Well-behaved At Infinity First Integrals of Polynomial Vector Fields

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Contents

The context

Basic tools

A planar polynomial differential system X of degree d in $\mathbb{C}^2$:

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y). \tag{1}$$

A first integral is H such that

$$XH = p\frac{\partial H}{\partial x} + q\frac{\partial H}{\partial y} = 0.$$

An invariant algebraic curve if $f = 0$ such that

$$Xf = p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = kf.$$

k is the cofactor of $f = 0$.

One place at infinity

Definitions

- Let $L : \{Z = 0\}$ be the infinity line.
- Let $C : \{F(X, Y, Z) = 0\}$. C has only one place at infinity if $C \cap L = \{P\}$ and C is reduced and unibranch at P.
- $H = \prod_{i=1}^{r} f_i^{n_i}$ is a well-behaved at infinity (WAI) function if $F_i = Z^{d_i} f_i(X/Z, Y/Z)$ has only one place at infinity.
- We define

$$\bar{H}(X, Y, Z) = \frac{\prod_{i=1}^{r} F_i^{n_i}}{Z^n},$$

where $n = \sum_{i=1}^{r} d_i n_i$.

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The question(s)

The Catalan way of asking about things

1. Does X has a WAI polynomial first integral? (Y/N)
2. In the affirmative case, can we compute a minimal WAI polynomial first integral? (Y/N)

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NOTE: Even if we answer YES to both questions, nothing seems to happen.

Contents

Vector fields and invariant algebraic curves

The 1-form $\Omega = AdX + BdY + CdZ$ of degree $d+1$ is projective if $XA + YB + ZC = 0$.

Let P, Q, and R of degree d such that

$$A = ZQ - YR, \quad B = XR - ZP, \quad C = YP - XQ.$$

(P, Q, R) can be thought of as a homogeneous polynomial vector field in $\mathbb{CP}^2$ of degree d:

$$\mathcal{X} = P\frac{\partial}{\partial X} + Q\frac{\partial}{\partial Y} + R\frac{\partial}{\partial Z},$$

$F(X, Y, Z) = 0$ is invariant for $\mathcal{X}$ if

$$\mathcal{X}F = P\frac{\partial F}{\partial X} + Q\frac{\partial F}{\partial Y} + R\frac{\partial F}{\partial Z} = KF.$$

Vector fields and invariant algebraic curves

- The 1-form $p(x,y)dy - q(x,y)dx$ can be extended to $\mathbb{CP}^2$:

$$Z^d\left(p(YdZ - ZdY) - q(XdZ - ZdX)\right).$$

- $\begin{cases} f(x,y)=0 \\ n=\deg f \in \mathbb{N} \\ k(x,y) \end{cases} \Rightarrow \begin{cases} F = Z^n f(X/Z, Y/Z) = 0 \\ K = Z^{d-1} k(X/Z, Y/Z) \end{cases}$

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- $(P(x,y,1) - xR(x,y,1))\,dy - (Q(x,y,1) - yR(x,y,1))\,dx.$

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The blow-up technique

Blowing-up a singular point P

$$V_1^P: \ \dot{x} = p(x, xz), \quad \dot{z} = \frac{q(x, xz) - zp(x, xz)}{x};$$

$$V_2^P: \ \dot{z} = \frac{p(yz, y) - zq(yz, y)}{y}, \quad \dot{y} = q(yz, y).$$

The exceptional divisor $E_P : \{x = 0\}$ (resp. $\{y = 0\}$).

The projection map

$$\begin{aligned} \Pi_P : BL_P(M) &\to M \\ (x, z) &\mapsto (x, xz) \end{aligned}$$

from which $E_P = \Pi_P^{-1}(P)$.

Reduction of singularities

- From $\omega = p\,dy - q\,dx$ we have $\omega_m = p_m\,dy - q_m\,dx$.
- Let $\Pi_O : Bl_O(\mathbb{C}^2) \to \mathbb{C}^2$ and charts (V_i^O, ϕ_i).
- The total transform by Π_O of w in V_1^O is

$$\omega^*|_{V_1^O} := x^m\big[(\alpha(1,z) + x\beta(x,z))dx + x(p_m(1,z) + x\gamma(x,z))dz\big],$$

  where $\alpha(x,y) = yp_m(x,y) - xq_m(x,y)$.
- The strict transform by Π_O of w in V_1^O is $\tilde{\omega}|_{V_1^O} := \omega^*|_{V_1^O}/x^{m+1}$ if $\alpha \equiv 0$ (resp. $= \omega^*|_{V_1^O}/x^m$ if $\alpha \not\equiv 0$).
- From $\tilde{\omega}|_{V_i^O}$ we construct a 1-form $\tilde{\omega}$ on $Bl_O(\mathbb{C}^2)$.
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Reduction of singularities

