On the wave length of smooth periodic traveling waves of the Camassa-Holm equation

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Introduction

The Camassa-Holm equation

\[ u_t + 2\kappa u_x - u_{xxt} + 3u u_x - 2u_x u_x + u u_{x} = 0, \quad (CH) \]

arises as a two-dimensional shallow water approximation of the Euler equations, where \( u(x,t) \) describes the horizontal velocity component \([1, 2, 4]\). For traveling wave solutions \( u(x,t) = \phi(x - ct) \) the equation takes the form

\[ \phi''(x - ct) + \frac{(\phi')^2}{2} + r + (c - 2\kappa) \phi_x - \frac{3}{2}\phi_x = 0, \quad (1) \]

where \( c \) is the wave speed and \( r \in \mathbb{R} \) is a constant of integration \([5]\). We are interested in smooth periodic traveling wave solutions \( \phi \) of \( (CH) \), which have a unique maximum (crest) and minimum (trough) per period.

**Q:** Is there a relationship between the length \( \lambda \) and the height \( a \)?

**A:** Yes! The function \( \lambda(a) \) is well-defined and either unimodal or monotone.

Main Result

Given \( c,\kappa \) with \( c \neq -\kappa \), there exist real numbers \( r_1 < r_0 < r_2 \) such that the Camassa-Holm equation \( (CH) \) has smooth periodic TWS of the form \( \phi(x - ct) \) if, and only if, the integration constant \( r \) belongs to the interval \( (r_1, r_2) \). For such \( r \in (r_1, r_2) \), the set of smooth periodic TWS form a continuous family \( \{\phi_r\}_r \) parameterized by the wave height \( a \).

The wave length \( \lambda = \lambda(a) \) of \( \phi_r \) satisfies the following:

- If \( r \in (r_1, r_0) \), then \( \lambda(a) \) is monotonically increasing.
- If \( r \in (r_0, r_2) \), then \( \lambda(a) \) has a unique critical point (maximum).
- If \( r \in [r_2, r_0) \), then \( \lambda(a) \) is monotonically decreasing.

Waves \( \leftrightarrow \) Orbits

\( \phi \) is a smooth periodic solution of \( (1) \) if and only if \( \gamma_a := (w, v) = (\phi - c, \phi') \) is a periodic orbit of the planar system

\[ \frac{dw}{dt} = v, \quad \frac{dv}{dt} = \frac{\alpha + 2\beta w - \frac{3}{2}v^2 + \frac{1}{4}v^4}{w}, \quad (2) \]

where \( \alpha := r - 2\kappa \) and \( \beta = -(c + \kappa) \). Every periodic orbit belongs to the period annulus of a center of \( (2) \), which exists if and only if \( -2\beta^2 < 3\alpha < 0 \). The set of periodic orbits is parametrized by the energy levels of the first integral of \( (2) \), which are diffeomorphic to the wave height \( a \) of \( \phi \). Hence, the set of smooth periodic solutions of \( (1) \) forms a continuous family \( \{\phi_a\}_a \) parameterized by \( a \) and the function \( a\lambda(a) \) is well-defined.

The wave length \( \lambda \) of \( \phi \) equals the period \( T \) of a periodic orbit \( \gamma_a \) of \( (2) \), and the wave height \( a \) is diffeomorphic to the energy levels \( h \) of the first integral. Therefore, the qualitative properties of the function \( \lambda(a) \) can be deduced from the period function \( T(h) \) of the center of \( (2) \).

Qualitative Study of the Period Function

Consider an analytic planar differential system satisfying these hypotheses:

- The system has a center at the origin, \( (\ast) \).
- An analytic first integral of the form
  \[ H(x, y) = A(x) + B(x)y + C(x)y^2 \]
  with \( A(0) = 0 \), and its integrating factor \( K \) depends only on \( x \).

The function \( M - \frac{\lambda(cK - B)}{\sqrt{4K^3}} \) defines a unique involution \( \sigma \) satisfying \( M \circ \sigma = M \) on the projection \((x, x)\) on the \( x \)-axis of the period annulus around the center of the system. Given an analytic function \( f \) on \((x, x) \) \( \setminus \{0\} \) one can define its \( \sigma \)-balance

\[ \sigma_{\mu}(f)(x) = \frac{f(x) - f(\sigma(x))}{2} \]

Criterion to bound the number of critical periods (see [3]):

Under hypotheses \( (\ast) \) let \( \mu_0 = -1 \) and define

\[ \mu_i := \left( \frac{1}{4} + \frac{1}{2^{i+1}} \right) \mu_{i-1} + \frac{\sqrt{i+1}}{\sqrt{4i^2 + 1}} \]

and \( \ell_i := \frac{K\mu_i}{\sqrt{4i^2 + 1}} \) for \( i \geq 1 \).

If the number of zeros of \( \sigma_{\mu_0}(u)(0, x_i) \) is \( n \geq 0 \) and it holds that \( i > n \), then the number of critical periods of the center at the origin of the system is at most \( n \) (counted with multiplicities).

To apply the above criterion, we move the center of \((2) \) to the origin via a homothetic coordinate transformation and obtain the differential system

\[ \begin{align*}
  x' &= y \\
  y' &= \frac{1}{2} - 3x^2 - y^2, \quad \text{with } \phi := \frac{1}{4} - \frac{1}{2} \left( 1 + \frac{1}{\sqrt{4i^2 + 1}} \right) > 0.
\end{align*} \]

System \((3) \) is one-parametric and satisfies hypotheses \( (\ast) \) with \( B = 0 \) away from \( x = -\theta \). The criterion stated above facilitates bounds on the number of critical periods, which vary with \( \phi \). Monotonicity or unimodality of the period function \( T(h) \) then follows from the sign of \( T(h) \) near its endpoints.

### References


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