Invariant manifolds in some three dimensional piecewise smooth differential systems

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Introduction

Over the last years a renewed interest has appeared in the mathematical community working in differential equations for understanding the dynamical richness of piecewise smooth systems, because these systems are widely used to model processes appearing in electronics, mechanics, economy, etc.

However, the study of the minimal and invariant sets in such systems have been massively restricted to periodic orbits and other objects of higher dimension have been poorly studied.

In this work we present a result which provides a sufficient condition to the existence of bi-dimensional manifolds in a three-dimensional piecewise smooth system and present a simple case of a piecewise linear smooth system presenting invariant cylinder and cones filled up by periodic orbits.

1. Main Results

We consider the differential system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x, \\ \dot{z} &= h(x, y), \end{aligned} \qquad (1)$$

and note that the cylinders $C_\rho = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = \sqrt{\rho}\}$ are invariant sets for system (1) for all positive values of ρ.

By performing a small perturbation of system (1) we verify the existence of cylinder, cones, as well as some compact bi-dimensional manifolds as spheres and torus. The results are obtained using the averaging theory. Moreover, in order to obtain more than invariance but periodic behavior on the cylinders we will assume that function h satisfies

$$h(x, y) = \varphi(x^2 + y^2)\overline{h}(x, y),$$

where $\overline{h}(x, y) = x\,\phi(x^2, y^2) + xy\,\chi(x^2, y^2) + y\,\psi(x^2, y^2)$ with ϕ, χ and ψ are polynomial functions and $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is any continuous function.

Consider now a perturbation of system (1) into the polynomials $g^\pm = (p^\pm, q^\pm, r^\pm)$ given by

$$\begin{aligned} p^\pm(x, y, z) &= \sum_{i+j+k \le m} a^\pm_{ijk} x^i y^j z^k, \\ q^\pm(x, y, z) &= \sum_{i+j+k \le n} b^\pm_{ijk} x^i y^j z^k, \\ r^\pm(x, y, z) &= \sum_{i+j+k \le p} c^\pm_{ijk} x^i y^j z^k, \end{aligned} \qquad (2)$$

with $i, j, k \in \mathbb{N}$ and $a_{ijk}, b_{ijk}, c_{ijk} \in \mathbb{R}$, $\forall i, j, k \in \mathbb{N}$. Consider also the function $g(x, y, z)$ given by

$$\frac{1}{2}(g^+(x, y, z) + g^-(x, y, z)) + \frac{\mathrm{sgn}(y)}{2}(g^+(x, y, z) - g^-(x, y, z)).$$

Of course, the expression of function g depends on the region which we are dealing with. Indeed, consider the codimension one manifold $\mathcal{M}$ of $\mathbb{R}^2$ given by $\mathcal{M} = H^{-1}(0)$, where $H : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is given by $H(x, y, z) = y$. Consider also the sets $\mathcal{M}^+ = \{(x, y, z)\mathbb{R}^3; y \ge 0\}$ and $\mathcal{M}^+ = \{(x, y, z)\mathbb{R}^3; y \le 0\}$. In this case we get $g(p) = g^+(p)$ if $p \in \mathcal{M}^+$ and $g(p) = g^-(p)$ if $p \in \mathcal{M}^-$ for each $p \in \mathbb{R}^3$. Thus the perturbation of system (1) through the non-smooth function g leads to the piecewise smooth system

$$\dot{X}_\varepsilon = f(t, X) + \varepsilon g(X), \qquad (3)$$

where $f(t, X)$ is the vector field of system (1), $X = (x, y, z)$ and ε is a small parameter. We consider the function

$$F(r, z) = \int_0^\pi [p^+(\vartheta)\cos\theta + q^+(\vartheta)\sin\theta]\, d\theta + \int_\pi^{2\pi} [p^-(\vartheta)\cos\theta + q^-(\vartheta)\sin\theta]\, d\theta, \qquad (4)$$

where

$$\vartheta = \vartheta(r, z, \theta) = \left(r\cos\theta, r\sin\theta, z + \int_0^\theta h(\cos s, \sin s)ds\right).$$

Then we have the following results. We denote by $d_B(f, V, b)$ the Brouwer degree of function f in a neighborhood V of the point b.