Types of singular points

- O is a dicritical singular point if $\alpha \equiv 0$.
- O is non-dicritical if and only if E_O is invariant.
- O is simple if $m = 1$ and $\begin{pmatrix} p_{1x} & p_{1y} \\ q_{1x} & q_{1y} \end{pmatrix}$ has EV λ_1, λ_2 s.t. $\lambda_1 = 0 \neq \lambda_2$ or $\frac{\lambda_1}{\lambda_2} \notin \mathbb{Q}^+$.
- O is ordinary if it is not simple (includes dicritical).

Some observations

- Simple singular points cannot be reduced.
- Ordinary singular points can be reduced (by a finite sequence of BU) s.t. the strict transform $\mathcal{X}$ in the last obtained complex manifold has no ordinary singularities.

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Reduction of singularities

Neighbors

- E_P is the first infinitesimal neighborhood of P.
- The i-th infinitesimal neighborhood of P is formed by the points on the first infinitesimal neighborhood of some point in the $(i-1)$-th infinitesimal neighborhood of P.
  They are infinitely near to P.
- Q is proximate to P if it belongs to the strict transform of E_P.
- Q is a satellite if it is proximate to two points. Otherwise it is free.
- R precedes Q if Q is infinitely near to R.

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- The i-th infinitesimal neighborhood of P is formed by the points on the first infinitesimal neighborhood of some point in the $(i-1)$-th infinitesimal neighborhood of P. They are infinitely near to P.
- Q is proximate to P if it belongs to the strict transform of E_P.
- Q is a satellite if it is proximate to two points. Otherwise it is free.
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Reduction of singularities

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Reduction of singularities

Configurations of infinitely near points

- A configuration is $C = \{Q_0, \ldots, Q_n\}$ such that $Q_0 \in X_0 = M, Q_i \in Bl_{Q_{i-1}}(X_{i-1}) =: X_i \to X_{i-1}$.
- We can construct the proximity graph Γ_C.
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An example!

Example

Let $\mathcal{X}$ be the vector field

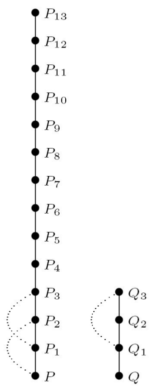
$$2XZ^4\,dX + 5Y^4Z\,dY - (5Y^5 + 2X^2Z^3)dZ,$$

with singularities $\begin{cases} P = (1:0:0), \\ Q = (0:0:1). \end{cases}$

We have

$\mathcal{S}(\mathcal{X}) = \{P, Q\} \cup \{P_i\}_{i=1}^{13} \cup \{Q_i\}_{i=1}^{3}$,

$\mathcal{D}(\mathcal{X}) = \{P\} \cup \{P_i\}_{i=1}^{13}$.



Contents

Linear systems

- A linear system on $\mathbb{CP}^2$ is the set of algebraic curves given by a linear subspace of $\mathbb{C}_m[X, Y, Z] \cup \{0\}$.

  If it has dimension 1, then it is a pencil.
- A cluster of $\mathbb{CP}^2$ is $(\mathcal{C}, \mathbf{m})$ where $\mathcal{C} = (Q_0, \ldots, Q_n)$ is a configuration and $\mathbf{m} = (m_0, \ldots, m_n)$, $m_i \in \mathbb{N}$.

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Linear systems constructed from clusters

Virtual transform

Set $\mathcal{K} = (\mathcal{C}, \mathbf{m})$ a cluster and $C : \{f = 0\}$ an algebraic curve.

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- Case $\ell(Q_k) = 1$: the virtual transform $C^{\mathcal{K}}_{Q_k}$ is $f(x, y) = 0$.
- C passes virtually through Q_k if $m_{Q_k}(C^{\mathcal{K}}_{Q_k}) \geq m_k$.
- Case $\ell(Q_k) > 1$: Q_k in the 1IN of $Q_j \in \mathcal{C}$ and C passes virtually through Q_j.
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Linear systems constructed from clusters

The strict transform $\tilde{C}$ of C is the global curve given by the virtual transform through the cluster of points and multiplicities defined by the curve.

The linear system $\mathcal{L}_m(\mathcal{K})$ determined by $m \in \mathbb{N}$ and $\mathcal{K}$ is the linear system on $\mathbb{CP}^2$ given by those curves defined by polynomials in $\mathbb{C}_m[X, Y, Z] \cup \{0\}$ that pass virtually through $\mathcal{K}$.

