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Variform Exact One-Peakon Solutions for Some Singular Nonlinear Traveling Wave Equations of the First Kind*

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In this paper, we consider variform exact peakon solutions for four nonlinear wave equations. We show that under different parameter conditions, one nonlinear wave equation can have different exact one-peakon solutions and different nonlinear wave equations can have different explicit exact one-peakon solutions. Namely, there are various explicit exact one-peakon solutions, which are different from the one-peakon solution $pe^{-\alpha|x-ct|}$. In fact, when a traveling system has a singular straight line and a curve triangle surrounding a periodic annulus of a center under some parameter conditions, there exists peaked solitary wave solution (peakon).

Keywords: Peakon; nonlinear wave equation; exact solution; smoothness of wave.

1. Introduction

In recent years, nonlinear wave equations with non-smooth solitary wave solutions, such as peaked solitons (peakons) and cusped solitons (cuspons), have attracted much attention in the literature. Peakon was first proposed by [Camassa & Holm, 1993; Camassa *et al.*, 1994] and thereafter other peakon equations were developed (see [Degasperis & Procesi, 1999; Degasperis *et al.*, 2002; Qiao, 2006, 2007; Li & Dai, 2007; Novikov, 2009], and cited references therein). Peakons are the so-called peaked solitons, i.e. solitons with discontinuous first-order derivative at the peak point. Usually, the profile of a wave function is called a peakon if at a continuous point its left and right derivatives are finite and have different signs [Fokas, 1995]. But if its left and right derivatives are positive and negative infinities, respectively, then the wave profile is called a cuspon.

In our paper [Li & Chen, 2007] and book [Li & Dai, 2007] (or more recent book [Li, 2013]), using the dynamical system approach, it has been theoretically proved that there exists a curve triangle including one singular straight line in a phase portrait of the traveling wave system corresponding to some nonlinear wave equation such that the traveling wave solutions have peaked profiles and lose their smoothness. In fact, the existence of a singular straight line leads to a dynamical behavior with two scale variables in a period annulus of a center. For a singular nonlinear traveling wave system of the first kind, the following two results hold (see [Li, 2013]).

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Theorem A (The Rapid-Jump Property of the Derivative Near the Singular Straight Line). *Suppose that in a left (or right) neighborhood of a singular straight line there exist a family of periodic orbits. Then, along a segment of every orbit near the straight line, the derivative of the wave function jumps down rapidly on a very short time interval.*

Theorem B (Existence of Finite Time Interval of Solution with Respect to Wave Variable in the Positive or Negative Direction). *For a singular nonlinear traveling wave system of the first class with possible change of the wave variable, if an orbit transversely intersects with a singular straight line at a point or it approaches a singular straight line, but the derivative tends to infinity, then it only takes a finite time interval to make the moving point of the orbit arrive on the singular straight line.*

These two theorems tell us that for a nonlinear wave equation, a peakon solution has a determined geometric property. It depends on the existence of a curve triangle surrounding a period annulus of a center of the corresponding traveling wave system, in the neighborhood of a singular straight line (see [Li, 2013]). In fact, the curve triangle are the limit curves of a family of periodic orbits of the traveling wave system. It gives rise to a peakon profile of the nonlinear wave equation.

For an example, as a shallow water model, the generalized Camassa–Holm (CH) equation with real parameters k, α

$$u_t + ku_x - u_{xxt} + \alpha u u_x = 2u_x u_{xx} + u u_{xxx} \quad (1)$$

has a one-peakon solution

$$u(x,t) = u(x-ct) = \phi(\xi) = ce^{-\sqrt{\frac{\alpha}{3}}|\xi|}, \quad (2)$$

when $\alpha = \frac{3}{c}(c-k)$ with $c > 0$, $k < c$, where c is the wave velocity. Equation (1) has the traveling system

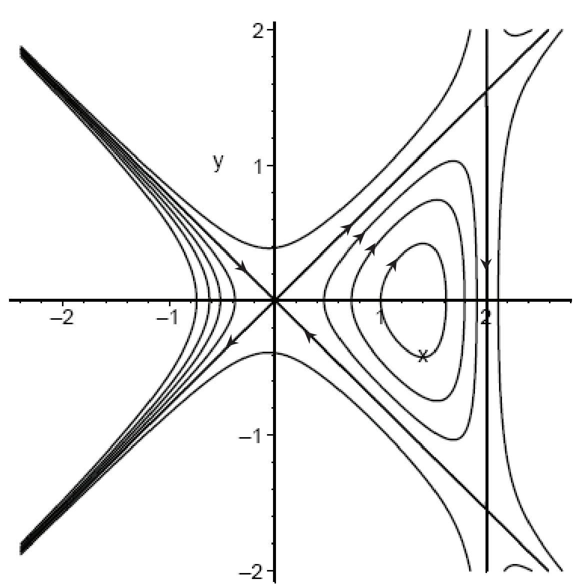
$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-y^2 + 2(k-c)\phi + \alpha\phi^2}{2(\phi - c)}, \quad (3)$$

which has the following first integral:

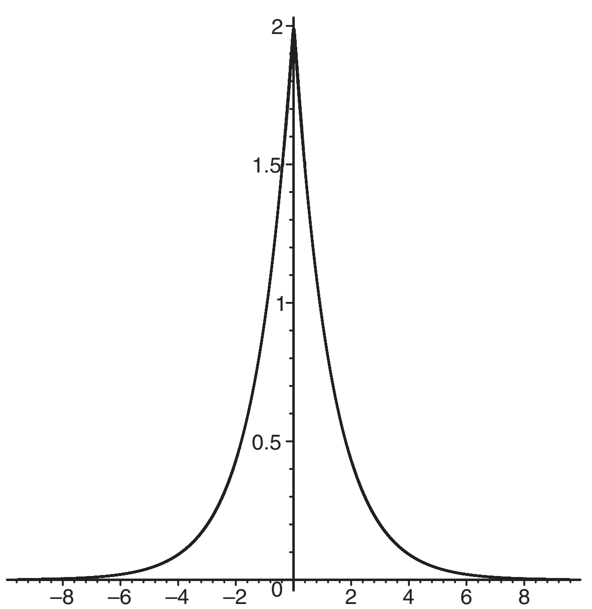
$$H(\phi, y) = (\phi - c)y^2 - \left[(k-c)\phi^2 + \frac{1}{3}\alpha\phi^3\right] = h. \quad (4)$$

Figure 1(a) shows the phase portrait of system (3) when $\alpha = \frac{3}{c}(c-k)$. Corresponding to the curve triangle enclosing the period annulus of the center $E_1(\frac{2(c-k)}{\alpha}, 0)$, Fig. 1(b) shows the peakon profile of Eq. (1) given by (2).

When $k = 0$, $\alpha = 3$, Eq. (1) is the original Camassa–Holm equation, it has one-peakon solution $u(x,t) = ce^{-|x-ct|}$. On the basis of this solution form, in [Beals *et al.*, 1999], the authors investigated



(a)



(b)

Fig. 1. The phase portraits of (3) and a peakon when $\alpha = \frac{3}{c}(c-k)$. (a) Phase portrait of system (3) when $\alpha = \frac{3}{c}(c-k)$ and (b) peakon solution.

the N-soliton solution of CH-equation of the form

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{5}$$

where the positions q_j and amplitudes p_j satisfy the following system:

$$\begin{aligned} \dot{q}_j &= \sum_{k=1}^{N} p_k e^{-|q_j-q_k|}, \\ \dot{p}_j &= p_j \sum_{k=1}^{N} p_k \operatorname{sgn}(q_j - q_k) e^{-|q_j-q_k|}, \\ &\qquad \text{for } j = 1, \ldots, N. \end{aligned} \tag{6}$$

In [Hone & Wang, 2008], the authors considered the N-soliton solution of form (5) of the Novikov equation [Novikov, 2009]:

$$u_t - u_{xxt} + 4u^2 u_x = u u_x u_{xx} + u^2 u_{xxx}, \tag{7}$$

where

$$\begin{aligned} \dot{q}_j &= \sum_{k=1}^{N} p_k p_l e^{-|q_j-q_k|-|q_j-q_l|}, \\ \dot{p}_j &= p_j \sum_{k=1}^{N} p_k p_l \operatorname{sgn}(q_j - q_k) e^{-|q_j-q_k|-|q_j-q_l|}, \\ &\qquad \text{for } j = 1, \ldots, N. \end{aligned} \tag{8}$$

Unfortunately, we have showed in [Li, 2014] that even though $\phi = pe^{(x-ct)}$ and $\phi = pe^{-(x-ct)}$ are two traveling wave solutions of Eq. (7), they cannot be combined to become the solution $\phi = pe^{-|x-ct|}$, i.e. an one-peakon solution of Eq. (7).

In this paper, we shall show the following two conclusions:

(1) Under different parameter conditions, one nonlinear wave equation can have different exact one-peakon solutions.
(2) Different nonlinear wave equations can have different explicit exact one-peakon solutions.

Namely, there are various exact explicit one-peakon solutions, which are different from the one-peakon solution given by (2). Therefore, to investigate N-peakon solutions for a given nonlinear wave equation, we may need to consider other forms of exact solutions, which is different from (5).