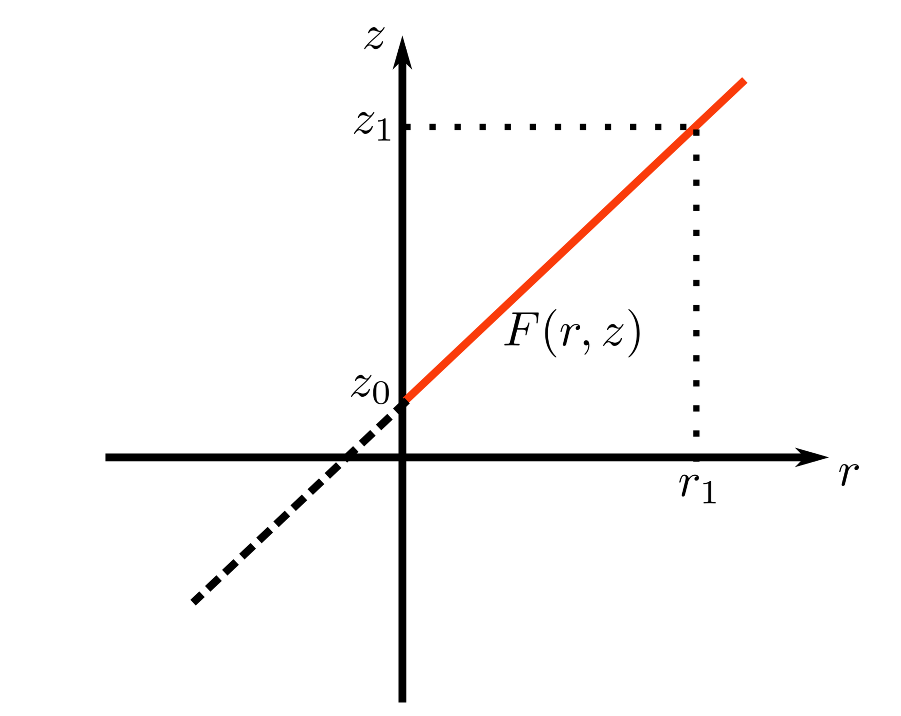
Theorem 1. *Consider system (3) and suppose that function h satisfies $(\partial h/\partial r)(r\cos\theta, r\sin\theta) \equiv 0$. Moreover, assume that for every $z_0 \in \mathbb{R}$, the function $F(r, z)$ has a zero $r_0 = r(z_0)$ such that there exist a neighborhood V of r_0 such that $F(r_0) \neq 0$ for all $r_0 \in \overline{V} \setminus \{r_0\}$ and $d_B(F, V, r_0) \neq 0$. Then, for $|\varepsilon|$ sufficiently small, system (3) has a bi–dimensional manifold $\mathfrak{M} = \mathbb{S}^1 \times \mathcal{L}$, where $\mathcal{L} = graph(F) \cap \mathbb{R}^2_+$ is a curve defined in the rz-plane.*

