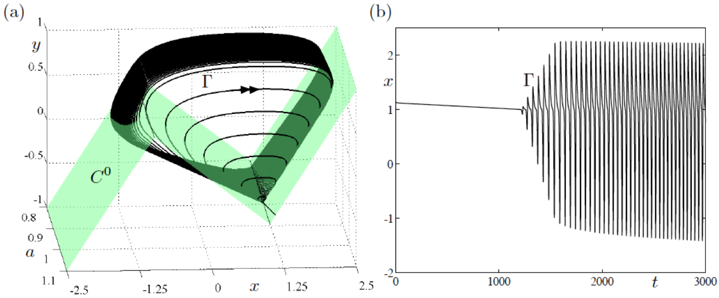
Generating mechanism: quasi-canard explosion

$$f(x) = x + \frac{1}{2}(1+k)(|x-1| - |x+1|)$$



Transient MMO in a three-dimensional version of the two-piece local system (1). Parameter values for this transient MMO trajectory are: $\varepsilon = 0.1$, $k = 0.5$, $c = -0.001$.

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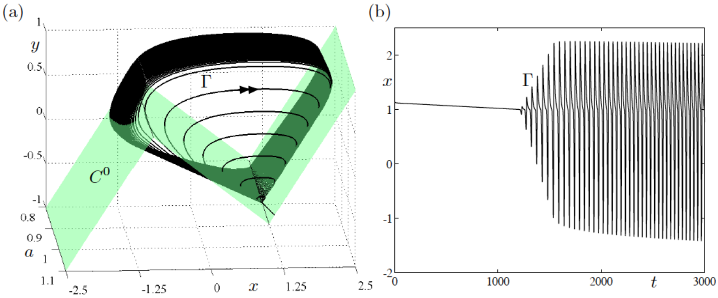

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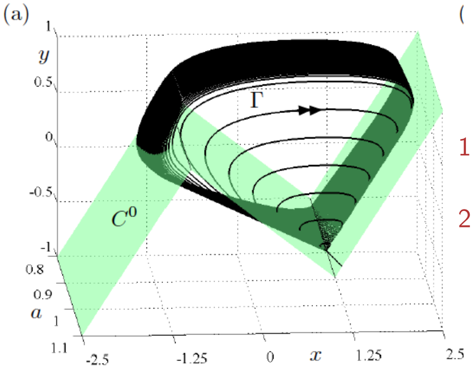

(

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Hence, one can create transient MMO dynamics but not of true canard type.

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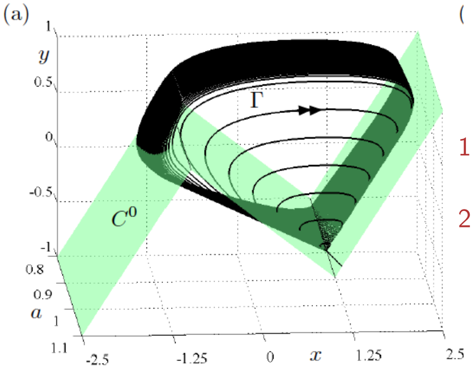

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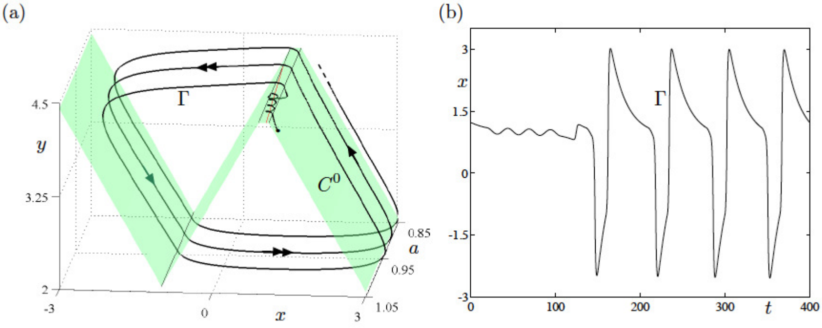
Generating mechanism: canard explosion

$$f(x) = F_\delta(x) = \begin{cases} -x + (\beta+1)\delta & \text{if} \quad x \geq \delta, \\ \beta x & \text{if} \quad |x| \leq \delta, \\ \ \ x - (\beta-1)\delta & \text{if} \quad x_0 < x < -\delta, \\ -x + 2x_0 - (\beta-1)\delta & \text{if} \quad x \leq x_0. \end{cases}$$