We consider the following four nonlinear wave equations as examples.

(i) The generalized Camassa–Holm equation

$$\begin{aligned} u_t + 2ku_x - u_{xxt} + \frac{1}{2}[\alpha u^2 + \beta u^3]_x \\ = 2u_x u_{xx} + u u_{xxx}. \end{aligned} \tag{9}$$

When $\beta = 0$, Eq. (9) is just Eq. (1).

(ii) The nonlinear dispersion equation $K(m,n)$, i.e.

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \quad m, n \geq 1, \tag{10}$$

where m, n are integers, a is a real parameter (see [Rosenau, 1997; Li & Liu, 2002]).

(iii) The two-component Hunter–Saxton (HS) system with real parameters A, σ (see [Moon, 2013]):

$$\begin{aligned} &u_{txx} + 2\sigma u_x u_{xx} + \sigma u u_{xxx} - \rho\rho_x + Au_x = 0, \\ &\rho_t + (\rho u)_x = 0, \end{aligned} \tag{11}$$

where $\sigma \in R$ and $A \geq 0$. System (11) is the short wave (or high-frequency) limit of the generalized two-component form of the Camassa–Holm shallow water equations.

(iv) The two-component Camassa–Holm system with real parameters $k, \alpha, e_0 = \pm 1$ (see [Olver & Rosenau, 1996; Chen *et al.*, 2006; Chen *et al.*, 2011; Li & Qiao, 2013]):

$$\begin{aligned} &m_t + \sigma u m_x - Au_{xx} + 2\sigma m u_x \\ &\qquad + 3(1-\sigma) u u_x + e_0 \rho\rho_x = 0, \\ &\rho_t + (\rho u)_x = 0, \end{aligned} \tag{12}$$

where $m = u - \alpha^2 u_{xx} - \frac{k}{2}$.

The corresponding traveling wave systems of Eqs. (9)–(12) have one or two singular straight lines, respectively (see next sections below). Under some particular parameter conditions, there exist at least one family of periodic orbits surrounding a center such that the boundary curves of the period annulus are a curve triangle including a singular straight line (see the phase portraits in the next sections). Applying the classical analysis method, we can obtain the parametric representations for these boundary curves. When we take these curve triangles into account as the limit curves of period annulus, these exact parametric representations provide very good understanding of the occurrence of peaked traveling wave solutions.

Namely, the curve triangle gives rise to a solitary cusp wave (*peakon*) solution.

This paper is organized as follows. In Secs. 2–5, we discuss respectively the exact peakon solutions for Eqs. (9)–(12).

2. Peakon Solutions of the Generalized Camassa–Holm Eq. (9)

Let $u(x,t) = \phi(x - ct)$. Then, Eq. (9) has the traveling system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{-y^2 + 2(k-c)\phi + \alpha\phi^2 + \beta\phi^3}{2(\phi - c)}. \tag{13}$$

Making the transformation $d\xi = (\phi - c)d\zeta$ for $\phi \neq c$, system (13) becomes

$$\frac{d\phi}{d\zeta} = y(\phi - c), \qquad \frac{dy}{d\zeta} = \frac{1}{2}[-y^2 + 2(k-c)\phi + \alpha\phi^2 + \beta\phi^3]. \tag{14}$$

System (14) is an integrable cubic system, which has the same invariant curve solutions as system (13) and the same first integral

$$H(\phi, y) = (\phi - c)y^2 - \left[(k-c)\phi^2 + \frac{1}{3}\alpha\phi^3 + \frac{1}{4}\beta\phi^4\right] = h. \tag{15}$$

Denote that

$$f(\phi) = \phi(\beta\phi^2 + \alpha\phi + 2(k-c)).$$

We assume that $\beta \neq 0$. Then, for $\beta > 0$, $\alpha = 0$ and $0 < k < c$, $f(\phi)$ has three zeros at $\phi_0 = 0$ and $\phi_{b\pm} = \pm\sqrt{\frac{2(c-k)}{\beta}}$. For $\beta > 0$, $\alpha \neq 0$, $f(\phi)$ has three zeros at

$$\phi_0 = 0, \quad \phi_{1,2} = \frac{1}{2\beta}[-\alpha \pm \sqrt{\alpha^2 - 8\beta(k-c)}], \quad (\phi_1 > \phi_2),$$

when $\Delta = \alpha^2 - 8\beta(k-c) > 0$. Thus, system (14) has three equilibrium points $E_1(\phi_1, 0)$, $O(0,0)$ and $E_2(\phi_2, 0)$ for $\Delta > 0$. In the straight line $\phi = c$, there are two equilibrium points $(c, Y_\pm)$, where $Y_\pm = \sqrt{c(c\alpha + 2(k-c) + \beta c^2)}$. For $c > 0$, $\Delta > 0$, the condition $c = \phi_{1,2}$ implies that for a fixed pair (c, k),

$$(c^2\beta + 2(k-c) + c\alpha)(c^2\beta + 2(k-c) - c\alpha) = 0,$$

$$\text{i.e. } \alpha < 0, \quad \beta = -\frac{1}{c}\alpha + \frac{2}{c^2}(c-k).$$

This equality follows also that $Y_\pm = 0$.

Let $M(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (14) at an equilibrium point (ϕ_i, y_i). We have

$$\begin{aligned} J(0,0) &= \det M(0,0) = c(k-c), \\ J(\phi_{1,2}, 0) &= \det M(\phi_{1,2}, 0) = \frac{1}{2}(c - \phi_{1,2})f'(\phi_{1,2}), \\ J(c, Y_\pm) &= \det M(c, Y_\pm) = -Y_\pm^2. \end{aligned} \tag{16}$$

By the theory in the planar dynamical systems, we see from (16) that the equilibrium points $(c, Y_\pm)$ are saddle points.

We denote that

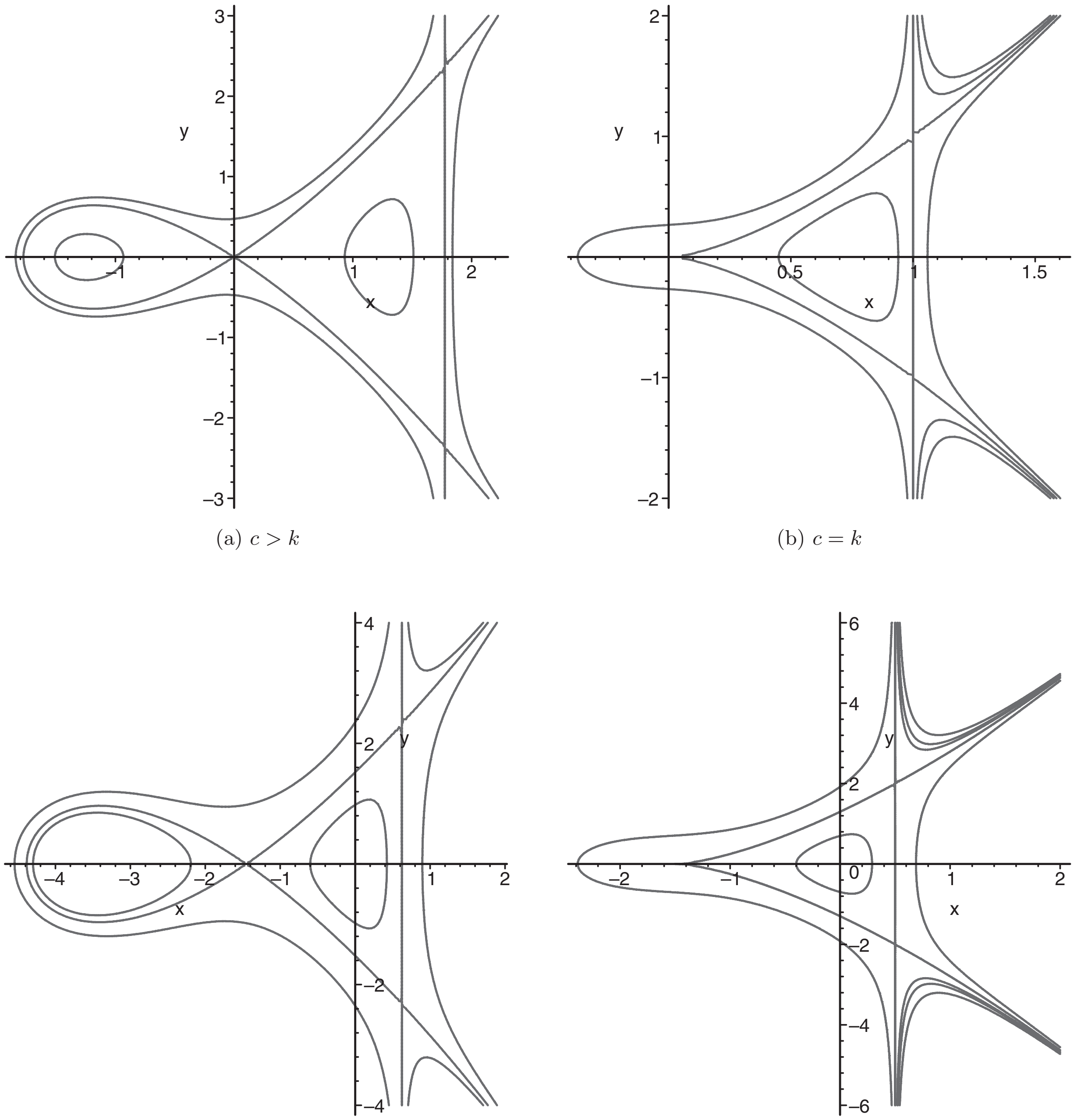
$$h_i = H(\phi_i, 0) = \frac{1}{12}\phi_i^3(2\alpha + 3\beta\phi_i), \quad (i = 1,2), \quad h_a = H(\phi_a, 0),$$

$$h_0 = H(0,0) = 0,$$

$$h_s = H(c, Y_\pm) = -c^2\left[(k-c) + \frac{1}{3}\alpha c + \frac{1}{4}\beta c^2\right].$$

Thus, for $c \neq 0$ and a fixed pair (c,k), $h_s = 0$ if and only if $\beta = -\frac{4}{3c}\alpha + \frac{4}{c^2}(c-k)$.

