Advances on the topological characterization of ω-limit sets for analytic flows on the sphere

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Abstract

In 2007, V. Jiménez and J. Llibre presented (in [1]) topological characterizations of the omega-limit sets for analytic flows on (open subsets of) the plane, the sphere and the projective plane. Their proof is based on an auxiliary lemma stating that analytic flows on arbitrary analytic surfaces have the following property: if an orbit meets both sides of an arc of singular points contained in its omega-limit set, then the flow must be equally oriented in both sides.

Despite the validity of the statement of the lemma above for the plane, the sphere and the projective plane, the statement is no longer true for general surfaces. Some examples on proper open subset of the plane are shown. Therefore, the characterizations given in [1] are incomplete; in this poster we present an alternative lemma which allows as to prove the characterization for the case of the whole plane (and the whole sphere and projective plane).

1. Introduction

We start listing some basic and well-known notation and results on analytic plane vector fields (that is, analytic functions mapping open subsets of $\mathbb{R}^2$ into $\mathbb{R}^2$) and their associated flows.

In what follows, the analytic vector field $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ will remain fixed.

Consider the autonomous system

$$\dot{z} = f(z). \tag{1}$$

The zeros of f are also called the **singular points** of (1); the rest of the points are called **regular**. Given any $z \in U$, the system (1) admits a unique maximal solution $\varphi_z : I_z \to U$ satisfying $\varphi_z(0) = z$. The interval I_z is open and the function φ_z is analytic; moreover, the **flow** of (1) defined by (whenever it makes sense) $\varphi(t, z) = \varphi_z(t)$ is analytic as well. The image of φ_z, $\gamma_z = \varphi_z(I_z)$, is called the **orbit of (1) through z**. If z is a singular point of (1), then $I_z = \mathbb{R}$ and $\gamma_z = \{z\}$. If z is regular and φ_z is a periodic function, then we say that γ_z is a **periodic orbit**. The orbits of the system (1) foliate the phase space U, that is, two distinct orbits do not intersect each other.

For any $z \in U$, $I_z = (a_z, b_z)$, the **ω-limit set** of the point z (or of the orbit γ_z) is the set

$$\omega_f(z) = \omega_f(\gamma_z) = \{w \in U : \exists\, t_n \to b_z \text{ such that } \gamma_z(t_n) \to w\}.$$

The previous definition is a good one: if $\gamma_{z'} = \gamma_z$, then $\omega_f(z) = \omega_f(z')$. The *$\alpha$-limit set* $\alpha_f(z) = \alpha_f(\gamma_z)$ is analogously defined replacing $t_n \to b_z$ by $t_n \to a_z$. Points in $\alpha_f(z)$ and in $\omega_f(z)$ are called **limit points** of the orbit γ_z.

An orbit is said to be **nonrecurrent** if it does not meet with any of its limit points. The only nonrecurrent orbits for continuous flows on the plane are the trivial ones: the periodic and the singular orbits.

One of the landmarks of bidimensional qualitative theory of differential equations is the famous Poincaré-Bendixson theorem. It deals with the asymptotical behaviour of the orbits of (1).

Theorem 1 (The Poincaré-Bendixson theorem) *Let $z \in U$ and suppose that the set $\gamma_z^+ = \{\varphi_z(t) : t \geq 0\}$ (respectively $\gamma_z^- = \{\varphi_z(t) : t \leq 0\}$) is contained in a compact subset of U. Then either the set $\omega_f(z)$ (respectively $\alpha_f(z)$) contains some singular point or it is a periodic orbit itself.*

2. Omega-limit sets for analytic flows on the plane

A topological space A is said to be a **cactus** if it is a simply connected union of finitely many disks (notice that each pair of these disks can have at most one common point). We say that $A \subset \mathbb{R}^2$ is a **half-plane** if both A and $\mathbb{R}^2 \setminus \operatorname{Int} A$ are homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$. We say that $A \subset \mathbb{R}^2$ is a **chain** if there are disks $\{D_i\}_{i\in\mathbb{N}}$ such that: $A = \bigcup_{i\in\mathbb{N}} D_i$, every bounded set of $\mathbb{R}^2$ intersects finitely many disks D_i and for any i and j either $D_i \cap D_j$ consists of exactly one point (if $|i - j| = 1$) or $D_i \cap D_j = \emptyset$ (otherwise).