Canard-induced MMOs in transient dynamics exactly as in the smooth case



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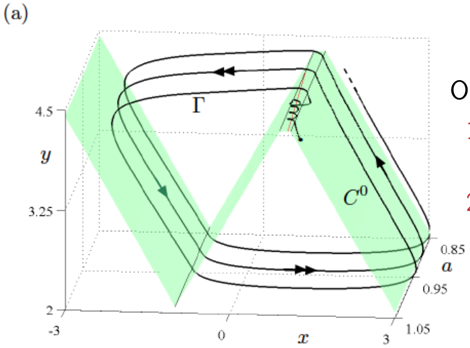


Observe:

1) the four-piece PWL critical manifold
2) a dynamic canard explosion, and hence, folded node type dynamics

The parameter values of the critical manifold are the same as in [AON97], and the speed of the drift is $c = -0.01$.

Therefore, this could be -in PWL systems- the correct framework to find where the equivalent of the folded node is.

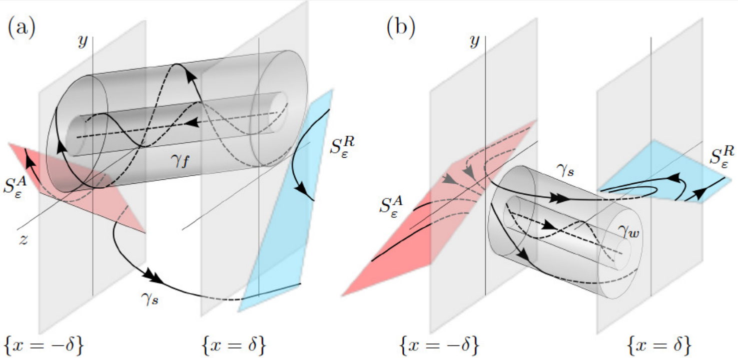
Dynamics near the PWL equivalent of folded singularities

Three-piece local system,

$$\begin{aligned} \varepsilon\dot{x} &= -y + f(x) \\ \dot{y} &= p_1 x + p_2 z \\ \dot{z} &= p_3, \qquad \text{where} \qquad f = f_\delta. \end{aligned}$$

$$f_\delta(x) = \begin{cases} 0 & \text{if} \quad |x| \leq \delta, \\ |x| - \delta & \text{if} \quad |x| \geq \delta. \end{cases}$$

Dynamics near the PWL equivalent of folded singularities



Case $p_1 > 0$: (a) folded saddle, (b) folded node .

In the central zone, $H(x,y,z) = \varepsilon p_1(p_1 x + p_2 z)^2 + (p_1 y - \varepsilon p_2 p_3)^2$, is a first integral. It is either a hyperbola ($p_1 < 0$) or a cylinder ($p_1 > 0$), with axis $x = -\frac{p_2}{p_1} z, \quad y = \frac{\varepsilon p_2 p_3}{p_1}$. If $p_1 < 0$ no rotation can happen in this region.

Dynamics near the PWL equivalent of folded singularities

If $p_1 > 0$, the eigenvalues are $\pm i\sqrt{\varepsilon p_1}$, therefore trajectories do rotate in this region.

The line segment organises the dynamics of the full system by acting as an axis of rotation for trajectories that display Small-Amplitude Oscillations (SAOs) in the central zone, which corresponds to the so-called *weak canard* in the smooth case.

It can be proved that the associated maximal winding number μ is obtained as

$$\mu = \frac{\delta}{\pi\sqrt{\varepsilon}} \frac{p_1\sqrt{p_1}}{|p_2 p_3|}$$

Note that μ is reminiscent of the eigenvalue ratio at a folded singularity in the smooth setting

Dynamics near the PWL equivalent of folded singularities

In the smooth case, this maximal winding number is independent of ε.