Assume that $\beta > 0$, $c > 0$. For a fixed pair (c,k), we consider three cases $c > k$, $c = k$ and $c < k$, respectively. When $c > k$, we know that $\phi_2 < 0 < \phi_1$ and the origin $O(0,0)$ is a saddle point. When $c = k$, if $\alpha \neq 0$, the origin $O(0,0)$ is a cusp point. There is another equilibrium point $E_2(-\frac{\alpha}{\beta}, 0)$ of system (14). When $c < k$, if $\alpha \neq 0$, the origin $O(0,0)$ is a center point. Under different parameter conditions, we have six phase portraits of system (14) shown in Figs. 2(a)–2(f), for which there exist heteroclinic triangle loops of system (14) surrounding a period annulus of a center. By Theorem A, near the straight line $\phi = c$, the variable "ζ" is a fast variable while the variable "ξ" is a



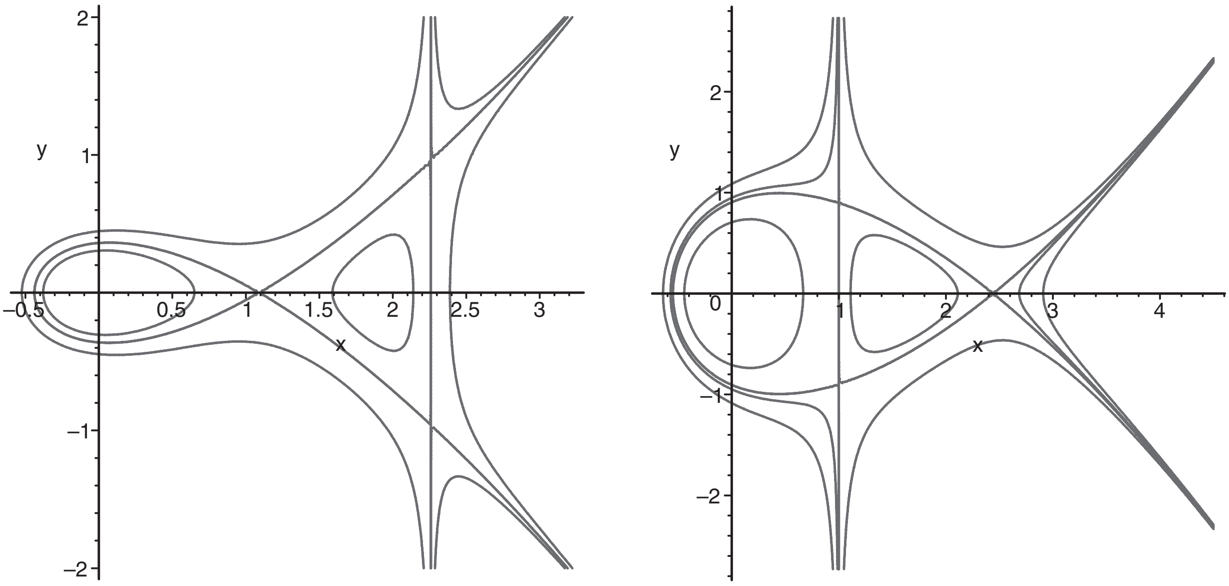
(a) $c > k$

(b) $c = k$

(c) $c < k$

(d) $c < k$

Fig. 2. Some phase portraits of system (14) when $\beta > 0$ and $c > 0$. Parameter conditions: (a) $\beta = -\frac{4\alpha}{3c} + \frac{4}{c^2}(c-k)$, (b) $\alpha < 0$, $\beta = -\frac{4\alpha}{3c}$, (c) $Y_+ > 0$, $H(\phi_1, 0) = H(c, Y_\pm)$, (d) $\Delta = 0$, $H(\phi_1, 0) = H(c, Y_\pm)$, (e) $\alpha < -\frac{4(k-c)}{c}$, $Y_+ > 0$, $H(\phi_2, 0) = H(c, y_\pm)$ and (f) $-\frac{4(k-c)}{c} < \alpha < 0$, $Y_+ > 0$, $H(\phi_1, 0) = H(c, Y_\pm)$.



(e) $c < k$

(f) $c < k$

Fig. 2. (*Continued*)

slow variable in the sense of the geometric singular perturbation theory.

We now discuss the exact peakon solutions of Eq. (9).

(i) When $c > k$, $\beta = \frac{4(c\alpha+3(k-c))}{3c^2}$, two heteroclinic orbits of system (14) in Fig. 2(a) given by $H(\phi, y) = 0$ can be written as $y^2 = \frac{1}{4}\beta\phi^2(\phi + \phi_m)$, where ϕ_m is the ϕ-coordinate of the intersection point of the homoclinic orbit defined by $H(\phi, y) = 0$ with the ϕ-axis. If $\alpha = 0$, $\phi_m = -\frac{4(c-k)}{\beta}$, by using the first equation of (13) to integrate and taking the initial value as $\phi(0) = c$ by Theorem B, we obtain the following peakon solution of Eq. (9):

$$\phi(\xi) = (-\phi_m)\mathrm{csch}^2\left(\frac{1}{2}\sqrt{c-k}\xi - \Omega_0\right), \quad \text{for } \xi \in (-\infty, 0),$$
$$\phi(\xi) = (-\phi_m)\mathrm{csch}^2\left(\frac{1}{2}\sqrt{c-k}\xi + \Omega_0\right), \quad \text{for } \xi \in (0, \infty), \tag{17}$$

where $\Omega_0 = \mathrm{ctnh}^{-1}\sqrt{\frac{c-\phi_m}{(-\phi_m)}}$.

(ii) When $\alpha < 0$, $c = k > 0$, $\beta = -\frac{4\alpha}{3c}$, two heteroclinic orbits of system (14) in Fig. 2(b) given by $H(\phi, y) = 0$ can be written as $y^2 = \frac{1}{4}\beta\phi^3$. Thus, corresponding to this curve triangle, we have the peakon solution of Eq. (9) as follows:

$$\phi(\xi) = \frac{4}{\left(\frac{2}{\sqrt{c}} - \frac{1}{2}\beta\xi\right)^2}, \quad \text{for } \xi \in (-\infty, 0),$$
$$\phi(\xi) = \frac{4}{\left(\frac{2}{\sqrt{c}} + \frac{1}{2}\beta\xi\right)^2}, \quad \text{for } \xi \in (0, \infty). \tag{18}$$

Clearly, by moving the saddle to the origin, for the three cases in Figs. 2(c), 2(e) and 2(d), we can obtain similar results as (17) and (18).

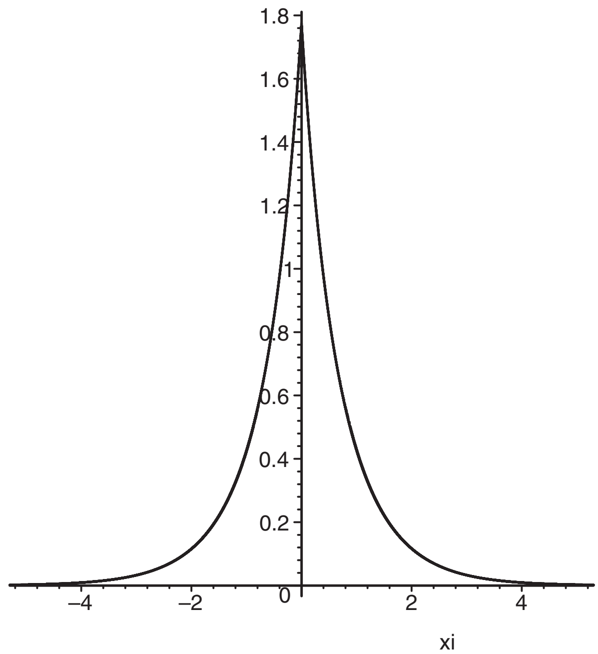
(iii) When $c < k$, $-\frac{4(k-c)}{c} < \alpha < 0$, $Y_+ > 0$, $H(\phi_1, 0) = H(c, Y_\pm)$, two heteroclinic orbits of system (14) in Fig. 2(f) given by $H(\phi, y) = h_s = h_1$ can be written as

$$y^2 = \frac{1}{4}\beta\left(\phi^3 + \frac{4\alpha + 3c\beta}{3\beta}\phi^2 + a_1\phi + ca_1\right) = \frac{1}{4}\beta(\phi_1 - \phi)^2(\phi - \phi_m),$$

