Advances in Qualitative Theory of Differential Equations,
20–24 April 2015, Tarragona, Spain

Looking for simultaneous local and global bifurcations in piecewise linear Filippov systems

E. Freire*, E. Ponce*, A. Rivera† and F. Torres*

*Departamento de Matemática Aplicada II, Universidad de Sevilla, SPAIN
†Departamento de Ciencias Naturales y Matemáticas, Pontificia Universidad Javeriana Cali, COLOMBIA

Introduction

- We deal with the dynamical richness of planar discontinuous piecewise smooth systems, that is Filippov systems in the plane with two zones and a straight line as the discontinuity manifold.
- In a recent work [6], it was reported the possibility of concurrent homoclinic bifurcation and Hopf bifurcation in the piecewise smooth system

$$(\dot{x},\dot{y}) = \begin{cases} (1, -2x+3lx^2+\varepsilon_1), & \text{if } y>0, \\ (-1+mx+ny, -x+\varepsilon_2), & \text{if } y>0, \end{cases} \tag{1}$$

constituted by a linear plus a quadratic vector field, leading to the simultaneous generation of two limit cycles. It is assumed $n<0$, $l, m \in \mathbb{R}$ being $\varepsilon_1, \varepsilon_2$ perturbation terms.
- Here, we show that a richer dynamics can be obtained by considering just a discontinuous piecewise linear system, where in a half-plane the dynamics is of focus type while there is a saddle in the other. Namely, **a simultaneous generation of three limit cycles surrounding the sliding set is shown**: one of the limit cycles comes from a homoclinic connection and the other two arise from a local bifurcation related to a boundary focus, in a similar way as it was done in [2].

1

Introduction (continued)

- We assume that $\Sigma = \{(x,y) : x=0\}$ is the separation line, so that the two linearity regions in the phase plane are

$$S^- = \{(x,y) : x<0\}, \quad S^+ = \{(x,y) : x>0\}.$$

The system becomes

$$\dot{\mathbf{x}} = A^-\mathbf{x} + \mathbf{b}^-, \text{if } x \in S^-, \quad \dot{\mathbf{x}} = A^+\mathbf{x} + \mathbf{b}^+, \text{if } x \in S^+, \tag{2}$$

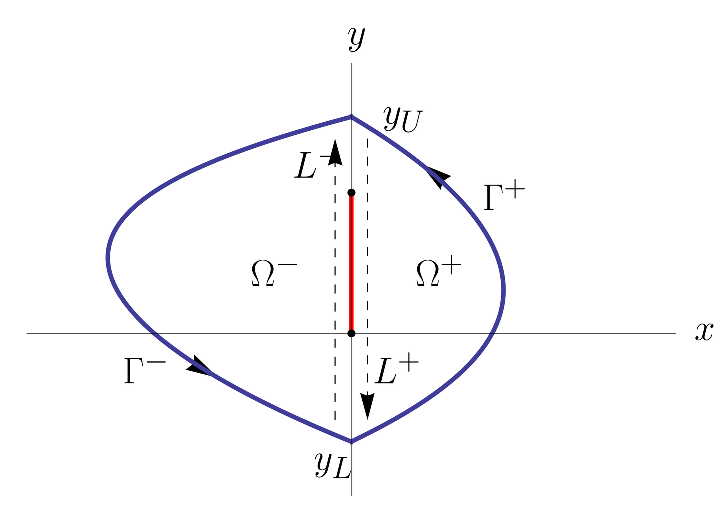
where $\mathbf{x} = (x,y)^T \in \mathbb{R}^2$, $A^- = (a^-_{i,j})$ and $A^+ = (a^+_{i,j})$ are real matrices and $\mathbf{b}^-$, $\mathbf{b}^+$ are planar vectors. From the initial 12 entries we can pass to a canonical form with 5 parameters, see [1].
- **Proposition 1** *If in system* (2) $a^+_{12}a^-_{12} < 0$, $\det A^+ < 0$, *and* $4\det A^- - tr(A^-)^2 > 0$, *then after some continuous change of variables we arrive at the normalized canonical form*

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2+1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix}, \quad \text{if } \mathbf{x} \in S^-, \\ \dot{\mathbf{x}} &= \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2-1 & 0 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix}, \quad \text{if } \mathbf{x} \in S^+, \end{aligned} \tag{3}$$

where $|\gamma_R| < 1$.