Thus, in order to reproduce quantitatively the behaviour observed in the smooth context, we choose

$$\delta = \pi\sqrt{\varepsilon},$$

and hence, the maximal winding number is

$$\mu = \frac{p_1\sqrt{p_1}}{|p_2 p_3|}$$

This choice gives a complete match (qualitative and quantitative) with the behaviour of smooth slow-fast systems near folded singularities. That is,

Dynamics near the PWL equivalent of folded singularities

the ε-dependence of δ given by

$$\delta = \pi\sqrt{\varepsilon},$$

forces the central zone collapse to a single corner-line in the singular limit $\varepsilon = 0$, that is, the three-piece local system for $\varepsilon > 0$ converges, in the singular limit, to a two-piece local system. Hence,

- one can see the central zone, needed to obtain canard dynamics, as a blow-up of the corner-line that exists in the singular limit.
- the size of this blow-up, $O(\sqrt{\varepsilon})$, matches that of the smooth case.

Recall: when blow-up is performed near non-hyperbolic points in smooth slow-fast systems, it can be proven that the region of hyperbolicity, where canards are shown to exist, is extended in the blown-up locus by a size of $O(\sqrt{\varepsilon})$.

Maximal canards and weak canards

But, are there maximal canards? i.e. explicit solutions passing from the attracting slow manifold to the repelling one.

It can be shown that, yes.

In particular it can be shown that, indeed, there is a unique maximal canard, that passes from one side to the other without completing a full rotation; by definition, this special solution is the *primary canard* or strong canard.

There are, also, maximal canards completing full k rotations, for some values of k, named *secondary canards*.

Proposition

Consider $p_3 > 0$, $\delta = \pi\sqrt{\varepsilon}$ and ε small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

a) If $p_1 > 0$ and $p_2 < 0$, for every integer k with $0 \leq k \leq [\mu]$, there exists a maximal canard γ_k intersecting the switching plane $\{x = -\delta\}$ at $\mathbf{p}_k = (-\delta, y_k, z_k)$ where

$$\begin{aligned} y_k &= -\left(\left(k+\frac{1}{2}\right)\frac{p_2 p_3}{\sqrt{p_1}} + p_1\right)\pi\varepsilon^{\frac{3}{2}} - p_2 p_3 \varepsilon^2 + O(\varepsilon^{\frac{5}{2}}), \\ z_k &= -\left(k+\frac{1}{2}\right)\frac{p_3}{\sqrt{p_1}}\pi\sqrt{\varepsilon} + O(\varepsilon). \end{aligned} \tag{3}$$

Moreover, γ_k turns k times around the weak canard γ_w, therefore γ_0 is the strong canard.

Proposition

Consider $p_3 > 0$, $\delta = \pi\sqrt{\varepsilon}$ and ε small enough, and assume that every maximal canard with a given flight time, between the switching planes, is unique. The following statements hold.

b) If $p_1 > 0$ and $p_2 > 0$, there exists a unique maximal canard γ_0 intersecting the switching plane at $\mathbf{p}_0 = (-\delta, y_0, z_0)$ where the coordinates y_0 and z_0 satisfy equation (3) with $k = 0$. Since, γ_0 turns less that one time around the faux canard γ_f, therefore γ_0 is the strong canard.

c) If $p_1 < 0$, there are no maximal canards.

Finally, we show on an example how to construct a (linear) global return and obtain PWL MMOs.

A PWL example with MMOs

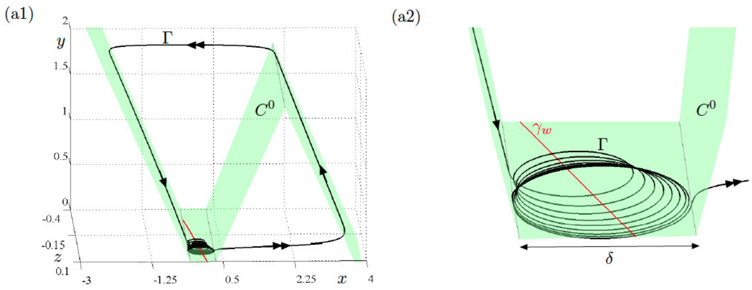
A global return near a PWL folded node, so that we can create canard-induced MMOs.

