Ultra fast deformation of periodic orbit

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The integrable system

We propose to study perturbation of the generalized Liénard system:

$$\begin{aligned} \varepsilon\dot{x} &= y - f(x) \\ \dot{y} &= -f'(x) \end{aligned} \tag{1}$$

Which after the rescaling $(x, y, t) \to (\varepsilon^{1/2}x, y\varepsilon, \varepsilon^{1/2}t)$ becomes

$$\begin{aligned} \dot{x} &= y - f_\varepsilon(x) \\ \dot{y} &= -f'_\varepsilon(x) \end{aligned} \tag{2}$$

This system is integrable of integrand factor e^{-y}.

$$H(x,y) = e^{-y}[f_\varepsilon(x) - y - 1] = h \in [-1, h_{max}].$$

and

$$y(x) = f_\varepsilon(x) - 1$$

is a solution. To simplify the visualization, we represent the level set of H for $\varepsilon = 1$ in the two following examples: a) $f(x) = (x+1)x(x-1)(x-3/2)(x-3)$, b) $f(x) = \frac{x^2}{2} - \frac{x^4}{4}$.

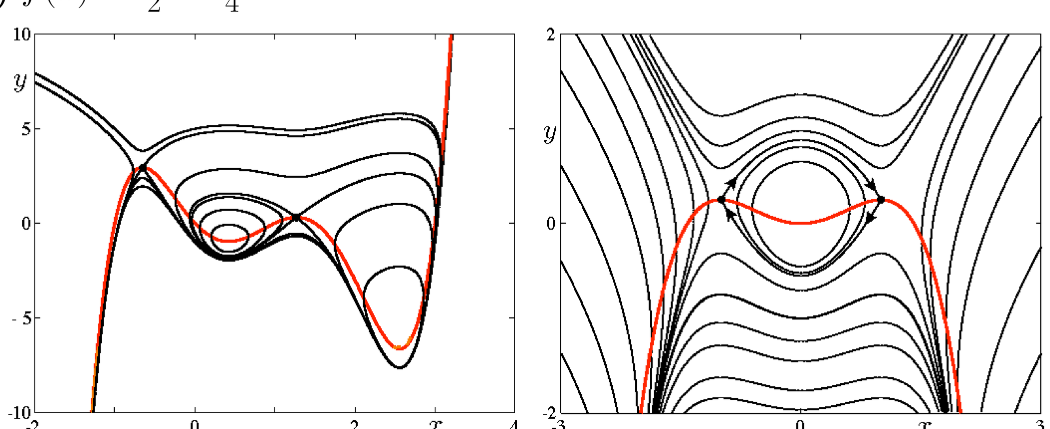

FIGURE 1: a) Three homoclinic loops bounding three nests, b) A nest bounded by an heteroclinic loop. Figure retrieved from [1]

Lambert function: It is the reciprocal of the function $x \to xe^x$. This function is not injective but the Lambert function has only two real branches: the principal branch W_0 and the other W_{-1}

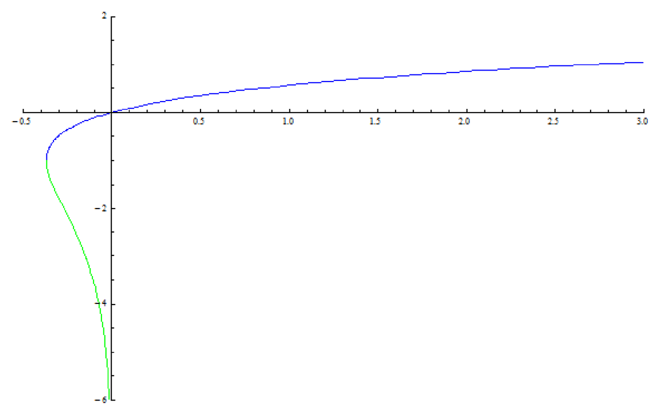

FIGURE 2: The two real branches of the Lambert function.

