# Classification and counting of planar quasi-homogeneous but nonhomogeneous differential systems

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## ABSTRACT

***The planar differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, with $P, Q \in \mathbb{C}[x,y]$ is called quasi–homogeneous if there exist $s_1$, $s_2$, $d \in \mathbb{N}$ such that for an arbitrary $\alpha \in \mathbb{R}^+$, it is verified that $P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x,y)$ and $Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x,y)$. The quasi-homogeneous systems have important properties (for example, all of them are integrable) and they have been studied from many different points of view (integrability, centers, normal forms, limit cycles). But until recently there was not an algorithm for constructing all the quasi-homogeneous polynomial differential systems of a given degree which was obtained in [1] that we recall here. Using this algorithm the same authors obtained the classification of the quasi-homogeneous planar systems of degree 2 and 3, and later other authors solve the case of degrees 4 (see [3]) and 5 (see [4]). In our work we obtain the exact number of different forms of quasi-homogeneus but nonhomogeneous planar differential systems of an arbitrary degree $n$, proving a nice relation between this number and the Euler's totient function whose definition and properties can be seen in [2].***


## 1. BASIC DEFINITIONS

A polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y),$$

of degree $n = \max\{deg(P), deg(Q)\}$ is called ***quasi–homogeneous*** (in short, QH-system) if there exist $s_1$, $s_2$, $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R},\ a > 0\}$, it is verified that

$$P(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_1-1+d}P(x,y),$$

$$Q(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^{s_2-1+d}Q(x,y).$$

We call the vector $\mathrm{w} = (s_1, s_2, d)$ ***weight vector*** of the system, where $s_1$ and $s_2$ are *weight exponents* of the system, and $d$ *weight degree* with respect to $s_1$ and $s_2$.

We say that the weight vector $\mathrm{w}_m = (s_1^*, s_2^*, d^*)$ is the ***minimun weight vector*** of the system if any weight vector $\mathrm{w} = (s_1, s_2, d)$ verifies $s_1^* \leq s_1$, $s_2^* \leq s_2$ and $d^* \leq d$.

## 2. FIRST RESULTS ABOUT QH-SYSTEMS

We consider a vector field $X = (P, Q)$ associated to a QH-system of degree $n$ and let $\mathrm{w} = (s_1, s_2, d)$ be a weight vector of $X$. We have proved that:

- If $s_1 = s_2$, then $X$ is a homogeneous system and $\mathrm{w}_m = (1, 1, n)$.
- If $s_1 \neq s_2$, there is $p \in \{0, \ldots, n\}$ such that the homogeneous part of highest degree of $X$ is

$$X_n^p = (a_{p,n-p}x^p y^{n-p}, b_{p-1,n-p+1}x^{p-1}y^{n-p+1})$$

and the following equation holds:

$$e_p^0[0] \equiv (p-1)s_1 + (n-p)s_2 + 1 - d = 0.$$

Moreover, if the homogeneous part of $X$ of degree $n - t$ is nonzero, then there is $k \in \{1, \ldots, n-t-p+1\}$ such that

$$X_{n-t}^{p,k} = (a_{p+k,n-p-k-t}x^{p+k}y^{n-p-k-t}, b_{p+k-1,n-p-k-t+1}x^{p+k-1}y^{n-p-k-t+1})$$

and the next equation is verified:

$$e_p^t[k] \equiv (p+k-1)s_1 + (n-t-p-k)s_2 + 1 - d = 0.$$

Since the homogeneous systems are well known, in order to obtain all the QH-systems of a fixed degree, we need to obtain all the quasi-homogeneous but nonhomogeneous systems (hereafter, QHNH-systems) and hence we consider only the case $s_1 \neq s_2$. Moreover, through an adequate change of variables, we can restrict our study to the case $s_1 > s_2$.

## 3. PROPERTIES OF THE QHNH-SYSTEMS

Let $\mathrm{w} = (s_1, s_2, d)$ be a weight vector of a QHNH-system.

**Case 1: d=1.** We have proved that the general expression of the $n$-degree system in this case is

$$\dot{x} = a_{1,0}x + a_{0,n}y^n, \quad \dot{y} = b_{0,1}y.$$

Furthermore it is verified that $\mathrm{w}_m = (n, 1, 1)$.