First, adding a fourth zone to allow for LAOs

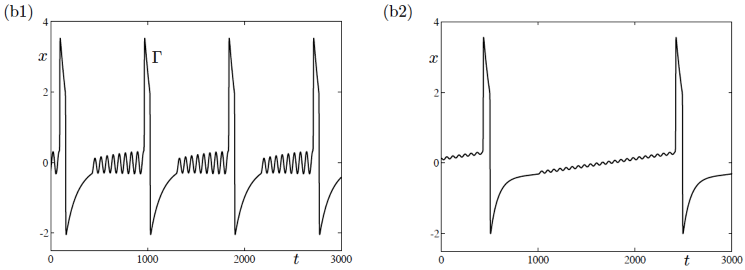
$$\widetilde{f}_\delta(x) = \begin{cases} -x-\delta & \text{if} \quad x \leq -\delta, \\ \;\; 0 & \text{if} \quad |x| \leq \delta, \\ \;\; x-\delta & \text{if} \quad \delta < x < x_0, \\ -x+2x_0-\delta & \text{if} \quad x \geq x_0. \end{cases}$$

Then, add linear terms to the z equation in order to obtain a global return mechanism.

$$\begin{aligned} \varepsilon \dot{x} &= -y + \widetilde{f}_\delta(x) \\ \dot{y} &= p_1 x + p_2 z \\ \dot{z} &= p_3 + \alpha_1(x-\kappa) + \alpha_2(y-\zeta) + \alpha_3(z-\xi). \end{aligned}$$



Periodic PWL MMO Γ near a folded node. Panels (a1) and (a2) show a phase-space representation of Γ together with the 4-piece PWL critical manifold C^0; panel (a2) is a zoom of panel (a1) near the central flat zone, highlighting the SAOs



Panel (b1) shows the time profile of Γ for the fast variable x. Panel (b2) shows a similar MMO obtained by imposing conditions so that Γ has SAOs with a constant amplitude.

MMO in this model have SAOs with increasing amplitude as the trajectory travels through the central zone. This is simply due to the fact that the eigenvalues in the central zone have non-zero real part because of the new terms in the z-equation.

Application: Estimation of the synaptic conductance in a McKean-model neuron

A. Guillamon, R. P., A.E. Teruel and C. Vich, *Estimation of the synaptic conductance in a McKean-model neuron*. Preprint 2015.

To understand the flow of information in the brain,

estimating the synaptic conductances impinging on a single neuron, directly from its membrane potential, is one of the open problems.

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Estimation of the synaptic conductance in a McKean-model neuron

Simplified model of neuronal activity, namely a piecewise linear version of the Fitzhugh-Nagumo model, the McKean model

$$C\frac{dv}{dt} = f(v) - w - w_0 + I - I_{\mathsf{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0,$$

where f is a 3-zone piecewise linear function,

$$f(v) = \begin{cases} -v & v < a/2, \\ v - a & a/2 \leq v \leq (1+a)/2, \\ 1 - v & v > (1+a)/2. \end{cases}$$

- variables: membrane potential, v, the fast variable and w the slow component,
- parameters: membrane capacitance, C, $0 < C < 0.1$; total current that the neuron is receiving from non-synaptic inputs, I; v_0, w_0, γ and a conductance properties and combinations of membrane reversal potentials.

$$C\frac{dv}{dt} = f(v) - w - w_0 + I - I_{\text{syn}}, \quad \frac{dw}{dt} = v - \gamma w - v_0,$$

- we consider the synaptic current[3] $I_{\text{syn}} = g_{\text{syn}}(v - v_{\text{syn}})$ apart from the total one
- g_{syn} stands for the synaptic conductance and is considered to be constant

Therefore, I_{syn} can be understood as a representation of the mean field of the synaptic inputs.