Theorem 2 (see [1, The $\mathbb{R}^2$-analytic theorem, pp. 680–681]) *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic vector field and consider the omega-limit set $\Omega = \omega_f(z)$ for some $z \in \mathbb{R}^2$. Then $\Omega = \operatorname{Bd} A$, with A being*

(a) *the empty set;*

(b) *a single point;*

(c) *a cactus;*

(d) *the union of a circle C and finitely many pairwise disjoint cactuses, each of them contained in the disk enclosed by C and intersecting C at exactly one point;*

(e) *a union of countably many cactuses, half-planes and chains, which are pairwise disjoint except that each cactus intersects either one of the half-planes or one of the chains at exactly one point; moreover, every bounded set of $\mathbb{R}^2$ intersects finitely many of these sets.*

Conversely, for every set $A \subset \mathbb{R}^2$ as in (a)–(e) and $\Omega = \operatorname{Bd} A$, there are an analytic vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h(\Omega)$ is the ω-limit set for some orbit of the flow associated to f.

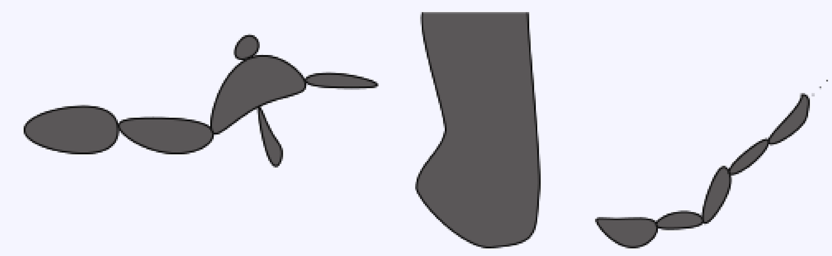

Figure 1: A cactus (left), a half-plane (center) and a chain (right).

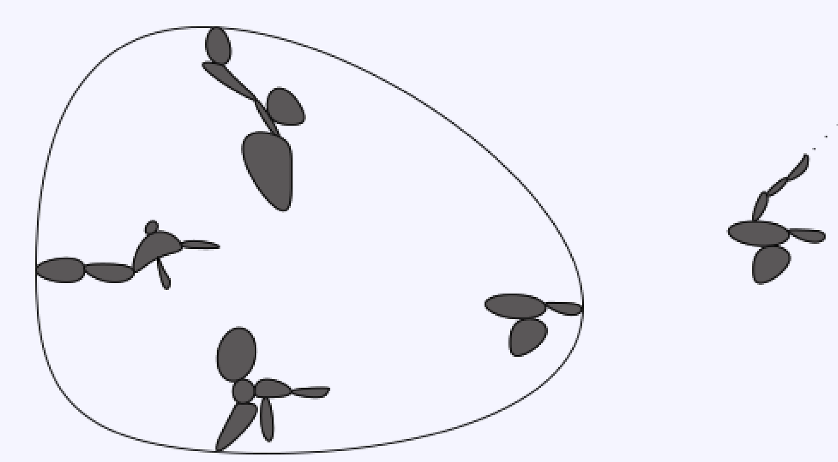

Figure 2: Some possible omega-limit sets on the plane.

3. What is it wrong on the proof by Llibre and Jiménez?

The transcendental property (for proving the theorem above) about the ω-limit sets for anlytic flows defined on the whole plane is that they are *tight generalized graphs* (we shall recall the definitions below). This fact is stated (but wrongly proved) in [1][Proposition 4.9]. If one accepts this property then all what is said in section 5 of [1] is perfectly correct and the desired characterization is deduced. Therefore, all one has to do is to give a proper proof of [1][Proposition 4.9] for the whole plane.