An example

Example

Consider the cluster $\mathcal{K} = (\mathcal{C}, \mathbf{m})$, where

- $\mathcal{C} = \{Q, P, P_1, P_2\}$, $\mathbf{m} = (2, 2, 1, 1)$;
- $P = (0:0:1)$, $Q = (1:0:1)$; or $(0,0)$, $(1,0)$ in $Z \neq 0$.
- $P_1 = (0,3) \in V_1^P$, $P_2 = (1,0) \in V_2^{P_1}$ infinitely near to P.

Let us compute $\mathcal{L}_3(\mathcal{K})$.

An example

Example

Let $C \in \mathcal{L}_3(\mathcal{K})$ be

$$aX^3 + bX^2Y + cX^2Z + dXY^2 + eXYZ + fXZ^2 + gY^3 + hY^2Z + iYZ^2 + kZ^3,$$

Consider it in the local chart $Z \neq 0$.

- The multiplicity of C at P must be at least 2, then $f = i = k = 0$.
- The multiplicity of C at Q must be at least 2, so $a = c = 0$ and $b = -e$.

An example

Example

The local equation defining the virtual transform of C at P_1, $C^{\mathcal{K}}_{P_1}$, is

$$3(e+3h)+(9d-3e+27g)x_1+(e+6h)y_1 \\ +(6d-e+27g)x_1y_1+hy_1^2+(d+9g)x_1y_1^2+gx_1y_1^3=0$$

in coordinates $(x_1 = x, y_1 = y/x)$.

- The multiplicity of $C^{\mathcal{K}}_{P_1}$ at P_1 must be at least 1, then $e = -3h$.

An example

Example

The local equation of the virtual transform of C at P_2 is

$$3h + (9d + 27g + 9h)x_2 + hy_2 \\ + (6d + 27g + 3h)x_2y_2 + (d + 9g)x_2y_2^2 + gx_2y_2^3 = 0,$$

where $x_2 = x_1/y_1$ and $y_2 = y_1$.

- $C_{P_2}^{\mathcal{K}}$ passes virtually through P_2 if and only if $h = 0$.

Hence the curves in $\mathcal{L}_3(\mathcal{K})$ are defined by $Y^2(\alpha X + \beta Y) = 0$, for $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

Cluster of base points

Let $\mathcal{BP}(\mathcal{L})$ be the configuration of points such that all the generic curves of $\mathcal{L}$ have the same multiplicities $\mathrm{mult}_Q(\mathcal{L})$ at every point $Q \in \mathcal{BP}(\mathcal{L})$ and empty intersection at the manifold obtained by blowing-up these points.

Let $\mathbf{m} = (\mathrm{mult}_Q(\mathcal{L}))_{Q \in \mathcal{BP}(\mathcal{L})}$.

We have the cluster of base points $(\mathcal{BP}(\mathcal{L}), \mathbf{m})$.

An example

Example

Back to

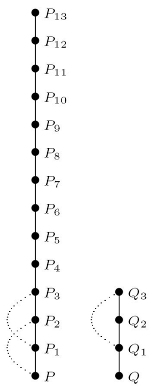
$$2XZ^4\,dX + 5Y^4Z\,dY - (5Y^5 + 2X^2Z^3)dZ,$$

consider $\mathcal{L}$ defined by

$$\alpha(X^2Z^3 + Y^5) + \beta Z^5 = 0.$$

The cluster of base points of $\mathcal{L}$ is

$$(\mathcal{D}(\mathcal{X}), (3, 2, 1, \ldots, 1)).$$



Cluster of base points

Proposition

If $\mathcal{L}$ is a pencil, then

$$\mathcal{BP}(\mathcal{L}) = \mathcal{D}(\mathcal{X}_{\mathcal{L}}),$$

where $\mathcal{X}_{\mathcal{L}}$ is the vector field with invariant curves given by $\mathcal{L}$.

- Let $\mathcal{P}_{\mathcal{X}} = \mathbb{P}\langle F_1^{n_1} \cdots F_r^{n_r}, Z^n \rangle$ ($\Leftrightarrow \bar{H}$).
- We have $\mathcal{P}_{\mathcal{X}} = \mathcal{L}_n(\mathcal{BP}_{\mathcal{X}})$.
- We can compute H from $\mathcal{BP}_{\mathcal{X}}$ and n.

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First main result

Theorem

Consider $\mathcal{X}$ having a WAI PFI $H = \prod_{i=1}^{r} f_i^{n_i}$.