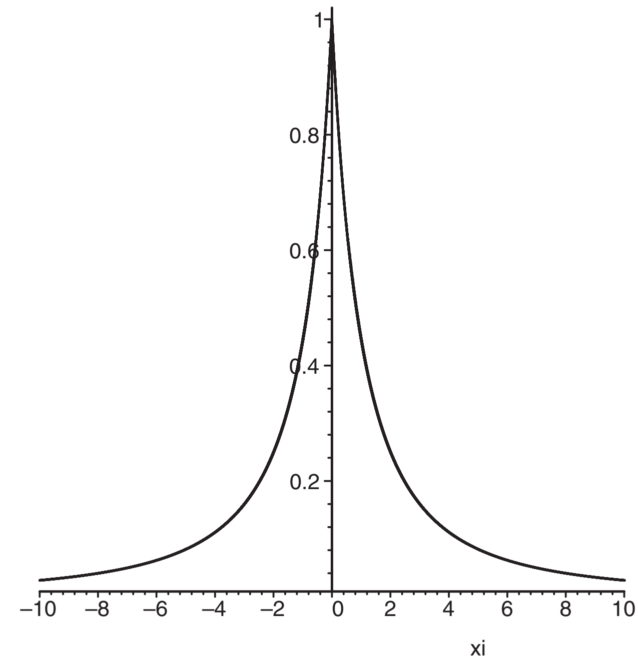
where

$$a_1 = c\left(\frac{4\alpha + 3c\beta}{3\beta}\right) + \frac{4(k-c)}{\beta},$$

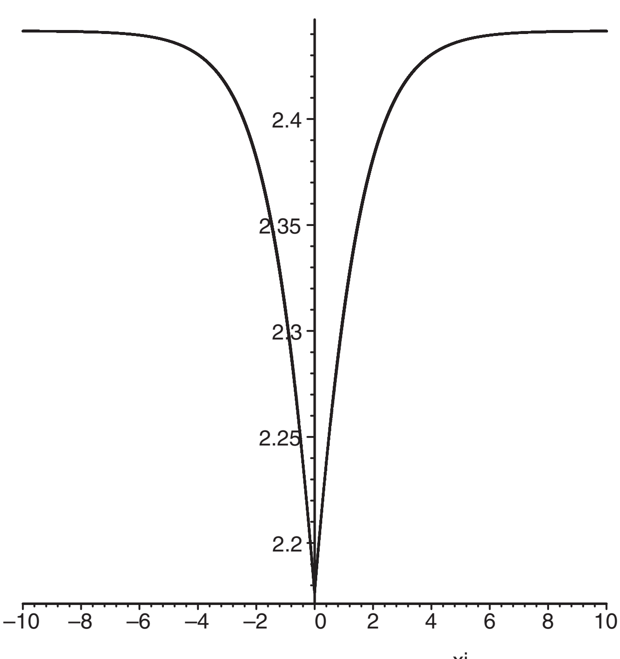
$$\phi_m = -\frac{\alpha + 3c\beta + 3\sqrt{\Delta}}{3\beta}.$$



(a) $c > k$



(b) $c = k$



(c) $c < k$

Fig. 3. Three peakon profiles of Eq. (9) when $\beta > 0$ and $c > 0$.

Hence, corresponding to this curve triangle, we have the peakon solution of Eq. (9) as follows:

$$\begin{aligned}\phi(\xi) &= \phi_1 + \phi_m \mathrm{sech}^2(\omega_0\xi + \Omega_1), \\ &\qquad \text{for } \xi \in (0,\infty), \\ \phi(\xi) &= \phi_1 + \phi_m \mathrm{sech}^2(\omega_0\xi - \Omega_1), \\ &\qquad \text{for } \xi \in (-\infty, 0),\end{aligned} \tag{19}$$

where

$$\omega_0 = \frac{1}{4}\sqrt{\beta(\phi_1 - \phi_m)},$$

$$\Omega_1 = \tanh^{-1}\sqrt{\frac{c-\phi_m}{\phi_1-\phi_m}}.$$

Figures 3(a)–3(c) show the peakon profiles given by (17)–(19), respectively.

From the above discussion, we have the following conclusion.

Theorem 1. *Equation (9) has three different exact explicit peakon solutions given by (17)–(19), respectively. The corresponding peakon profiles are shown in Figs. 3(a)–3(c).*

3. Peakon Solutions of the Nonlinear Dispersion Equation $K(m,n)$

Corresponding to Eq. (10), it has the following traveling system (see [Rosenau, 1997]):

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{-n(n-1)\phi^{n-2}y^2 - a\phi^m + c\phi + g}{n\phi^{n-1}}, \tag*{$(20)_{(m,n)}$}$$

which has the first integral

$$H(\phi,y) = \phi^n\left(n\phi^{n-2}y^2 + \frac{2a}{m+n}\phi^m - \frac{2c}{n+1}\phi - \frac{2g}{n}\right) = h. \tag{21}$$

Letting $d\xi = n\phi^{n-1}d\zeta$, system (20) becomes the following system

$$\frac{d\phi}{d\zeta} = ny\phi^{n-1}, \qquad \frac{dy}{d\zeta} = -n(n-1)\phi^{n-2}y^2 - a\phi^m + c\phi + g. \tag{22}$$

On the (ϕ,y)-phase plane, the abscissas of equilibrium points of system (22) on the ϕ-axis are the zeros of $E(\phi) = a\phi^m - c\phi - g$. When $n=2$, there are two equilibrium points of (22) at $Y_-(0,-\sqrt{0.5g})$ and $Y_+(0,\sqrt{0.5g})$ on y-axis if $g>0$. When $n>2$, system (22) has no equilibrium on the y-axis if $g \neq 0$. Noting that $E'(\phi) = am\phi^{m-1} - c$, for an odd m and $ac > 0$, $E'(\phi)$ has two zeros at $\tilde{\phi}_\pm = \pm(\frac{c}{am})^{\frac{1}{m-1}}$; for an even m, $E'(\phi)$ has only one zero at $\tilde{\phi}_+$. Clearly, $E(\tilde{\phi}_+) = -(\frac{m-1}{m}c\tilde{\phi}_+ + g)$. By using this information, we know the distributions of the zeros of $E(\phi)$ on the ϕ-axis. Let(ϕ_e, y_e) be an equilibrium of system (22). At this point, the determinant of the linearized system of system (22) has the form

$$J(\phi_e, y_e) = -n^3(n-1)\phi_e^{2(n-2)}y_e^2 + n\phi_e^{n-1}E'(\phi_e).$$

It is clear that for $n=2$, two equilibrium points on the y-axis are saddle points. As to the equilibrium $(\phi_e, 0)$ on the x-axis, it is a center (or a saddle point), if $\phi_e^{n-1}E'(\phi_e) > 0$(or <0). When $E(\phi)$ has two zeros on the ϕ-axis, we denote them as $\phi_{ej},\ j=1,2,\ \phi_{e1} < \phi_{e2}$. Write that

$$h_1 = H(\phi_{e1}, 0), \quad h_2 = H(\phi_{e2}, 0),$$

$$h_s = H(0, \pm\sqrt{0.5g}) = 0,$$

where H is defined by (21).

By using the above facts to do qualitative analysis, we obtain the following results.

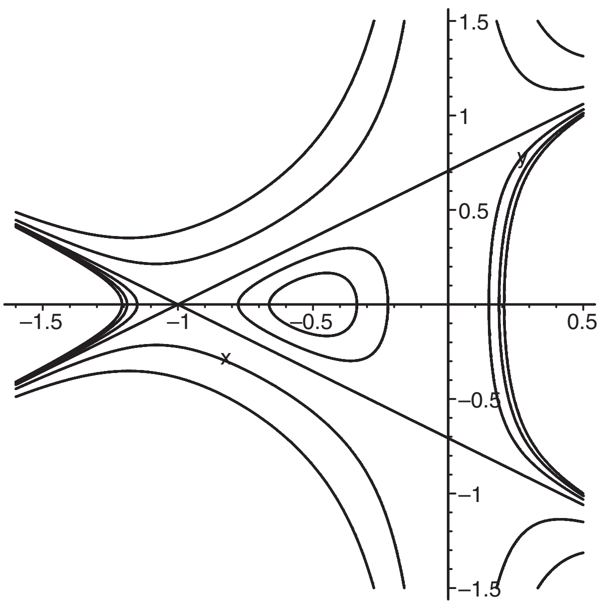
(1) For equation $K(2,2k)$, when $a<0$, $g>0$, $c = 6k(\frac{|a|}{2(k+1)})^{\frac{1}{2k}}(\frac{g}{2(2k-1)})^{\frac{2k-1}{2k}}$, there exist a heteroclinic loop of system (22). Taking $k=1,2$, we have the two phase portraits of system (22) shown in Figs. 4(a) and 4(b).