[3] Synaptic current is the movement of charge through the postsynaptic membrane due to synaptic transmission. The post-synaptic membrane is the membrane of the nerve after the synapse.

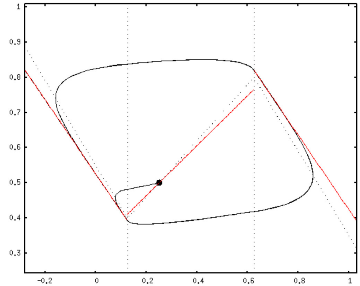
Existence and uniqueness of the periodic orbit

Llibre J, Ordóñez M, Ponce E. *On the existence and uniqueness of limit cycles in planar continuous piecewise linear systems without symmetry*. (2013). Nonlinear Anal. Real World Appl.

If

$$g_{syn} > 1 - \frac{1}{\gamma}, \quad I_1 < I < I_2 \quad \text{and} \quad |g_{syn} + C\gamma| < 1,$$

Th.1 gives that there exists a limit cycle which is unique and stable.



⇒ At a first step, we infer steady synaptic conductances from the cell's oscillatory activity.

The idea is to get g_{syn} as follows:

- once we get the analytical expression of the period of oscillation $T(g_{syn})$, and

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$$T(g_{syn}) = \widetilde{T} \qquad \text{(inverse problem)}$$

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In practice...

to solve

$$T(g_{syn}) = \widetilde{T}, \qquad \text{(inverse problem)}$$

we approximate $T(g_{syn})$ and $\widetilde{T}$ by

- T_a an analytical approximation of $T(g_{syn})$
- $\widetilde{T_a}$ a numerical approximation of the period of oscillation of $\widetilde{T}$.

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and then we solve

$$T_a(g_{syn}) = \widetilde{T_a} \to g_{syn,a}$$

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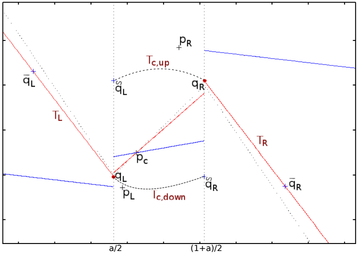
Steady synaptic conductances estimation

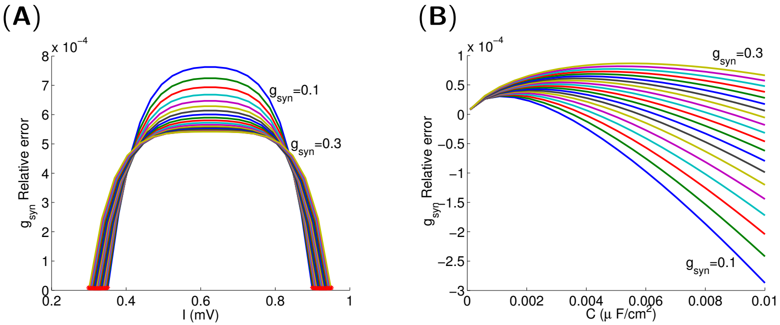


$$T_{lateral} = \frac{1}{2\lambda_s} \ln \left(\left| \frac{\gamma(I - I_i)}{\gamma(I - I_i) - K} \right| \right)$$

$$T_{central} = \frac{1}{2\lambda_q} \ln \left(\left| \frac{\gamma(I - I_i) + K_{1,i}}{\gamma(I - I_i) + K_{2,i}} \right| \right), \quad i = 1, 2.$$

$T_{lateral}$ stands for T_L when $i = 1$ and for T_R when $i = 2$. $T_{central}$ stands for $T_{c,up}$ when $i = 1$ and for $T_{c,down}$ when $i = 2$. K, $K_{1,i}$ and $K_{2,i}$ are functions depending on the system parameters and they have a non-linear dependence with g_{syn}.

Relative error when we estimate the synaptic conductance. Different traces correspond to different values of g_{syn}



(**A**) versus the applied current, I, for $C = 10^{-4}$.