2

Introduction (continued)



- A direct consequence of Proposition 3.7 in [1] is that a necessary condition for having a periodic orbit Γ crossing the line $x=0$ through the points $(0,y_L)$ and $(0,y_U)$ where $y_U - y_L = h > 0$, is

$$2\gamma_L\sigma^- + 2\gamma_R\sigma^+ + bh = 0, \tag{4}$$

being $\sigma^\pm = \text{area}(\Omega^\pm)$. As we intend to work for small values of b, henceforth we assume $\gamma_L\gamma_R < 0$. To avoid ambiguity, we take $-1 < \gamma_R < 0$ and $\gamma_L > 0$.

3

Equilibrium points and Homoclinic connections

- The (real or virtual) equilibrium points are a focus at (x_L, y_L) and a saddle at (x_R, y_R), where

$$x_L = \frac{a_L}{\gamma_L^2+1}, \quad x_R = \frac{a_R}{\gamma_R^2-1},$$

and $y_L = 2\gamma_L x_L$ and $y_R = 2\gamma_R x_R + b$.
- The linear invariant manifolds of the right saddle point intersect Σ at the points

$$y_u = b + (\gamma_R+1)x_R, \quad y_s = b + (\gamma_R-1)x_R.$$

- We assume in the sequel $a_R < 0$ so that $x_R > 0$.
- For $x_L = 0$, there exists a homoclinic connection if $y_s = -e^{\pi\gamma_L}y_u$, that is for

$$b = b^0_H := -x_R\frac{e^{\pi\gamma_L}(\gamma_R+1)+\gamma_R-1}{1+e^{\pi\gamma_L}}.$$

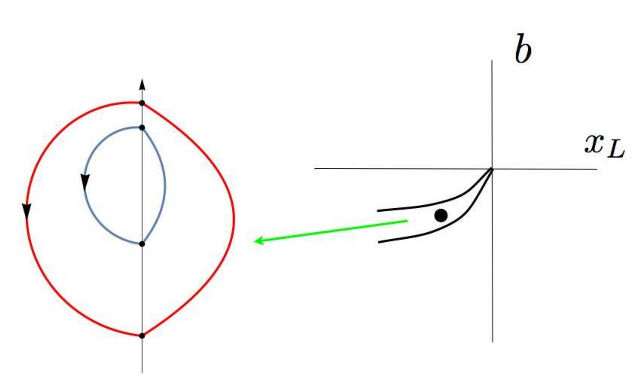
- For $x_L \neq 0$, it is easy to derive the existence of a bifurcation curve of homoclinic connections in the parameter plane (x_L, b) passing through the point $(0, b^0_H)$.

4

Previous Results (Local bifurcation)

- Next result was proved in [2], improving some results in [4, 5].

Proposition 2 *Assume* $\gamma_L > 0, a_R < 0, \gamma_R < 0$, *in* (3).
Then there exist $\xi > 0$ *and two continuous functions* η_1, η_2, *satisfying* $\eta_1(\varepsilon) < \eta_2(\varepsilon) < 0$ *for* $-\xi < \varepsilon < 0$, *and* $\eta_1(0) = \eta_2(0) = 0$, *such that for the parameter sector defined by* $-\xi < x_L < 0$ *and* $\eta_1(x_L) < b < \eta_2(x_L)$ *system* (3) *has at least* **two nested crossing periodic orbits** *that surround the sliding segment* $\{(0,y) : b \le y \le 0\}$.
The periodic orbits have opposite stabilities, the bigger one being unstable and including in its interior the stable one. When $(x_L, b) \to (0,0)$ *within the above sector, both periodic orbits decrease in size, eventually shrinking to the origin.*



5

Main Results (Global bifurcation)

Theorem 1 *Considering system* (3) *with*

$$-1 < \gamma_R < 0, \quad x_R > 0, \quad \gamma_L = \frac{1}{\pi}\ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right),$$

the following statements hold.

(a) *For* $(x_L, b) = (0,0)$ *the origin of the phase plane is an unstable boundary focus surrounded by an homoclinic connection and there are no periodic orbits.*

(b) *The above homoclinic connection persists on the graph of a curve defined by* $b = b_H(x_L)$ *in a neighborhood of the origin in the parameter plane* (x_L, b). *The local expansion of the function* $b_H(x_L)$ *is given by*

$$b = b_H(x_L) = 2\gamma_L x_L - \frac{(1+\gamma_L^2)\sinh(\pi\gamma_L)}{2x_R}x_L^2 + O(x_L^3).$$

(c) *There exists* $\delta^* > 0$ *such that if* $|x_L| < \delta^*$ *then in the transition from* $b = b_H(x_L)$ *to* $b < b_H(x_L)$ *we pass from the homoclinic orbit to* **a stable crossing periodic orbit**.