(2) For equation $K(2,2k+1)$, when $a>0$, $c>0$, $g = \frac{4k}{2k+3}(a)^{-\frac{1}{2k}}(\frac{(2k+3)c}{3(2k+1)})^{\frac{2k+1}{2k}}$, there exist a heteroclinic loop of system (22). Taking $k=1$, we have the phase portraits of system (22) shown in Fig. 4(c).

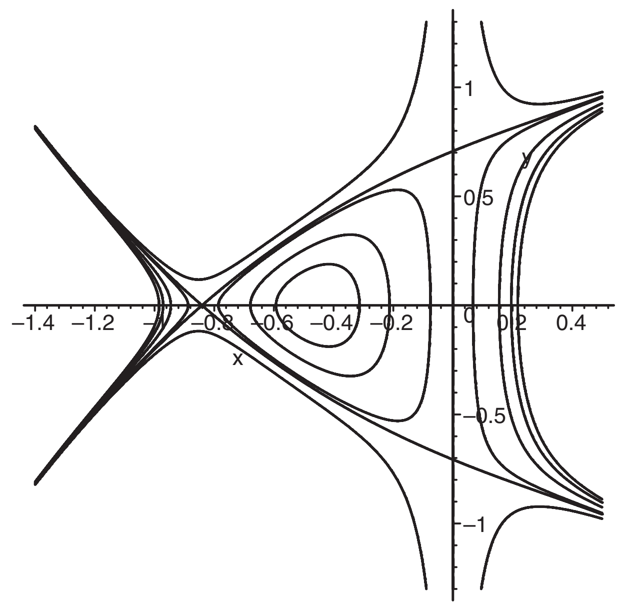
We next consider the exact peakon solutions.

(i) $K(2,2)$ peakon.

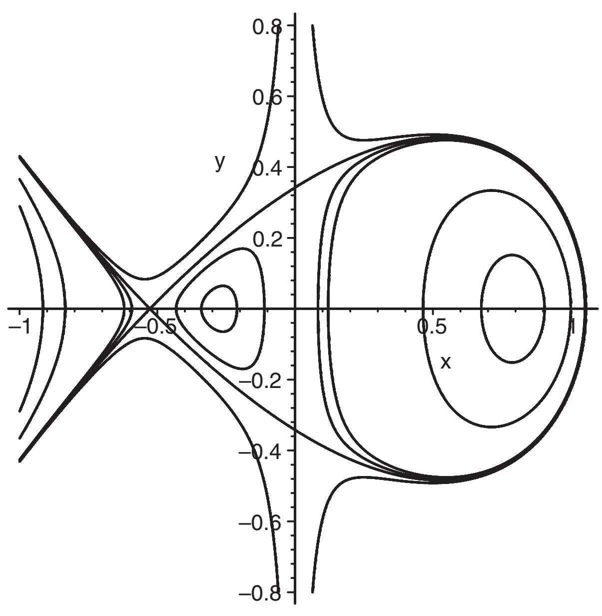
For $m=n=2$, when $a<0$, $g>0$ and $c = \frac{3}{2}\sqrt{2|a|g}$, we have the phase portrait Fig. 4(a). By (21) with $h=0$, we know that the upper and lower straight lines of the boundary triangle of the periodic annulus with center $C(\frac{c}{3a}, 0)$ are $y = \pm\frac{\sqrt{|a|}}{2}(\phi - \frac{2c}{3a})$. By using the first equation of



(a) $K(2,2)$

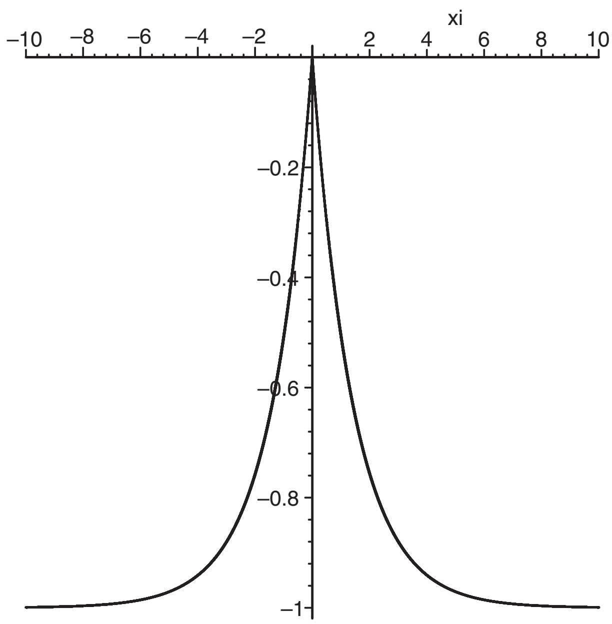


(b) $K(4,2)$

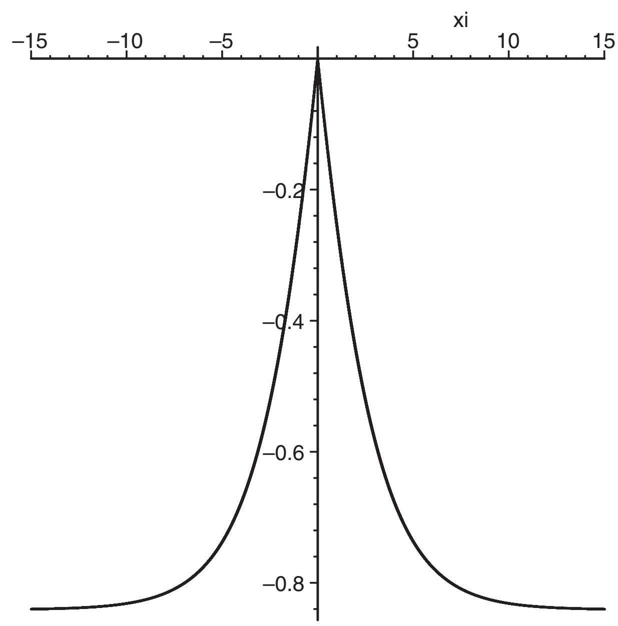


(c) $K(3,2)$

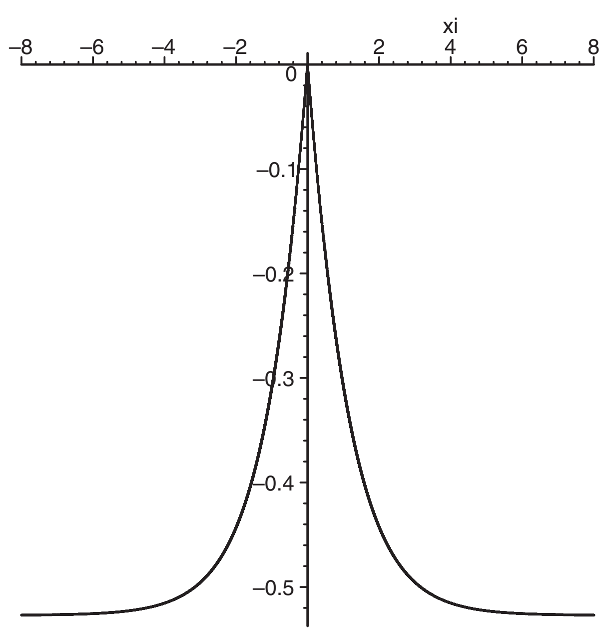
Fig. 4. Three phase portraits of system (22).



(a) $K(2,2)$ peakon



(b) $K(4,2)$ peakon



(c) $K(3,2)$ peakon

Fig. 5. Three peakon profiles of Eq. (10).

system (20), we obtain following parametric representations:

$$\phi(\xi) = \frac{2c}{3a}\left(1 - \exp\left(-\frac{\sqrt{|a|}}{2}\xi\right)\right), \quad \text{for } \xi \in (0, \infty),$$
$$\phi(\xi) = \frac{2c}{3a}\left(1 - \exp\left(\frac{\sqrt{|a|}}{2}\xi\right)\right), \quad \text{for } \xi \in (-\infty, 0). \tag{23}$$

(ii) $K(4,2)$ peakon.

When $m = 4$, $n = 2$, $a < 0$, $g > 0$ and $c = 2|a|^{\frac{1}{4}}g^{\frac{3}{4}}$, system $(20)_{(4,2)}$ has two connecting orbits to saddle point $S(\phi_1, 0)$, where $\phi_1 = -(\frac{g}{|a|})^{\frac{1}{4}}$ in Fig. 4(b). By (21) with $h = 0$, the upper and lower boundary curves of the period annulus of center C are $y = \pm\sqrt{\frac{|a|}{6}}(\phi - \phi_1)\sqrt{\phi^2 + 2\phi_1\phi + 3\phi_1^2}$. Thus, by using the first equation of system (20), we obtain the following parametric representations:

$$\phi(\xi) = \phi_1 + \frac{12(3\sqrt{2}+4)|\phi_1|}{(3\sqrt{2}+4)^2 e^{-\sqrt{\frac{|a|}{6}}|\phi_1|\xi} - 2e^{\sqrt{\frac{|a|}{6}}|\phi_1|\xi} + 4(3\sqrt{2}+4)}, \quad \text{for } \xi \in (-\infty, 0),$$
$$\phi(\xi) = \phi_1 + \frac{12(3\sqrt{2}+4)|\phi_1|}{(3\sqrt{2}+4)^2 e^{\sqrt{\frac{|a|}{6}}|\phi_1|\xi} - 2e^{-\sqrt{\frac{|a|}{6}}|\phi_1|\xi} + 4(3\sqrt{2}+4)}, \quad \text{for } \xi \in (0, \infty). \tag{24}$$

(iii) $K(3,2)$ peakon.