Solution of the integrable system:

$$e^{f_\varepsilon(x)-y-1}[f_\varepsilon(x) - y - 1] = he^{f_\varepsilon(x)-1}.$$

$$\begin{aligned} \hookrightarrow y_+ &= f_\varepsilon(x) - 1 - W_{-1}(\tfrac{h}{e}e^{f_\varepsilon(x)}), \\ y_- &= f_\varepsilon(x) - 1 - W_0(\tfrac{h}{e}e^{f_\varepsilon(x)}). \end{aligned} \tag{3}$$

Proposition 1 *Any periodic trajectory intersects transversally the critical curve in exactly two points.*

$\to y(x) = y_+(x)$ above the critical curve $y = f_\varepsilon(x)$ and $y(x) = y_-(x)$ below. We choose to study the system:

$$\begin{aligned} \varepsilon\dot{x} &= y - \frac{x^2}{2} - \alpha\frac{x^3}{3} \\ \dot{y} &= -x - x^2\alpha. \end{aligned} \tag{4}$$

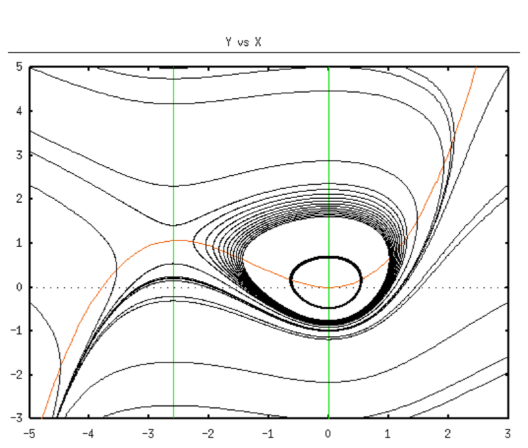

FIGURE 3: Phase portrait of system (4) for $\varepsilon = 1$.

Canard-induced loss of stability across a homoclinic bifurcation

We now consider the perturbed system

$$\begin{aligned} \varepsilon\dot{x} &= y - \frac{x^2}{2} - \alpha\frac{x^3}{3} \\ \dot{y} &= -x - x^2(\alpha - \beta) + \sqrt{\varepsilon}\mu. \end{aligned} \tag{5}$$

After rescaling this yields:

$$\begin{aligned} \dot{x} &= y - \frac{x^2}{2} - \sqrt{\varepsilon}\alpha\frac{x^3}{3} \\ \dot{y} &= -x - \sqrt{\varepsilon}x^2(\alpha - \beta) + \mu. \end{aligned} \tag{6}$$

Numerical simulation (done with XPPAUT (see [4] and [2])). For $0 < \alpha < 1$, $0 < \sqrt{\varepsilon} \ll 1$ and $0 < \beta < 1$ fixed, a small canard cycle is born by varying μ across a Hopf bifurcation, for variation of μ of the order of 10^{-7} the cycle explodes and disappears across a homoclinic bifurcation.

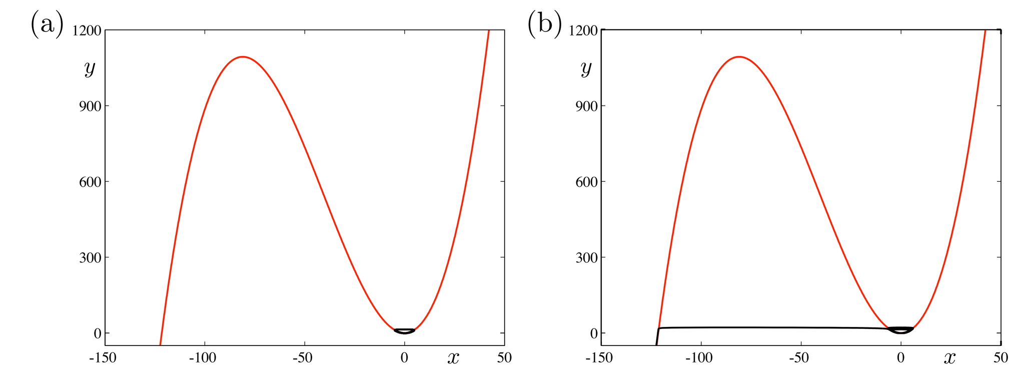

FIGURE 4: (a): Small canard cycle. (b): Trajectory with the same initial condition after explosion. Figure retrieved from [1].

