On the wave length of smooth periodic traveling waves of the Camassa-Holm equation

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Introduction

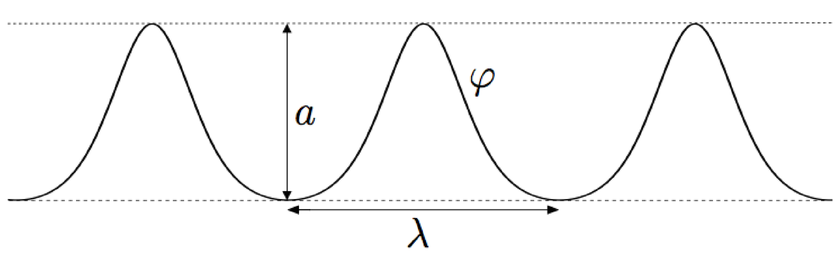
The **Camassa-Holm equation**

$$u_t + 2\kappa\, u_x - u_{txx} + 3\, u\, u_x = 2\, u_x u_{xx} + u\, u_{xxx}, \qquad \text{(CH)}$$

arises as a two-dimensional shallow water approximation of the Euler equations, where $u(x,t)$ describes the horizontal velocity component [1, 2, 4]. For traveling wave solutions $u(x,t) = \varphi(x - c\,t)$ the equation takes the form

$$\varphi''(\varphi - c) + \frac{(\varphi')^2}{2} + r + (c - 2\kappa)\,\varphi - \frac{3}{2}\varphi^2 = 0, \qquad (1)$$

where c is the wave speed and $r \in \mathbb{R}$ is a constant of integration [5]. We are interested in **smooth periodic traveling wave solutions** φ of (CH), which have a unique maximum (crest) and minimum (trough) per period:



wave length λ …the period of a smooth periodic traveling wave φ.
wave height a …the difference between the crest and trough of φ.

Q: Is there a relationship between the length λ and the height a?

A: Yes! The function $\lambda(a)$ is well-defined and either unimodal or monotonous.

Main Result

Given c, κ with $c \neq -\kappa$, there exist real numbers $r_1 < r_{b_1} < r_{b_2} < r_2$ such that the Camassa-Holm equation (CH) *has smooth periodic TWS of the form $\varphi(x - c\,t)$ if, and only if, the integration constant r in* (1) *belongs to the interval (r_1, r_2). For such $r \in (r_1, r_2)$, the set of smooth periodic TWS form a continous family $\{\varphi_a\}_a$ parametrized by the wave height a.*

The wave length $\lambda = \lambda(a)$ of φ_a satisfies the following:

- *If $r \in (r_1, r_{b_1}]$, then $\lambda(a)$ is monotonous increasing.*
- *If $r \in (r_{b_1}, r_{b_2})$, then $\lambda(a)$ has a unique critical point (maximum).*
- *If $r \in [r_{b_2}, r_2)$, then $\lambda(a)$ is monotonous decreasing.*

Waves ⟷ Orbits

φ is a smooth periodic solution of (1) if and only if $\gamma_\varphi := (w, v) = (\varphi - c, \varphi')$ is a periodic orbit of the planar system

$$w' = v, \quad v' = -\frac{\alpha + 2\beta w - \frac{3}{2}w^2 + \frac{1}{2}v^2}{w}, \qquad (2)$$

where $\alpha := r - 2\kappa c - \frac{1}{2}c^2$ and $\beta := -(c+\kappa)$. Every periodic orbit belongs to the period annulus of a center of (2), which exists if and only if $-2\beta^2 < 3\alpha < 0$. The set of periodic orbits is parametrized by the energy levels of the first integral of (2), which are diffeomorphic to the wave height a of φ. Hence, the set of smooth periodic solutions of (1) forms a continuous family $\{\varphi_a\}_a$ parametrized by a and the function $a\lambda(a)$ = wave length of φ_a is well-defined.

The wave length λ of φ equals the period T of a periodic orbit γ_φ of (2), and the wave height a is diffeomorphic to the energy levels h of the first integral. Therefore, the qualitative properties of the function $\lambda(a)$ can be deduced from the **period function** $T(h)$ of the center of (2).