(**B**) versus the membrane capacitance, C, for $I = I_1 + 10^{-3}$

$a = 0.25,\ v_0 = 0, \gamma = 0.5,\ w_0 = 0,\ v_{syn} = 0.25 + a/2.$

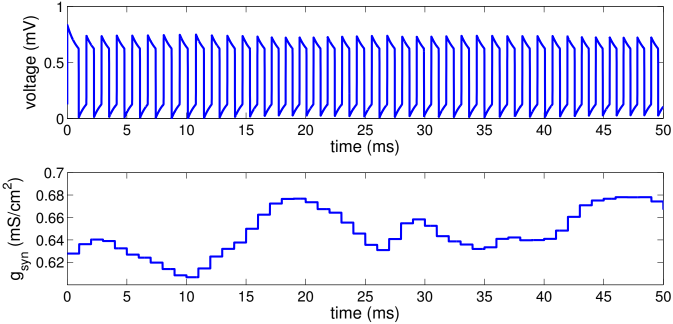
Variable synaptic conductances estimation

We want to estimate g_{syn} when the neuron is regularly spiking.

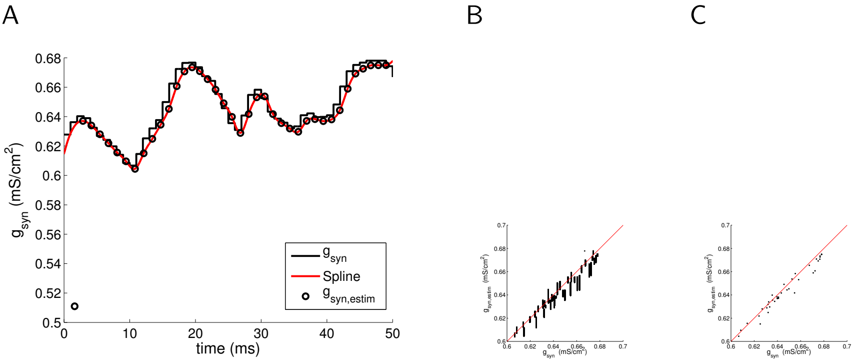
Idea:

1. Solve McKean system using the RK78 method.
2. Once we have $v(t)$, we find the different peaks of $v(t)$ and we compute the differences in time to obtain the sequence of periods $\{T_1, \ldots, T_k\}$.
3. For each T_k we get g_{syn}^k by using the steady synaptic conductance estimation for $T(g_{syn}^k, C, I) = T_k$.
4. We interpolate to obtain $g_{syn}(t)$.

Computational network that models layer $4C\alpha$ of primary visual cortex (McLaughlin et al (2000) and Tao et al (2004)).



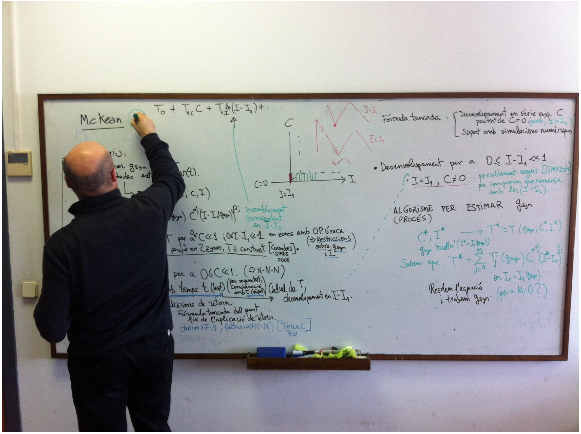
Estimation: Panel **A** shows the real and the estimated conductances vs time. The estimation fits the synaptic conductance with a small shift which is larger as C increase. $C = 0.001$.



B and **C**: scatter plot of the real vs the estimated
Panel B: after interpolation; Panel C: only with estimated values.

Parameters: $a = 0.25$, $v_0 = 0$, $\gamma = 0.5$, $w_0 = 0$, $v_{syn} = 0.25 + a/2$, $C = 0.001\ \mu F/cm^2$, $I = 0.625\ \mu A/cm^2$, $g_{syn}(t_0) = 0.6278$.

We still are working in these problems...



Thanks for your attention