When $m = 3$, $n = 2$, $a > 0$, $g > 0$, $c = \frac{4}{5}a^{-\frac{1}{2}} \times (\frac{5}{9}c)^{\frac{3}{2}}$, system $(20)_{(3,2)}$ has three equilibrium points $S_1(\phi_1, 0)$, $C_1(\phi_2, 0)$ and $C_2(\phi_3, 0)$ in Fig. 4(c), where $\phi_1 = -\frac{1}{3}\sqrt{\frac{5c}{a}}$, $\phi_2 = -\frac{1}{6}(\sqrt{21} - \sqrt{5})\sqrt{\frac{c}{a}}$, $\phi_3 = \frac{1}{6}(\sqrt{21} + \sqrt{5})\sqrt{\frac{c}{a}}$. By (21) with $h = 0$, the upper and lower boundary curves of the period annulus of center C_1 are $y = \pm\sqrt{\frac{a}{5}}(\phi - \phi_1)(\phi_M - \phi)$, where $\phi_M = \frac{2}{3}\sqrt{\frac{5c}{a}} = 2|\phi_1|$. By using the first equation of system (20), we obtain the following parametric representations:

$$\phi(\xi) = \phi_M - (\phi_M - \phi_1)\tanh^2\left(\frac{1}{2}\sqrt{c}\xi - \Omega_1\right), \quad \text{for } \xi \in (-\infty, 0),$$
$$\phi(\xi) = \phi_M - (\phi_M - \phi_1)\tanh^2\left(\frac{1}{2}\sqrt{c}\xi + \Omega_1\right), \quad \text{for } \xi \in (0, \infty), \tag{25}$$

where $\Omega_1 = \tanh^{-1}\sqrt{\frac{2}{3}}$.

We use Figs. 5(a)–5(c) to show the peakon profiles given by (23)–(25), respectively. Hence, we have

Theorem 2. *Corresponding to $K(2,2), K(4,2)$ and $K(3,2)$, Eq. (10) has three different exact explicit peakon solutions given by (23)–(25), respectively. The profiles of peakon solutions are shown in Figs. 5(a)–5(c), respectively.*

4. Peakon Solutions of the Two-Component Hunter–Saxton System (11)

Let $u(x,t) = \phi(x - ct) = \phi(\xi)$, $\rho(x,t) = v(x - ct) = v(\xi)$, where c is the wave speed. Then, the second equation of (11) becomes $-cv' + (v\phi)' = 0$, where "$'$" stands for the derivative with respect to ξ. Integrating this equation once and setting the integration constant as B, $B \neq 0$, it follows that $v(\xi) = \frac{B}{\phi - c}$. The first equation of (11) reads as

$$-c\phi''' + A\phi' + \sigma\left[\frac{1}{2}(\phi')^2 + \phi\phi''\right]' - vv' = 0.$$

Integrating this equation yields

$$(\sigma\phi - c)\phi'' = -\frac{1}{2}\sigma(\phi')^2 - A\phi + \frac{B^2}{2(\phi - c)^2} - \frac{1}{2}g,$$

where $\frac{1}{2}g$ is an integration constant. This equation is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi} = y,$$
$$\frac{dy}{d\xi} = \frac{-\sigma y^2(\phi-c)^2 - (\phi-c)^2(2A\phi+g)+B^2}{2(\phi-c)^2(\sigma\phi-c)}, \tag{26}$$

which has the following first integral:

$$H(\phi,y) = y^2(\sigma\phi-c) + A\phi^2 + g\phi + \frac{B^2}{(\phi-c)} = h. \tag{27}$$

Assume that $A>0$, $c>0$. Imposing the transformation $d\xi = (\phi-c)^2(\sigma\phi-c)d\zeta$ for $\phi \neq c$, $\frac{c}{\sigma}$ on system (26) leads to the following associated regular system:

$$\frac{d\phi}{d\zeta} = y(\phi-c)^2(\sigma\phi-c),$$
$$\frac{dy}{d\zeta} = -\frac{1}{2}\sigma y^2(\phi-c)^2 - \frac{1}{2}[(\phi-c)^2(2A\phi+g)-B^2]. \tag{28}$$

This system has the same first integral as (26). Apparently, two singular lines $\phi = c$ and $\phi = \frac{c}{\sigma}$ are two invariant straight line solutions of (28). To see the equilibrium points of (28), we write that

$$\begin{aligned} f(\phi) &= (\phi-c)^2(2A\phi+g) - B^2 \\ &= 2A\left(\phi^3 + \frac{g-4Ac}{2A}\phi^2 + \frac{2c^2A-2cg}{2A}\phi + \frac{c^2g-B^2}{2A}\right) \\ &\equiv 2A(\phi^3+a_2\phi^2+a_1\phi+a_0), \\ f'(\phi) &= 2(\phi-c)(3A\phi+g-Ac), \\ f''(\phi) &= 2(6A\phi+g-4Ac). \end{aligned}$$

Clearly, $f'(\phi)$ has two zeros at $\phi = \phi_{s1} = c$ and $\phi = \tilde{\phi} = \frac{Ac-g}{3A}$. In addition, we have $f(c) = -B^2$, $f'(c) = 0$ and $f''(c) = 2(2cA+g)$, $f(0) = gc^2 - B^2$.

Let $q = \frac{1}{3}a_1 - \frac{1}{9}a_2^2$, $r = \frac{1}{6}(a_1a_2 - 3a_0) - \frac{1}{27}a_2^3$. Then, the discriminant $S = q^3 + r^2$ of the cubic polynomial $f(\phi) = 0$ is $S = -\frac{B^2}{432A^4}\ S_1 = -\frac{B^2}{432A^4}(8A^3c^3 + g^3 + 12A^2c^2g + 6Acg^2 - 27A^2B^2)$.

It is easy to see that for given A, B^2, c, when $g > g_1 \equiv 3(A^2B^2)^{\frac{1}{3}} - 2Ac$, we have $S_1 > 0$. It follows that there exist three simple real roots ϕ_j $(j = 1, 2, 3)$ of $f(\phi)$ satisfying $\phi_1 < \tilde{\phi} < \phi_2 < c < \phi_3$. When $g = g_1$, there exist two real roots ϕ_{12} and ϕ_3 of $f(\phi)$ satisfying $\phi_{12} = \tilde{\phi} = c - A^{-\frac{1}{3}}B^{\frac{2}{3}} < c < \phi_3$.

In the ϕ-axis, the equilibrium points $E_j(\phi_j, 0)$ of (6) satisfy $f(\phi_j) = 0$. Obviously, system (28) has at most three equilibrium points at $E_j(\phi_j,0)$, $j = 1,2,3$. On the straight line $\phi = c$, there is no equilibrium point of (28) if $B \neq 0$. On the straight line $\phi = \frac{c}{\sigma}$, there exist two equilibrium points $S_{\mp}(\frac{c}{\sigma}, \mp Y_s)$ of (28) with $Y_s = \sqrt{\frac{-f(\frac{c}{\sigma})}{\sigma(\frac{c}{\sigma}-c)^2}}$, if $\sigma f(\frac{c}{\sigma}) < 0$.

Let $M(\phi_j, y_j)$ be the coefficient matrix of the linearized system of (28) at an equilibrium point $E_j(\phi_j, y_j)$. We have

$$\begin{aligned} J(\phi_j, 0) &= \det M(\phi_j, 0) \\ &= 2(\phi_j - c)^2(\sigma\phi_j - c)f'(\phi_j), \\ J\left(\frac{c}{\sigma}, \mp Y_s\right) &= \det M\left(\frac{c}{\sigma}, \mp Y_s\right) \\ &= -\sigma^2 Y_s^2\left(\frac{c}{\sigma} - c\right)^4. \end{aligned}$$

The sign of $f'(\phi_j)$ and the relative positions of the equilibrium points $E_j(\phi_j,0)$ of (28) with respect to two singular lines $\phi = c$ and $\phi = \frac{c}{\sigma}$ can determine the types (saddle points or centers) of the equilibrium points $E_j(\phi_j,0)$. When $\sigma \neq 0$, two equilibrium points $S_{\mp}(\frac{c}{\sigma}, \mp Y_s)$ are saddle points.