6

Main Results (Local and Global bifurcations)

- Recalling Proposition 2, we can state our final result.

Theorem 2 *Considering system* (3) *with*

$$-1 < \gamma_R < 0, \quad x_R > 0, \quad \gamma_L = \frac{1}{\pi}\ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right) > 0,$$

the following statements hold.

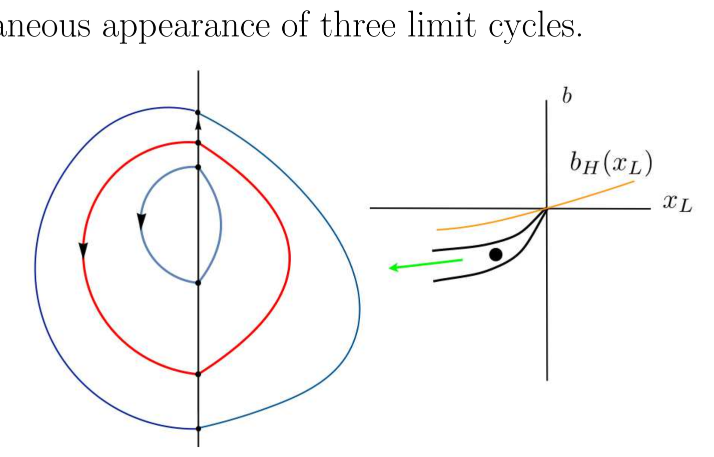
(a) *For* $(x_L, b) = (0,0)$ *the origin of the phase plane is an unstable boundary focus surrounded by an homoclinic connection and there are no periodic orbits.*

(b) *There exist* $\xi > 0$ *and two continuous functions* η_1, η_2, *satisfying* $\eta_1(\varepsilon) < \eta_2(\varepsilon) < 0$ *for* $-\xi < \varepsilon < 0$, *and* $\eta_1(0) = \eta_2(0) = 0$, *such that for the parameter sector defined by* $-\xi < x_L < 0$ *and* $\eta_1(x_L) < b < \eta_2(x_L)$ *system* (3) *has at least* **three nested crossing periodic orbits** *that surround the sliding segment* $\{(0,y) : b \le y \le 0\}$, *being the biggest and the smallest stable and a intermediate one unstable.*

7

Additional remarks

- Note that the choice of γ_L in Theorems 1 and 2 is made so that $b^0_H = 0$, and so the curve of homoclinic points passes through the origin in the parameter plane (x_L, b).
- In Proposition 2, the graph of function $b = \eta_1(x_L)$ corresponds to a standard saddle-node bifurcation of crossing periodic orbits, while the graph of function $b = \eta_2(x_L)$ corresponds to a Critical Crossing limit cycle (CC) bifurcation, leading to the transition of the inner limit cycle from a crossing to a sliding limit cycle, see [3].
- By moving adequately both parameters x_L and b, we can pass from the upper part of first quadrant (above the homoclinic curve) to the interior of the sector in the third quadrant, getting the simultaneous appearance of three limit cycles.



8

References

1. Freire, E. & Ponce, E. & Torres, F., *Canonical discontinuous planar piecewise linear systems*, SIAM J. Appl. Dyn. Sys. **11** (2012), 181–211.
2. Freire, E. & Ponce, E. & Torres, F., *A General Mechanism to Generate Three Limit Cycles in Planar Filippov Systems with Two Zones*, Nonlinear Dyn. DOI 10.1007/s11071-014-1437-7.
3. Freire, E. & Ponce, E. & Torres, F.: On the Critical Crossing Cycle Bifurcation in Planar Filippov Systems. Preprint. Submited for publication.
4. Guardia, M.& M.-Seara, T.& Teixeira, M.: Generic Bifurcations of Low Codimension of Planar Filippov Systems. J. Diff. Equations **250**, 1967–2023 (2011)
5. Han, M.& Zhang, W.: On Hopf Bifurcation in Non-smooth Planar Systems. J. Diff. Equations **248**, 2399–2416 (2010)
6. Li, L. & Huang, L. *Concurrent homoclinic bifurcation and Hopf bifurcation for a class of planar Filippov systems*, Journal of Mathematical Analysis and Applications, Vol. 411 (2014), 83–94.

9