Qualitative Study of the Period Function

Consider an analytic planar differential system satisfying these *hypotheses*:

$(*)$ The system has a center at the origin, an analytic first integral of the form

$$H(x,y) = A(x) + B(x)y + C(x)y^2 \text{ with } A(0) = 0,$$

and its integrating factor K depends only on x.

The function $M := \frac{4AC - B^2}{4|C|}$ defines a unique involution σ satisfying $M \circ \sigma = M$ on the projection (x_ℓ, x_r) on the x-axis of the period annulus around the center of the system. Given an analytic function f on $(x_\ell, x_r) \setminus \{0\}$ one can define its σ-*balance*

$$\mathscr{B}_\sigma(f)(x) := \frac{f(x) - f(\sigma(x))}{2}.$$

Criterion to bound the number of critical periods (see [3]):

Under hypotheses $(*)$ let $\mu_0 = -1$ and define

$$\mu_i := \left(\frac{1}{2} + \frac{1}{2i-3}\right)\mu_{i-1} + \frac{\sqrt{|C|}M}{(2i-3)K}\left(\frac{K\mu_{i-1}}{\sqrt{|C|}M'}\right)' \quad \text{and} \quad \ell_i := \frac{K\mu_i}{\sqrt{|C|}M'} \quad \text{for } i \geqslant 1.$$

If the number of zeros of $\mathscr{B}_\sigma(\ell_i)$ on $(0, x_r)$ is $n \geqslant 0$ and it holds that $i > n$, then the number of critical periods of the center at the origin of the system is at most n (counted with multiplicities).

To apply the above criterion, we move the center of (2) to the origin via a homothetic coordinate transformation and obtain the differential system

$$\begin{cases} x' = y, \\ y' = -\dfrac{x - 3x^2 + y^2}{2(x + \vartheta)}, \end{cases} \quad \text{with } \vartheta := \tfrac{1}{6}\left(\frac{2}{\sqrt{4 + \frac{6\alpha}{\beta^2}}} - 1\right) > 0. \qquad (3)$$

System (3) is one-parametric and satisfies hypotheses $(*)$ with $B = 0$ away from $x = -\theta$. The criterion stated above facilitates bounds on the number of critical periods, which vary with ϑ. Monotonicity or unimodality of the period function $T(h)$ then follows from the sign of $T'(h)$ near its endpoints.

