1. Introduction

For fixed but arbitrary integer $k \geq 1$, we aim at analytic understanding of

1. bifurcation of the separatrix skeleton for $X(m)$ in function of $m > 0$;
2. rule of separatrix skeleton in bifurcation of limit cycles;
3. Hilbert’s 16th Problem for $X(m)$.

Note: $(X(m))_{m \geq 0}$ is not a semi-complete family of rotated vector fields. There are three singularities: a nilpotent monodromic singularity and two symmetric hyperbolic saddles $p_{\pm}$ that move with $m \rightarrow 0$.

1. Known results from [3] for $k \geq 1$:
   1. for $m > m(k)$ the origin is a nilpotent attracting focus and for $m > m(k)$ it is a repelling focus, with $m(k) \equiv (2k+1)!/(4k+1)!$.
   2. Global phase portraits of $X(m)$ up to “similarity” of 2-saddle cycle.

2. Separatrix skeleton for $k \geq 1$

Let $k \geq 1$. The separatrix skeleton of $X(m)$ is the union of the singularities and separatrices of $X(m)$.

Theorem 1. [1] For $m < m(k)$ the origin is a global attractor of $X(m)$. For increasing $m \rightarrow m(k)$, the separatrix skeleton of $X(m)$ undergoes a separatrix bifurcation passing through a unique parameter value $m(k)$,

and $X(m)$ has no limit cycles.

3. Indefinite rotated vector fields

Lemma. For $m > 0$ the vector field $X(m)$ is topologically equivalent to $X_{k,R}^0$.

The family $(X_{k,R}^0)_{m \geq 0}$ is a semi-complete family of indefinite rotated vector fields, that is positively rotated in $\mathbb{R}^2$.

Localization of separatrices for $X_{k,R}^0$.

Monotonic movement of separatrices for $X_{k,R}^0$ with increasing $m > 0$.

Proposition. Let $k \geq 1$, $m > 0$. Polycycles and limit cycles of $X_{k,R}^0$ are created at the origin $p_{\pm}$.

Compactification of the family $(X_{k,R}^0)_{m \geq 0}$.

2. Small $m$

Theorem 2. [1] There exists $m(k)$ such that $X(m)$ does not have limit cycles of polycycles for $m > m(k)$. Furthermore, for $0 < m < m(k)$, this theorem determines the global phase portrait of $X(m)$, uniquely up to topological equivalence.

Proof. Define $V_{(x,y)}(x,y) = 2m^{2(k+1)}x^2 + y^4$ and $M(x,y,m) = (X_{k,R}^0(x,y),V_{(x,y)}(x,y) - m\eta_x(x,y)\Delta X_{k,R}^0(x,y),y)$.

For small enough $M(x,y,m) \geq 0$. Besides, the origin is the only maximal invariant set contained in $M(x,y,m) = 0$. Then by a generalization of Bendixson-Dulac criterion there exists at most one limit cycle or polycycle, and both cannot exist. Thus stability analysis of the origin/polecycle leads to the absence of limit cycles for $m$ small enough.

5. Large $m$

Let $k \geq 1$. For $m > 0$ the vector field $X(m)$ is topologically equivalent to

$X_{k,R}^1$ and $X_{k,R}^2$

Theorem 3. [1] For $m \rightarrow m(k)$, $X(m)$ exhibits a 2-saddle cycle, that is broken for $m < m(k)$.

6. Center/Focus Problem

Theorem 4. [1] For all $m > m(k)$, $X(m)$ has the bifurcation of small amplitude limit cycles for $m \rightarrow m(k)$ for $k \geq 2$ is a matter of a work in progress (in collaboration with Ilker Çolak).

7. Hilbert’s 16th Problem

Hilbert’s 16th Problem asks, if it exists, for the maximal number of limit cycles of a planar polynomial vector field $x = P(x,y), y = Q(x,y)$, only depending on the degree $n$ of the polynomials $P(x,y)$.

Theorem 5. [1] For all $m > m(k)$, the number of limit cycles of $X(m)$ in the global plane is bounded by $N(m)$. Furthermore, it is necessary that $N(m) \leq 1$.

8. Particular case $k = 1$

Theorem 8. [1] There exists a unique $5!/2000 < m(1) < 1/7$ such that the bifurcation diagram of global phase portraits of $X(m)$ in function of $m$ is given in next figure.

References