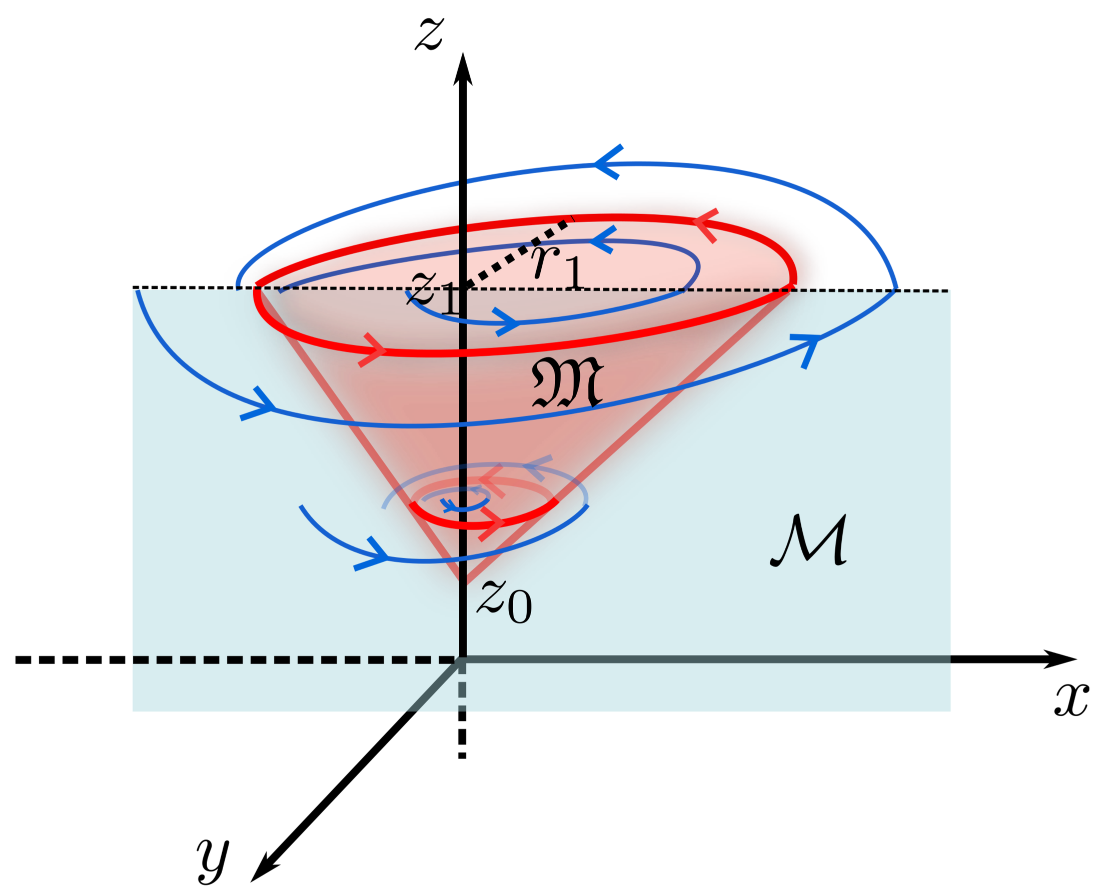
One should note that function F defined in (4) depend on the degrees m, n and p of the perturbation g of system (3). Nevertheless, while Theorem 1 provides a general result which does not depends on the values of m, n and p, next results give the explicit expression of the manifolds $\mathfrak{M}$ depending of these values.

Theorem 2. *Under the same hypotheses of Proposition 1, if $m = n = p = 1$, then $F(r, z)$ is a straight line in the rz-plane which writes $F(r, z) = Ar + Bz + C$, where A, B and C are real constants given by:*

$$\begin{aligned} A &= \frac{\pi}{2}(a^+_{100} + a^-_{100} + b^+_{010} + b^-_{010}), \\ B &= b^+_{001} - b^-_{001}, \\ C &= 2(b^+_{000} - b^-_{000}) + \frac{\pi}{2}(b^+_{001} + b^-_{001}). \end{aligned}$$

Moreover, if $A \neq 0$, then $\mathfrak{M}$ is a cone when $B \neq 0$ and a cylinder when $B = 0$ and $r > -C/A$. If $B = 0$ and $r < -C/A$, then $\mathfrak{M}$ is the trivial manifold $\mathfrak{M} = \emptyset$.





2. Sketch of the proof

Consider system (3). Since the periodic solutions of system (1), that we are perturbing, live on the cylinders C_ρ, we will perform a cylindrical change of coordinates in system (3) by introducing the new variables (z, r, θ) given implicitly by $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. In the new variables (z, r, θ) system (3) writes