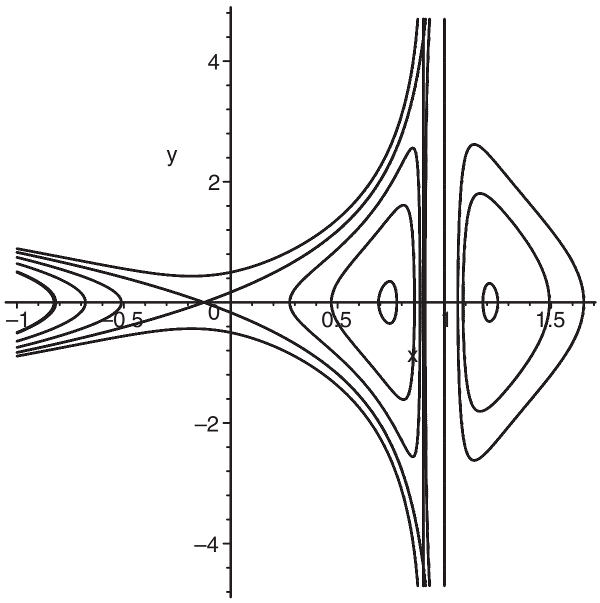
Let $h_i = H(\phi_i, 0)$ and $h_s = H(\frac{c}{\sigma}, \mp Y_s)$, where H is given by (27).

For a given wave speed $c > 0$ and parameters $A > 0$, $B^2 > 0$, we assume that the following condition holds: (H_1) $g > g_1 \equiv 3(A^2B^2)^{\frac{1}{3}} - 2Ac$.

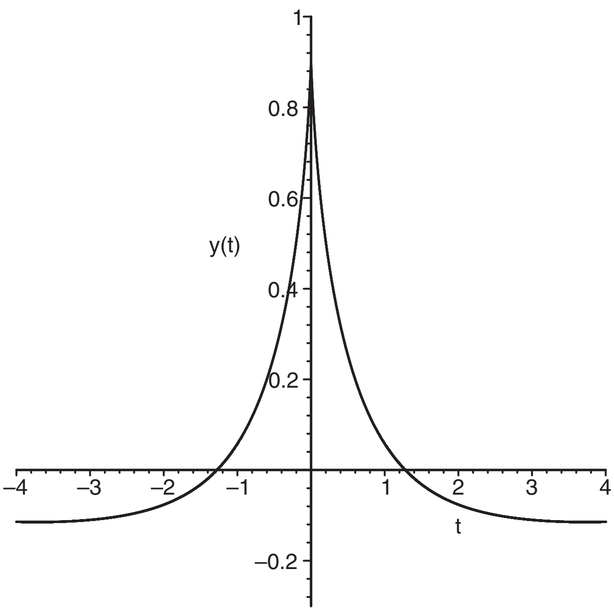
Under condition (H_1), system (6) has three simple equilibrium points $E_j(\phi_j, 0)$, $j = 1, 2, 3$ with $\phi_1 < \tilde{\phi} < \phi_2 < c < \phi_3$. Notice that for every $j = 1, 2, 3$, ϕ_j does not depend on the parameter σ.

It is easy to see that for a given positive parameter group of (A, B^2, c) and $g > 3(A^2B^2)^{\frac{1}{3}} - 2Ac$, under the parameter condition: $1 < \sigma = \sigma^* = \frac{Ac}{Ac-g-2A\phi_1}$, we have $h_s = h_1 < h_2 < h_3$. Thus, we obtain the phase portrait of (28) as shown in Fig. 6(a).

Corresponding to the heteroclinic orbit loop of (6) connecting three saddle points $E_1(\phi_1, 0), S_{\mp}$ and enclosing the center $E_2(\phi_2, 0)$ in Fig. 6(a), the first integral $H(\phi, y) = h_s = h_1$ can be written in



(a)



(b)

Fig. 6. A peakon solution defined by formula (29). (a) The phase portrait of system (28) and (b) a peakon solution of Eq. (11).

the form

$$y^2 = \frac{1}{\sigma(c-\phi)} \times \left(\frac{B^2}{\left(c-\frac{c}{\sigma}\right)} - (c-\phi)\left(A\phi + \frac{Ac}{\sigma} + g\right)\right) = \frac{A}{\sigma(c-\phi)}(\phi-\phi_1)^2.$$

Hence, by using the first equation of system (26) to integrate, along the heteroclinic orbits E_1S_+ and E_1S_-, we have

$$\int_\phi^{\frac{c}{\sigma}} \frac{(c-\phi)d\phi}{(\phi-\phi_1)\sqrt{c-\phi}} = \pm\sqrt{\frac{A}{\sigma}}\xi.$$

Thus, we obtain

$$\begin{aligned} \phi(\chi) &= c - (c-\phi_1)\tanh^2(\chi), \\ &\qquad \chi \in (-\infty, -\chi_0) \cup (\chi_0, \infty) \\ \xi(\chi) &= -\sqrt{\frac{\sigma}{A}}[\sqrt{c-\phi_1}(\chi - \tanh(\chi)) - \xi_0], \end{aligned} \tag{29}$$

where $\chi_0 = \operatorname{arctanh}\sqrt{\frac{c-\frac{c}{\sigma}}{c-\phi_1}}$, $\xi_0 = 2\sqrt{c-\phi_1}\chi_0 - 2\sqrt{c-\frac{c}{\sigma}}$. Equation (29) gives rise to peakon solution of Eq. (11). The wave profile is shown in Fig. 6(b).

To sum up, we have

Theorem 3. *For a given positive parameter group* (A, B^2, c), *when* $g > g_1 \equiv 3(A^2B^2)^{\frac{1}{3}} - 2Ac$, *system* (*28*) *has three real equilibrium points* $E_j(\phi_j, 0)$, $j = 1, 2, 3$ *satisfying* $\phi_1 < \tilde{\phi} < \phi_2 < c < \phi_3$. *When* $\sigma = \sigma^*$, *corresponding to the heteroclinic loop of system* (*28*), *Eqs.* (*11*) *has a peakon solution given by* (*29*).

5. Peakon Solutions of the Two-Component Camassa–Holm System (12)

Let $u(x,t) = \phi(x-ct) = \phi(\xi)$, $\rho(x,t) = v(x-ct) = v(\xi)$, where c is the wave speed. Then, the second equation of (12) becomes $-cv' + (v\phi)' = 0$, where "$\prime$" stands for the derivative with respect to ξ. Integrating this equation once and setting the integration constant as B, $B \neq 0$, it follows that $v(\xi) = \frac{B}{\phi - c}$. The first equation of (12) reads as

$$\begin{aligned} -c\phi''' &= -(A+c)\phi' + 3\phi\phi' \\ &\quad - \sigma\left[\frac{1}{2}(\phi')^2 + \phi\phi''\right]' + e_0vv'. \end{aligned}$$

Integrating this equation yields

$$(\sigma\phi - c)\phi'' = -\frac{1}{2}\sigma(\phi')^2 - (A+c)\phi + \frac{3}{2}\phi^2 + \frac{e_0 B^2}{2(\phi-c)^2} - \frac{1}{2}g,$$

where g is an integration constant. The above equation is equivalent to the following two-dimensional system:

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-\sigma y^2(\phi-c)^2 + (\phi-c)^2[3\phi^2 - 2(A+c)\phi - g] + e_0 B^2}{2(\phi-c)^2(\sigma\phi - c)}, \tag{30}$$

which admits the following first integral:

$$H(\phi,y) = y^2(\sigma\phi - c) - \phi^3 + (A+c)\phi^2 + g\phi + \frac{e_0 B^2}{(\phi-c)} = h. \tag{31}$$

For a given wave speed $c > 0$, system (31) is a four-parameter planar dynamical system with the parameter tuple (A, B, g, σ).

Assume $A > 0$. Imposing the transformation $d\xi = (\phi-c)^2(\sigma\phi - c)d\zeta$ for $\phi \neq c$, $\frac{c}{\sigma}$ on system (30) with $e_0 = \pm 1$, leads to the following regular system:

$$\frac{d\phi}{d\zeta} = y(\phi-c)^2(\sigma\phi - c),$$
$$\frac{dy}{d\zeta} = -\frac{1}{2}\sigma y^2(\phi-c)^2 + \frac{1}{2}[(\phi-c)^2 \times (3\phi^2 - 2(A+c)\phi - g) + e_0 B^2]. \tag{32}$$

Apparently, two singular lines $\phi = c$ and $\phi = \frac{c}{\sigma}$ are two invariant straight line solutions of (32).

To see the equilibrium points of (32), let us mark and calculate the following

$$f(\phi) = (\phi-c)^2(3\phi^2 - 2(A+c)\phi - g) + e_0 B^2,$$
$$f'(\phi) = 2(\phi-c)[6\phi^2 - 3(A+2c)\phi + c(A+c) - g],$$
$$f''(\phi) = 2(18\phi^2 - 6(A+4c)\phi + c(4A+7c) - 2g.$$

Apparently, $f'(\phi)$ has one zero at $\phi = \phi_{s1} = c$. When $\Delta = 9A^2 + 12Ac + 12c^2 + 24g > 0$, $f'(\phi)$ has two zeros at $\phi = \tilde{\phi}_{1,2} = \frac{1}{12}[3(A+2c) \mp \sqrt{\Delta}]$. So, we have $f(c) = e_0 B^2$, $f'(c) = 0$ and $f''(c) = 2(c^2 - 2cA - g)$, $f(0) = e_0 B^2 - gc^2$.