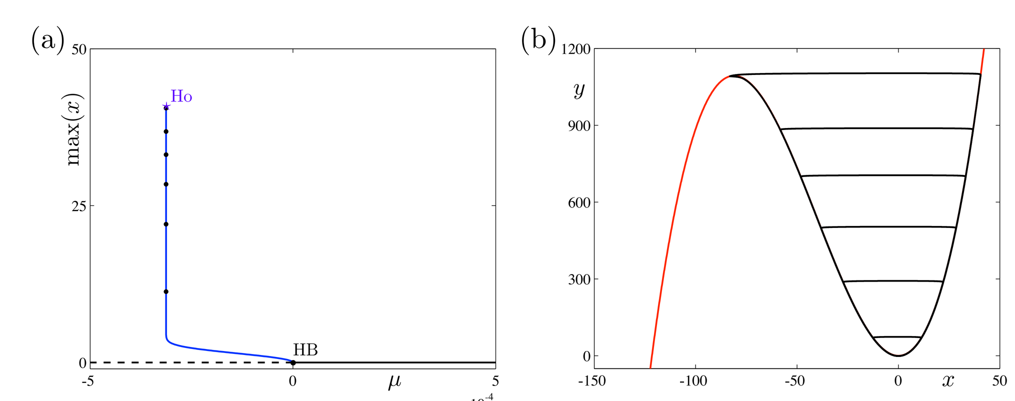

FIGURE 5: (a): Bifurcation diagram in μ. (b): a few limit cycles on the explosive branch (in blue) shown in panel (a), approaching the homoclinic connection. Figure retrieved from [1].

$\Rightarrow$ The value of μ for which there is a solution which follows the repulsive part of the critical manifold are given by:

$$\begin{aligned} \beta &= O(\varepsilon) \\ \mu &= \mu_c + \varepsilon^{5/2}\sigma e^{-\frac{k^2}{\varepsilon}}, \\ \mu_c &= -\sqrt{\varepsilon}\beta(1+d)[1+o(1)] \end{aligned} \tag{7}$$

for some $d \in [-2/3, 2/3]$. We prove it by following the Eckhaus methods (see [3]). We can compare this result with that obtained in the Van der Pol case: Let us consider the system

$$\begin{aligned} \varepsilon\dot{x} &= y - x^2/2 - \alpha x^3/3 \\ \dot{y} &= -x + \sqrt{\varepsilon}\mu. \end{aligned} \tag{8}$$

By following the Eckhaus methods we find:

$$\begin{aligned} \mu &= \mu_c + \varepsilon^{5/2}\sigma e^{-\frac{k^2}{\varepsilon}}, \\ \mu_c &= -\sqrt{\varepsilon}\alpha + O(\varepsilon^{3/2}) \end{aligned} \tag{9}$$

We now want to compute the value of μ for which the cycle explodes. We use a strategy based on the first return map and the derivative given by an integral of Lambert function.
Consider the following equations

$$\begin{aligned} h &= e^{-\frac{y}{\varepsilon}}\left[\frac{f(x)}{\varepsilon} - \frac{y}{\varepsilon} - 1\right] \\ \omega &= e^{-\frac{y}{\varepsilon}}\frac{y - f(x)}{\varepsilon}dy - e^{-\frac{-y}{\varepsilon}}(-f'(x) - \delta(x))dx \\ &= dh - e^{-\frac{y}{\varepsilon}}\delta(x)dx. \end{aligned} \tag{10}$$

The following integral equation holds:

$$\int_{\gamma_{\mu,\beta,h}} \omega = \int_{\gamma_{\mu,\beta,h}} dh - \int_{\gamma_{\mu,\beta,h}} e^{-\frac{y}{\varepsilon}}\delta(x)dx, \tag{11}$$