**Case 2: d>1.** If $X$ is a vector field associated to a QHNH-system of degree $n$, then there exist $p, t, k$ such that $X_n^p X_{n-t}^{p,k} \neq 0$. Therefore $e_p^0[0]$ and $e_p^t[k]$ holds and, by solving this linear system, it can be obtained that

$$s_1 = (t+k)(d-1)/D, \quad s_2 = k(d-1)/D,$$

where $D = (p-1)t + (n-1)k$.

Also, the minimum weight vector of $X$ is

$$\mathrm{w}_m = ((t+k)/s, k/s, 1 + D/s),$$

where $s$ is the greatest common divisor of $t$ and $k$.

In order that $X$ to have other nonzero homogeneous part we must consider other equation $e_p^{t^*}[k^*]$. We have proved that this third equation is compatible with the two previous equations if and only if

$$kt^* = k^*t.$$

## 4. THE ALGORITHM

This algorithm allow us to obtain all the QHNH-systems of a fixed degree $n$ that satisfy $d > 1$ for all its weight vectors. We remark that, taking into account the above results, each QHNH-system is associated to a linear system formed by those equations that correspond to nonzero homogeneous parts of the system.

- **Step 1**. We choose $p \in \{0, \ldots, n\}$ such that $X_n^p$ is the homogeneous part of degree $n$ of the vector field $X$. Therefore, the associated equation is $e_p^0[0]$.
- **Step 2**. We choose a value $t \in \{1, \ldots, n-p\}$ and a value $k \in \{1, \ldots, n-t-p+1\}$ such that the homogeneous part $X_{n-t}^{p,k}$ is also nonzero and hence the equation $e_p^t[k]$ holds. These values allow us to obtain the minimum weight vector $\mathrm{w}_m$, and also provide the weight exponents $s_1$ and $s_2$ as a function of $d$ (see Section 3).
- **Step 3**. In order to determine all the homogeneous parts that can be added to $X_n^p$ and $X_{n-t}^{p,k}$, we establish, for each $t^* \in \{1, \ldots, n-p\}$ with $t \neq t^*$, the value $k_{t^*} \in \{1, \ldots, n-t^*-p+1\}$, (if it exists), such that the equation $e_p^{t^*}[k_{t^*}]$ satisfies the compatibility condition with the equations $e_p^0[0]$ and $e_p^t[k]$.
- **Step 4**. We obtain the QHNH-vector field of degree $n$ formed by all the homogeneous parts of the above steps, namely

$$X = X_n^p + X_{n-t}^{p,k} + \sum_{t^* \in \{1,\ldots,n-p\} \setminus \{t\} \text{ and } k_{t^*}t = kt^*} X_{n-t^*}^{p,k_{t^*}},$$

where the explicit expression of each homogeneous part is defined in Section 2.
- **Step 5**. We go back to Step 2 and consider other choices of $t$ and $k$. As soon as that is not possible, we change in Step 1 the value of $p$ and repeat the whole process.

## 5. COUNTING OF QHNH-SYSTEMS

Let $c(n)$ the number of QHNH-systems of degree $n$. Then $c(1) = 0$, $c(2) = 3$, $c(3) = 8$ and for $n \geq 4$ we have the recursive form

$$c(n) = 2c(n-1) - c(n-2) + \varphi(n+1)$$

that can be written in analitic form as

$$c(n) = 5n - 7 + \sum_{k=4}^{n} \sum_{j=5}^{k+1} \varphi(j)$$

where $\varphi$ is the Euler´s totient function (see [4]) defined by:

$$\varphi(n) = |\{r \in \mathbb{N} \,:\, 1 \leq r \leq n\,,\ gcd(n,r) = 1\}|$$

The proof of this result takes into account the following aspects:

i) In [1] it has been proved that $c(1) = 0$, $c(2) = 3$ and $c(3) = 8$.

ii) We demonstrated that

$$c(n) = 1 + \sum_{a=1}^{n} m(a),$$

where $m(a)$ is the cardinal of the quotient set of

$$E(a) = \{(t,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \,:\, 2 \leq t + k \leq a + 1\}$$

with the equivalence relation defined by

$$(t,k) \approx (r,s) \iff ts = kr.$$

iii) $|\{(t,k) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \,:\, t + k = a + 1,\ \ gcd(n,r) = 1\}| = \varphi(a+1)$ and

$$m(a) = m(a-1) + \varphi(a+1) = c(a) - c(a-1).$$

According into the above result, with $a = n$ and $a = n - 1$ we obtain the recursive form for $c(n)$. Finally, if we sum both terms of the recursive form we get the analitic form.


## REFERENCES

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