We shall start recalling the definitions and concepts needed to understand properly what [1][Proposition 4.9] says.

Given any positive integer $r \in \mathbb{Z}^+$, we say that a topological space is an **r-star** if it is homeomorphic to $S_r = \{z \in \mathbb{C} : z^r \in [0, 1]\}$.If X is an r-star and $h : S_r \to X$ is an homeomorphism, then the image of the origin under h is called a **vertex of the star**. The vertex of a star is uniquely defined except in the cases $r = 1, 2$, when X is just a closed arc and the vertexes are, respectively, its endpoints (for $r = 1$) or its interior points (for $r = 2$).

A subset G of the plane is said to be a **generalized graph** if any of its points has a neighbourhood which is a star. Take a general graph G and consider its one-point compactification, G_∞. G is called a **tight generalized graph** if G_∞ is a topological circle or given any open arc $A \subset G_\infty$ with all its points being vertexes of 2-stars, then $G_\infty \setminus A$ is connected.

Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic vector field and consider the omega-limit set $\Omega = \omega_f(z)$ for some $z \in U$. Then Ω is a generalized graph.

Theorem 3 (see [1, Theorem 4.9]) *Let U be ~~(either a proper open subset of $\mathbb{R}^2$ or)~~ the whole $\mathbb{R}^2$ and $f : U \to \mathbb{R}^2$ be an analytic vector field. Then the ω-limit set of any orbit of the flow associated with f is not only a generalized graph but also a tight generalized graph.*

Theorem above is false in its more general statement (see counterexamples below) but it is true when we reduce ourselves to the case of the whole plane. The essential property about the limit sets for **analytic** flows on the plane which allows us to correct the proof for theorem above (once the correction in the statement in [1] has been taken into account) it is the following one.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic vector field and $u \in \mathbb{R}^2$ a point whose orbit is nonrecurrent. Let $A \subset \omega_f(u)$ be an open arc of singular points of f and take a point $w \in A$. If D is a disk neighbouring w such that $D \cap \omega_f(u) = D \cap A$ then one, and only one, of the components of $\operatorname{Int} D \setminus A$ meets with the orbit through u as long as D is taken sufficiently small.

This remark is only true for the case of the whole plane (but again not for its proper open subsets - see examples below). The critical property that is missed in [1], and that we have found as the key to amend the gap there, is the following obstruction for the sets of zeros of analytic functions on open subsets of the plane.

Theorem 4 (see [2, Theorem 3.1]) *Let $U \subset \mathbb{R}^2$ be open and connected and $f : U \to \mathbb{R}$ be an analytic function. Let $C = \{z \in U : f(z) = 0\}$ be the set of zeros of f. Then either $C = U$ or given any non-isolated point of C, $z \in C$, there exists a neighbourhood V of z and an $n \in \mathbb{Z}^+$ such that $V \cap C$ is a $2n$-star with vertex z.*

4. Counterexamples to [1, Theorem 4.9]

We have found some examples of omega-limit sets on some proper subsets of the plane which contradicts the general statement of [1, Theorem 4.9]. We have found explicit expression for analytical vector fields on the plane minus two or one point which have orbits whose behaviour is as we show in the following two figures.

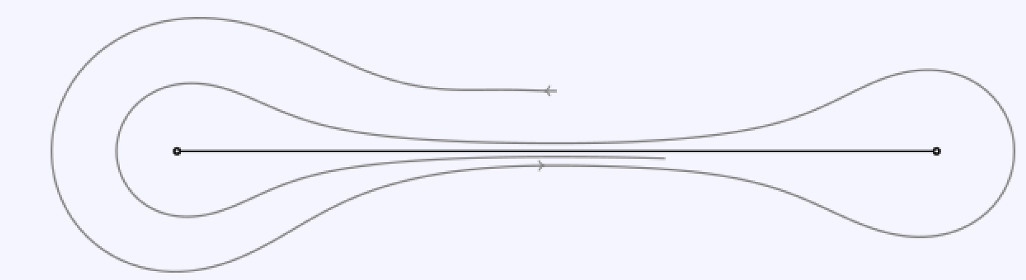

Figure 3: Possible ω-limit set on the plane minus two points.