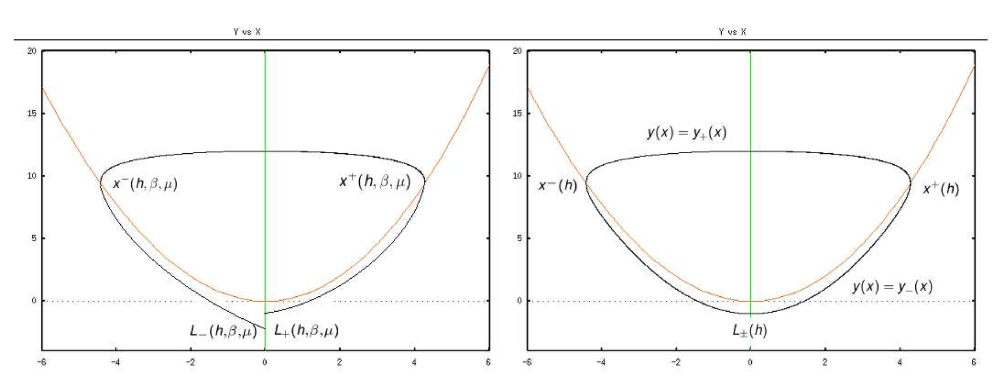

FIGURE 6: Schema of trajectories in positive and in negative time starting from an initial condition lying on the y-axis.

Using the parametrization with the Lambert function we obtain the condition:

$$\begin{aligned} \frac{L_+(h,\beta,\mu) - L_-(h,\beta,\mu)}{h} = \beta &\int_{x^-(h)}^{x^+(h)} x^2\left[\frac{1}{W_0(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})} - \frac{1}{W_{-1}(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})}\right]dx \\ &+ \sqrt{\varepsilon}\mu \int_{x^-(h)}^{x^+(h)} \frac{1}{W_0(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})} - \frac{1}{W_{-1}(\frac{h}{e}e^{\frac{f(x)}{\varepsilon}})}dx \\ &+ O\left((\sqrt{\varepsilon}\mu, \beta)^2\right). \end{aligned} \tag{12}$$

Solving this equation with MATHEMATICA for $h = e^{-\frac{1}{6\alpha^2\varepsilon}}$ (level set of the homoclinic loop) we find a very good approximation of the parameter for which the loss of stability happens.

Fast deformation of an explicit periodic orbit

Let us consider the integrable system:

$$\begin{aligned} \dot{x} &= \frac{1}{\psi}\frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{1}{\psi}\frac{\partial H}{\partial x}. \end{aligned} \tag{13}$$

$\to \psi$ is the integrating factor associated with the first integral of the system H. We suppose that for some I, the level set $H(x,y) = h \in I$ are closed. Let us consider the perturbed system

$$\begin{aligned} \dot{x} &= \frac{1}{\psi}\frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{1}{\psi}\frac{\partial H}{\partial x} + \beta[H - h], \end{aligned} \tag{14}$$

where $h \in I$. $\Rightarrow$ The curve $H(x,y) = h$ is a periodic orbit of the system.

Consider now the system

$$\begin{aligned} \varepsilon\dot{x} &= y - (\frac{x^2}{2} + \alpha\frac{x^3}{3}), \\ \dot{y} &= -x - \alpha x^2 + \varepsilon\beta\left[e^{-y/\varepsilon}(\frac{f(x)}{\varepsilon} - \frac{y}{\varepsilon} - 1) - h\right]. \end{aligned} \tag{15}$$

$\to$ For $h = -1$, the point $(0,0)$ is a center, $\to$ For $h = h_h = -e^{-\frac{1}{6\varepsilon\alpha^2}}$, the point $(-\frac{1}{\alpha}, f(-\frac{1}{\alpha}) = e^{-\frac{1}{6\alpha^2}})$ is a saddle-node For $\sqrt{\varepsilon} = 0.03125$, $\alpha = 0.3$: $h_h \simeq -2.811787299503 * 10^{-824}$.