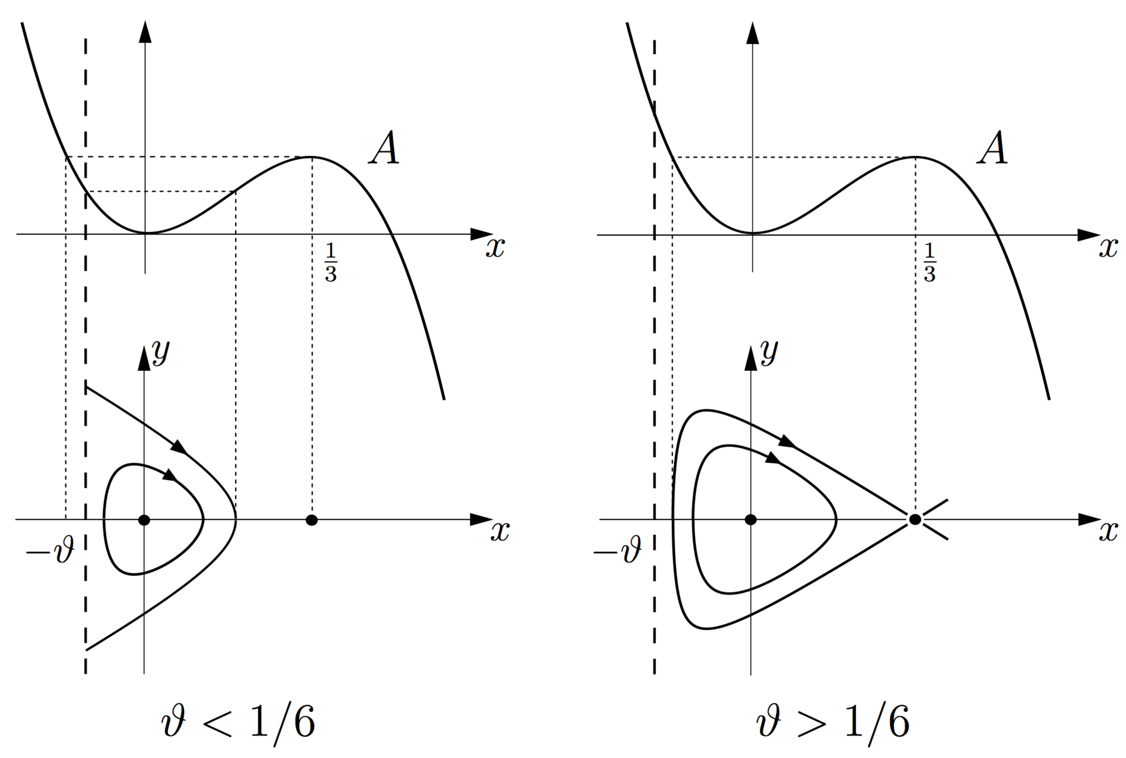


Figure 1: The period annulus of the center of system (3) with $A(x) = \frac{1}{2}x^2 - x^3$.

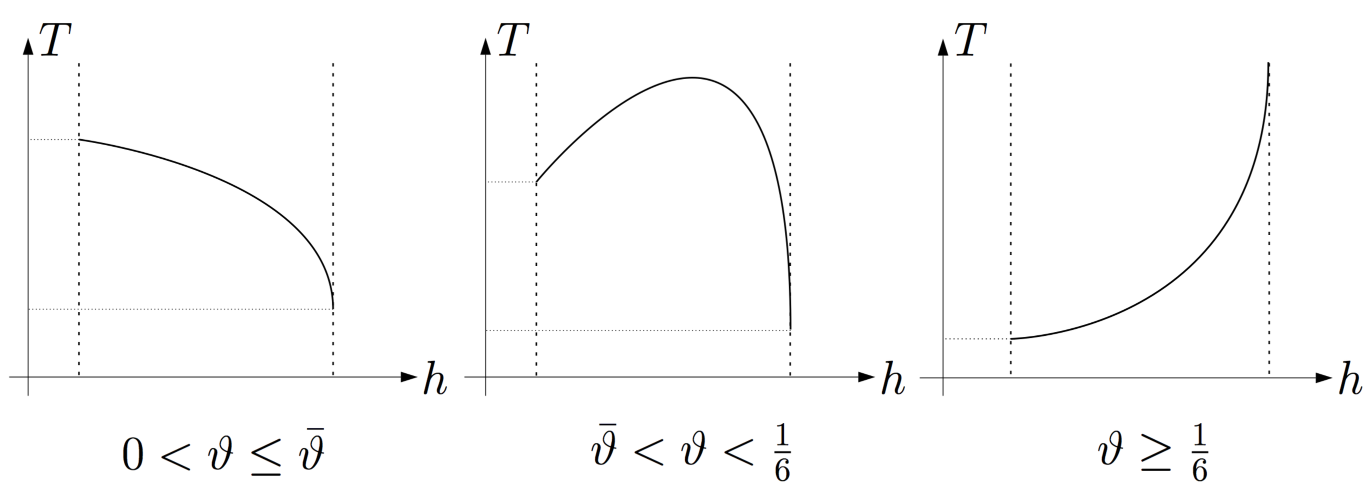


Figure 2: Sketch of the graph of the period function $T(h)$ with $\bar{\vartheta} = -\frac{1}{10} + \frac{1}{15}\sqrt{6}$.

References

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Acknowledgements

A.G. is supported by the FWF project J3452 "Dynamical Systems Methods in Hydrodynamics" of the Austrian Science Fund. J.V. is partially supported by the MEC/FEDER grant MTM2008-03437.

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The contents of this poster will appear in *J. Differ. Equ.* (2015).

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