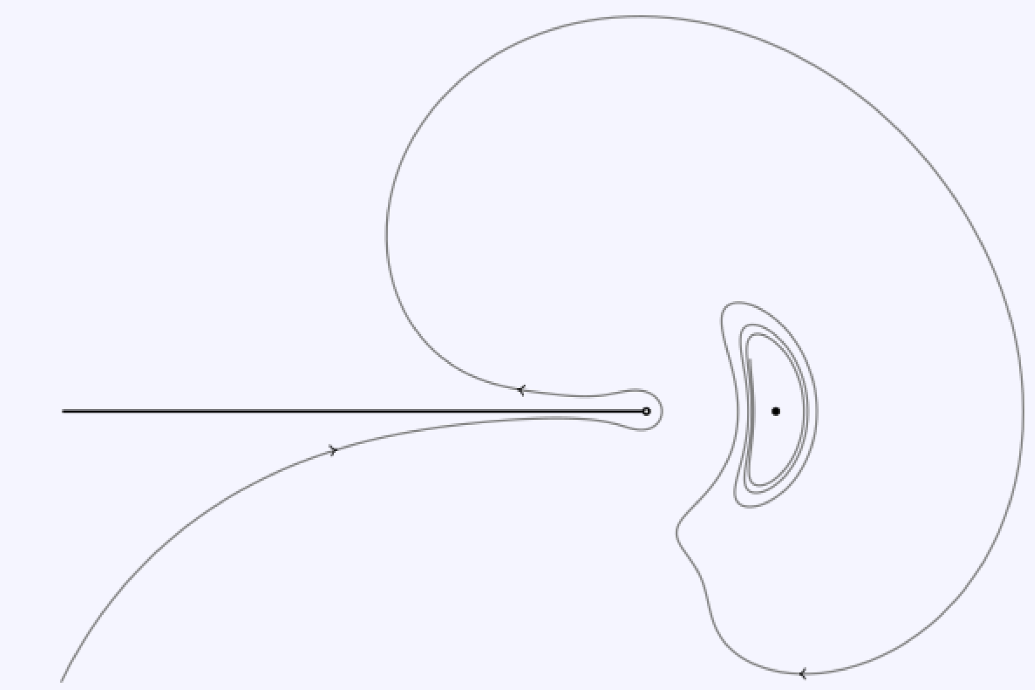

Figure 4: Possible ω-limit set on the plane minus one point.

5. ω-limit sets for analytic flows on proper subset of the plane

In section 7 of [1], the authors investigate the nature of the ω-limit sets of analytic vector fields on proper subsets of the plane and the projective plane. In [1, Theorem 7.1] a characterization of those limit sets is presented. In this case we could not success on amending the proof of that result: the examples above are themselves counterexamples for that characterization (they are not among the possible cases listed there).

Currently we are trying to find a correct characterization for the ω-limit sets on proper subset of the plane (and the projective plane). Once this problem is closed, we shall move to study ω-limit sets on the torus. By far, a more interesting (and difficult) analytic surface (as far as research is concerned): on the torus there can be non-trivial nonrecurrent orbits which we expect to produce a richer classification of its ω-limit sets. Besides, we recall that for the case of the whole plane, sphere and projective plane, the result [1, Theorem 4.9] (once the correction above has been taken into account) is the key to give the characterization of their ω-limit sets. We have also found a counterexample of that statement for the case of the torus so we should find a different tool to proceed in this case.

References

[1] V. Jiménez and J. Llibre, A topological characterization of the ω-limit sets for analytic flows on the plane, the sphere and the projective plane, *Adv. Math.* **206** (2007), 677–710.

[2] J.G. Espín and V. Jiménez, Local topological structure of analytic sets on the plane, *to appear in Appl. Math. Inf. Sci.* (2014).

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