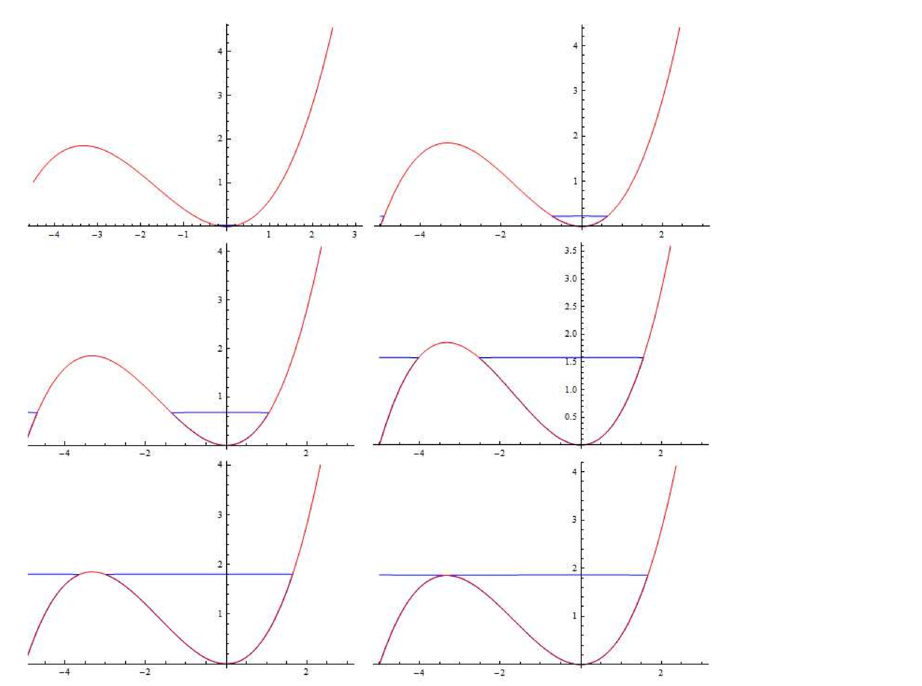

FIGURE 7: Evolution of the trajectory $H(x,y) = h$ for $\alpha = 0.3$, $\sqrt{\varepsilon} = 0.03125$ and h takes from left to right and top to bottom the value $h = -10^{-10}$, $h = -10^{-100}$, $h = -10^{-300}$, $h = -10^{-700}$, $h = -10^{-800}$ et $h = -2.9 \times 10^{-824}$

$\to$ The upper part of the limit cycle is given by $y(h,\varepsilon,x) = f(x) - \varepsilon[1 + W_{-1}(\frac{h}{e}e^{f(x)\varepsilon})]$
$\hookrightarrow$ the maximum height of the cycle $y(h,\varepsilon)$ $(y(h,\varepsilon,0))$ is given by:

$$y(h,\varepsilon) = -\varepsilon[1 + W_{-1}(he^{-1})].$$

$\to$ The sup of the limit cycle height (height of the homoclinic loop) is $L_{max} = -\varepsilon[1 + W_{-1}(-e^{-1-\frac{1}{6\alpha^2\varepsilon}})] \simeq \frac{1}{6\alpha^2}$.

$$\begin{aligned} -\varepsilon[1 + W_{-1}(he^{-1})] &\geq \frac{1}{600\alpha^2} \\ \Leftrightarrow \qquad h &\geq -(1 + \frac{1}{600\alpha^2\varepsilon})e^{-\frac{1}{600\alpha^2\varepsilon}} \end{aligned} \tag{16}$$

$\hookrightarrow$ The amplitude of the cycle increases by a ratio of 100 when h covers the interval $[-(1 + \frac{1}{600\alpha^2\varepsilon})e^{-\frac{1}{600\alpha^2\varepsilon}}, -e^{-\frac{1}{6\alpha^2\varepsilon}}]$.

References

[1] M. Desroches, J.-P. Françoise, L. Mégret, *Canard-Induced Loss of Stability Across a Homoclinic Bifurcation*, Revue africaine de la recherche en informatique et mathématiques appliquées, accepted.
[2] E. J Doedel, *Lecture Notes on Numerical Analysis of Nonlinear Equation*, Department Of Computer Science, Concordia University, Montreal, Canada.
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[4] B. Ermentrout, *Simulating, analysing, and animating dynamical systems: a guide to* XPPAUT *for researchers and students*, Software Environment and Tools vol. 14, SIAM, Philadelphia, 2002.