In the ϕ-axis, the equilibrium points $E_j(\phi_j, 0)$ of (32) satisfy $f(\phi_j) = 0$. Geometrically, for a fixed $c > 0$, the real zeros ϕ_j $(j = 1,2$ or $j = 1,2,3,4)$ of the function $f(\phi)$ can be determined by the intersection points of the quadratic curve $y = 3\phi^2 - 2(A+c)\phi - g$ and the hyperbola $y = -\frac{e_0 B^2}{(\phi-c)^2}$. Obviously, system (32) has at most four equilibrium points at $E_j(\phi_j, 0)$, $j = 1,2,3,4$. On the straight line $\phi = c$, there is no equilibrium point of (32) if $B \neq 0$. On the straight line $\phi = \frac{c}{\sigma}$, there exist two equilibrium points $S_{\mp}(\frac{c}{\sigma}, \mp Y_s)$ of (32) with $Y_s = \sqrt{\frac{f(\frac{c}{\sigma})}{\sigma(\frac{c}{\sigma}-c)^2}}$, if $\sigma f(\frac{c}{\sigma}) > 0$.

Next we assume that $e_0 = 1$. Let $h_i = H(\phi_i, 0)$ and $h_s = H(\frac{c}{\sigma}, \mp Y_s)$, where H is given by (31).

For a given wave speed $c > 0$, assume that one of the following two conditions holds:

(1) $g > 0$, $c < A + \sqrt{A^2 + g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.
(2) $g < 0$, $A^2 + 4g > 0$, $A - \sqrt{A^2+g} < c < A + \sqrt{A^2+g}$. For given A and g, $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$.

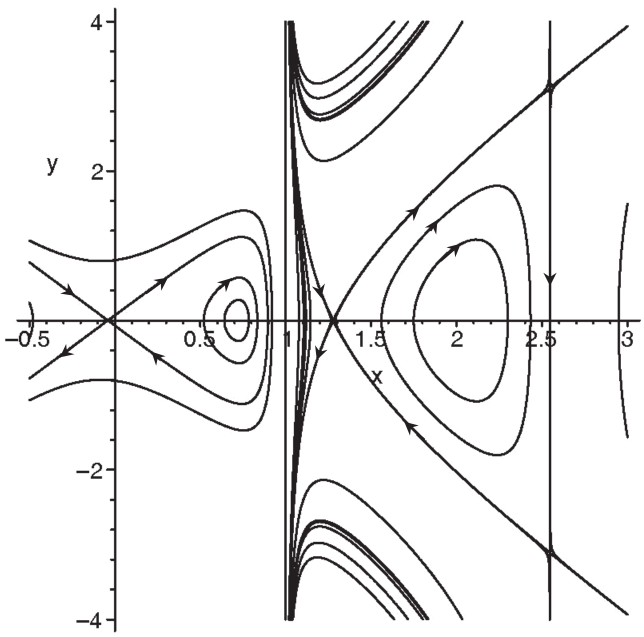
Then, Eq. (32) has four simple equilibrium points $E_j(\phi_j, 0)$, $j = 1,2,3,4$, satisfying $\phi_1 < \tilde{\phi}_1 < \phi_2 < c < \phi_3 < \tilde{\phi}_2 < \phi_4$.

Suppose that $\sigma < 1$. Under the conditions $h_1 < h_2 < h_s = h_3 < h_4$, $\phi_4 < \frac{c}{\sigma}$, we have the following phase portrait of Eq. (32) shown in Fig. 7(a).

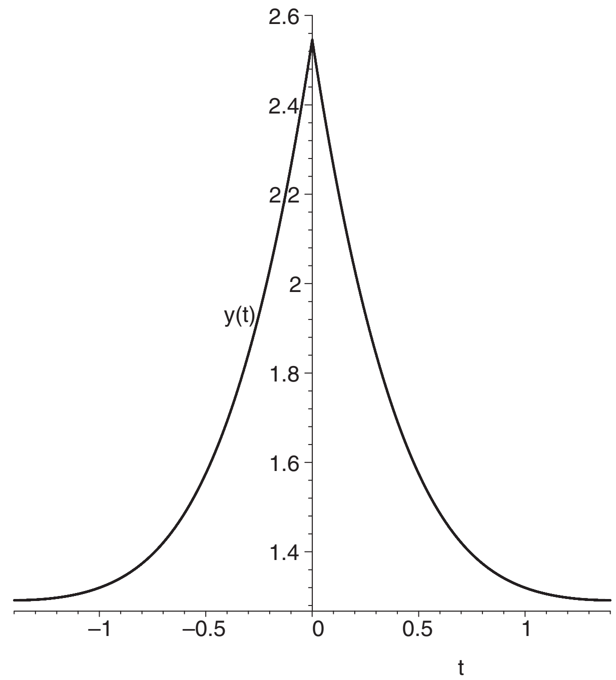
We now investigate exact parametric representations of the two heteroclinic orbits of (32) defined through $H(\phi, y) = h_3 = h_s$ in Fig. 7(a). By (31), we know that for a fixed integral constant h,

$$y^2 = \frac{(\phi-c)[\phi^3 - (A+2c)\phi^2 - g\phi + h] - eB^2}{(\phi-c)(\sigma\phi - c)}$$
$$\equiv \frac{G(\phi)}{(\phi-c)(\sigma\phi-c)} = \frac{\phi^4 - (A+2c)\phi^3 + (c^2 + Ac - g)\phi^2 + (h+cg)\phi - (ch + eB^2)}{(\phi-c)(\sigma\phi-c)}.$$

In the case of Fig. 7(a), function $G(\phi)$ can be written as $G(\phi) = (\frac{c}{\sigma} - \phi)(\phi - \phi_3)^2(\phi - \phi_l)$.



(a)



(b)

Fig. 7. A peakon solution defined by formulas (27). (a) The phase portrait of system (32) and (b) a peakon solution of Eq. (12).

Hence, taking integrals along the heteroclinic orbits E_3S_+ and E_3S_-, choosing initial value $\phi(0) = \frac{c}{\sigma}$, by Theorem B, we arrive at

$$\pm\frac{\xi}{\sqrt{\sigma}} = \int_{\frac{c}{\sigma}}^{\phi} \frac{d\phi}{\sqrt{(\phi - c)(\phi - \phi_l)}} + (\phi_3 - c)\int_{\frac{c}{\sigma}}^{\phi} \frac{d\phi}{(\phi - \phi_3)\sqrt{(\phi - c)(\phi - \phi_l)}}. \tag{33}$$

Thus, we obtain a new peakon solution of (12) as follows:

$$\begin{aligned} \phi(\chi) &= \frac{B_0}{2}\left[e^{\chi} + \left(\frac{c - \phi_l}{2B_0}\right)^2 e^{-\chi} + \frac{c + \phi_l}{B_0}\right], \quad \chi \in (-\infty, 0] \\ \xi(\chi) &= \sqrt{\sigma}\left[\chi - \sqrt{\frac{\phi_3 - c}{\phi_3 - \phi_l}}\ln\left(\frac{\sqrt{X(\phi(\chi) - \phi_3)} + \sqrt{X(\phi_3)}}{\phi(\chi) - \phi_3} + \frac{2\phi_3 - c - \phi_l}{2\sqrt{X(\phi_3)}}\right) + B_1\right] \end{aligned} \tag{34}$$

and

$$\begin{aligned} \phi(\chi) &= \frac{B_0}{2}\left[e^{-\chi} + \left(\frac{c - \phi_l}{2B_0}\right)^2 e^{\chi} + \frac{c + \phi_l}{B_0}\right], \quad \chi \in [0, \infty), \\ \xi(\chi) &= \sqrt{\sigma}\left[\chi + \sqrt{\frac{\phi_3 - c}{\phi_3 - \phi_l}}\ln\left(\frac{\sqrt{X(\phi(\chi) - \phi_3)} + \sqrt{X(\phi_3)}}{\phi(\chi) - \phi_3} + \frac{2\phi_3 - c - \phi_l}{2\sqrt{X(\phi_3)}}\right) - B_1\right], \end{aligned} \tag{35}$$

where

$$X(\phi) = (\phi - c)(\phi - \phi_l), \quad B_0 = \sqrt{X\left(\frac{c}{\sigma}\right)} + \frac{c}{\sigma} - \frac{1}{2}(c + \phi_l),$$

$$B_1 = \sqrt{\frac{\phi_3 - c}{\phi_3 - \phi_l}} \ln\left(\frac{\sqrt{X\left(\frac{c}{\sigma} - \phi_3\right)} + \sqrt{X(\phi_3)}}{\frac{c}{\sigma} - \phi_3} + \frac{2\phi_3 - c - \phi_l}{2\sqrt{X(\phi_3)}}\right).$$

In a summary, we obtain the following result.

Theorem 4. *Suppose that the traveling wave system (30) of Eqs. (12) satisfies the parameter condition* $\sigma < 0$, $g > 0$, $c < A + \sqrt{A^2 + g}$ *and for given A and g,* $f(\tilde{\phi}_1) < 0$, $f(\tilde{\phi}_2) < 0$. *Then, when* $h_1 < h_2 < h_s = h_3 < h_4$, $\phi_4 < \frac{c}{\sigma}$, *corresponding to the heteroclinic loop of system (32) defined by* $H(\phi, y) = h_s$ *in (31), formulas (34) and (35) give rise to* a peakon solution *of Eqs. (12).*

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