$$\begin{aligned} \dot{r} &= \varepsilon\frac{1}{2}\left[\cos\theta[p^+(\vartheta) + p^-(\vartheta)] + \sin\theta[q^+(\vartheta) + q^-(\vartheta)] + \right. \\ &\quad \left. \cos\theta[p^+(\vartheta) - p^-(\vartheta)] + \sin\theta[q^+(\vartheta) - q^-(\vartheta)]\,\mathrm{sgn}(r\sin\theta)\right], \\ \dot{\theta} &= 1 + \varepsilon\frac{1}{2r}\left[\cos\theta[q^+(\vartheta) + q^-(\vartheta)] - \sin\theta[p^+(\vartheta) + p^-(\vartheta)] + \right. \\ &\quad \left. \cos\theta[q^+(\vartheta) - q^-(\vartheta)] + \sin\theta[-p^+(\vartheta) + p^-(\vartheta)]\mathrm{sgn}(r\sin\theta)\right] \\ \dot{z} &= h(r\cos\theta, r\sin\theta) + \varepsilon\frac{1}{2}\left[r^+(\vartheta) + r^-(\vartheta) + (r^+(\vartheta) - r^-(\vartheta))\right. \\ &\quad \left. \mathrm{sgn}(r\sin\theta)\right] \end{aligned} \qquad (5)$$

where $\vartheta = (r\cos\theta, r\sin\theta, z)$.

Now we change the independent variable t of system (5) to the new variable θ and obtain the following equivalent system

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon\frac{1}{2}\left[\cos\theta[p^+(\vartheta) + p^-(\vartheta)] + \sin\theta[q^+(\vartheta) + q^-(\vartheta)] + \right. \\ &\quad \left. \cos\theta[p^+(\vartheta) - p^-(\vartheta)] + \sin\theta[q^+(\vartheta) - q^-(\vartheta)]\mathrm{sgn}(r\sin\theta)\right] + O(\varepsilon^2) \\ &= \varepsilon F_1(\theta, r, z) + O(\varepsilon^2) \\ \frac{dz}{d\theta} &= h(r\cos\theta, r\sin\theta) + \varepsilon\frac{1}{2}\left[r^+(\vartheta) + r^-(\vartheta) + (r^+(\vartheta) - r^-(\vartheta))\right. \\ &\quad \left. \mathrm{sgn}(r\sin\theta)\right] + O(\varepsilon^2) \\ &= h(r\cos\theta, r\sin\theta) + \varepsilon F_2(\theta, r, z) + O(\varepsilon^2), \end{aligned} \qquad (6)$$

where again $\vartheta = (r\cos\theta, r\sin\theta, z)$ and $\mathrm{sgn}(\cdot)$ means the sign function.

From the second equation of system (6), since $(\partial h/\partial r)(r\cos\theta, r\sin\theta) \equiv 0$ we have $h = h(\theta)$. Thus we obtain the solution of this equation, as follows.

$$z(\theta) = z_0 + \int_0^\theta h(\cos s, \sin s)ds + O(\varepsilon).$$

Consequently, replacing this expression into the first equation of system (6) we get

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon F_1\left(\theta, r, z_0 + \int_0^\theta h(\cos s, \sin s)ds + O(\varepsilon)\right) + O(\varepsilon^2) \\ &= \varepsilon F_1\left(\theta, r, z_0 + \int_0^\theta h(\cos s, \sin s)ds\right) + O(\varepsilon^2) \end{aligned} \qquad (7)$$

since F_1 is polynomial. Finally, in order to apply Theorem 3, we observe that system (7) satisfies its hypotheses, once every function of the correspondent vector field is a polynomial and 2π-periodic. For the same reason, item (i) of Theorem 3 holds.

Now we call

$$F(r, z) = \int_0^{2\pi} F_1\left(s, r, z + \int_0^s h(\cos v, \sin v)dv\right) ds.$$

and observe that function F_1 depends on the sign of $r\sin\theta$. Consequently, since $r > 0$, we have

$$\mathrm{sgn}(r\sin\theta) = \mathrm{sgn}(\sin\theta) = \begin{cases} 1, & 0 < \theta < \pi, \\ -1, & \pi < \theta < 2\pi. \end{cases}$$