- $\mathcal{D}(\mathcal{X}) = \mathcal{BP}(\mathcal{P}_{\mathcal{X}})$.
- $\mathcal{D}(\mathcal{X})$ has exactly r maximal points R_i. They are the unique dicritical singularities of $\mathcal{X}$.
- The set $\mathrm{Fr}(\mathcal{D}(\mathcal{X}))$ of free points of $\mathcal{D}(\mathcal{X})$ has exactly r maximal elements M_i. Moreover, R_i is infinitely near to M_i.
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Consider $\mathcal{X}$ having a WAI PFI $H = \prod_{i=1}^{r} f_i^{n_i}$.

- $\mathcal{D}(\mathcal{X}) = \mathcal{BP}(\mathcal{P}_{\mathcal{X}})$.
- $\mathcal{D}(\mathcal{X})$ has exactly r maximal points R_i. They are the unique dicritical singularities of $\mathcal{X}$.
- The set $\mathrm{Fr}(\mathcal{D}(\mathcal{X}))$ of free points of $\mathcal{D}(\mathcal{X})$ has exactly r maximal elements M_i. Moreover, R_i is infinitely near to M_i.
- The degree of F_i can be obtained from:
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Second main result

Theorem

- L is invariant and contains $\mathcal{D}(\mathcal{X}) \cap \mathbb{P}^2$.
- R_i are the unique IN dicritical singularities of $\mathcal{X}$.
- $\mathrm{MFr}(\mathcal{D}(\mathcal{X})) = \{M_1, \ldots, M_r\}$.
- For each i there exists C_i associated to M_i of degree d_i computable.
- After some computations (skipped), $n_i \in \mathbb{N}$ are obtained.
- If $C_i : \{f_i(x, y) = 0\}$ then $\prod_{i=1}^{r} f_i^{n_i}$ is a WAI PFI.

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Corollary

- n and n_i can be computed from the proximity graph of $\mathcal{D}(\mathcal{X})$ and the points in $\mathcal{D}(\mathcal{X})$ through which the strict transform of the infinity line passes.
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The algorithm

1. Compute $\mathcal{D}(\mathcal{X})$.
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An example

Example

$$(10x^7 - 9x^6 + 6x^5y + 9x^4y - 6x^3y + 6x^2y^2 + 2xy^2)dx$$
$$+(2x^6 - x^4 + 6x^3y - x^2y + 4y^2)dy.$$

We have

- $\mathcal{D}(\mathcal{X}) = \{P_i\}_{i=0}^{28}$.
- $r = 3$,
  $R_1 = M_1 = P_{13}$,
  $R_2 = M_2 = P_{23}$,
  $R_3 = M_3 = P_{28}$.

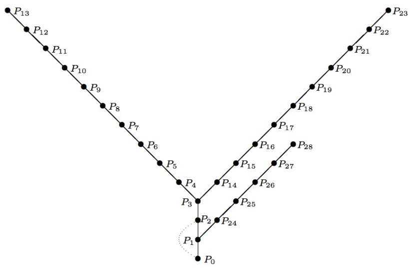

Figure: $\Gamma_{\mathcal{D}(\mathcal{X})}$.

An example

$$
\begin{pmatrix}
3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}
$$

An example

Example

After some technical stuff we compute

$$\mathbf{R} = (10; 6, 4, 2, 2, 1, \ldots, 1, 2, 2, 2, 2, 2).$$

After this we know that $n = 10$.

From the three first rows we can compute the three curves $X^3 - X^2Z + YZ^2 = 0$, $X^3 + YZ^2 = 0$, $X^2 + YZ = 0$.

Moreover,

$$\mathbf{R} = \mathbf{c}_1 + \mathbf{c}_2 + 2\mathbf{c}_3,$$

where $\mathbf{c}_i$ is the i-th row of the matrix.

$H = (y - x^2 + x^3)(y + x^3)(x^2 + y)^2$ is a first integral of $\mathbf{X}$.

An alternative step 4

Compute k_i the cofactor of $f_i = 0$ and solve $\sum_{i=1}^{r} n_i k_i(x,y) = 0$.

Example

Let

$$\begin{array}{ll} f_1 = y - x^2 + x^3, & k_1 = 2x(-x^2 - 4x^3 + 3x^4 - 5y + 3xy); \\ f_2 = y + x^3, & k_2 = 2x(3x^2 - 5x^3 + 3x^4 - y + 3xy); \\ f_3 = x^2 + y, & k_3 = x(-2x^2 + 9x^3 - 6x^4 + 6y - 6xy). \end{array}$$

Solving the linear system $\sum_{i=1}^{3} n_i k_i(x,y) = 0$ we get $n_1 = n_2 = 1$ and $n_3 = 2$.

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