The discontinuity in cylindrical coordinates is $\mathcal{M} = H^{-1}(0)$, where now $H(r, z, \theta) = \sin\theta$, that is, $\theta = \pi$. Therefore function F writes

$$\begin{aligned} F(r, z) &= \int_0^\pi F_1\left(s, r, z + \int_0^s h(\cos v, \sin v)dv\right) ds \\ &\quad + \int_\pi^{2\pi} F_1\left(s, r, z + \int_0^s h(\cos v, \sin v)dv\right) ds. \end{aligned} \qquad (8)$$

Following the list of hypotheses of Theorem 3, we see that hypothesis (ii) is also true since $F = F(r, z)$ is polynomial, so it is C^1 (which is a sufficient condition for bullet (ii), see remark after Theorem 3). Finally, bullet (iii) of Theorem 3 is also true since $(\partial H/\partial\theta)(r, z, \theta) = \cos\theta$ which does not vanish on $\theta = \pi$.

Then, by applying Theorem 3 for the z_0-parametric system (7), it follows that, for $|\varepsilon| > 0$ sufficiently small and for each $z_0 \in \mathbb{R}$, system (7) has a 2π-periodic orbit $\varphi(\theta; r(z_0))$ such that $\varphi(0; r(z_0)) \to r(z_0)$ when $\varepsilon \to 0$.

Therefore, coming back to system (6), we obtain that for $|\varepsilon| > 0$ sufficiently small we have a family of periodic orbits depending on the parameter z_0. Nevertheless, once $r = \sqrt{x^2 + y^2}$, we obtain a bi-dimensional manifold of codimension one filled by periodic orbits of the family C_ρ of cylinders of system (1).

4. Averaging theory

Theorem 3. *We consider the following discontinuous differential system*

$$x'(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \qquad (9)$$

with

$$\begin{aligned} F(t, x) &= F_1(t, x) + \mathrm{sgn}(h(t, x))F_2(t, x), \\ R(t, x, \varepsilon) &= R_1(t, x, \varepsilon) + \mathrm{sgn}(h(t, x))R_2(t, x, \varepsilon), \end{aligned}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T–periodic in the variable t and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having 0 as a regular value. Denote by $\mathcal{M} = h^{-1}(0)$, by $\Sigma = \{0\} \times D \not\subseteq \mathcal{M}$, by $\Sigma_0 = \Sigma \backslash \mathcal{M} \neq \emptyset$, and its elements by $z \equiv (0, z) \notin \mathcal{M}$.

Define the averaged function $f : D \to \mathbb{R}^n$ as

$$f(x) = \int_0^T F(t, x)dt.$$

We assume the following three conditions.

(i) F_1, F_2, R_1, R_2 and h are locally L–Lipschitz with respect to x;

(ii) for $a \in \Sigma_0$ with $f(a) = 0$, there exist a neighborhood V of a such that $f(z) \neq 0$ for all $z \in \overline{V} \backslash \{a\}$ and $d_B(f, V, a) \neq 0$, (i.e. the Brouwer degree of f at a is not zero).

(iii) If $\partial h/\partial t(t_0, z_0) = 0$ for some $(t_0, z_0) \in \mathcal{M}$, then $(\langle\nabla_x h, F_1\rangle^2 - \langle\nabla_x h, F_2\rangle^2)(t_0, z_0) > 0$.

Then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x(\cdot, \varepsilon)$ of system (9) such that $x(t, \varepsilon) \to a$ as $\varepsilon \to 0$.

We observe that if function $f(z)$ is of class C^1 and the Jacobian $Jf(a)$ is not zero, then $d_B(f, V, a) \neq 0$.

References

[1] C.A. Buzi, R.D. Euzébio and A.C. Mereu *Bifurcation of limit cycles from a non-smooth perturbation of a two-dimensional isochronous cylinder.* http://arxiv.org/pdf/1404.2630.pdf

[2] C.A. Buzi, R.D. Euzébio and A.C. Mereu *Birth of bi-dimension invariant manifolds for piecewise smooth vector fields in $\mathbb{R}^3$.* Work in progress.

[3] J. Llibre, D.D. Noaves and M.A. Teixeira *Averaging methods for studying the periodic orbits of discontinuous differential systems.* http://arxiv.org/pdf/1205.4211. pdf