Separatrix skeleton and limit cycles in some 1-parameter families of planar vector fields

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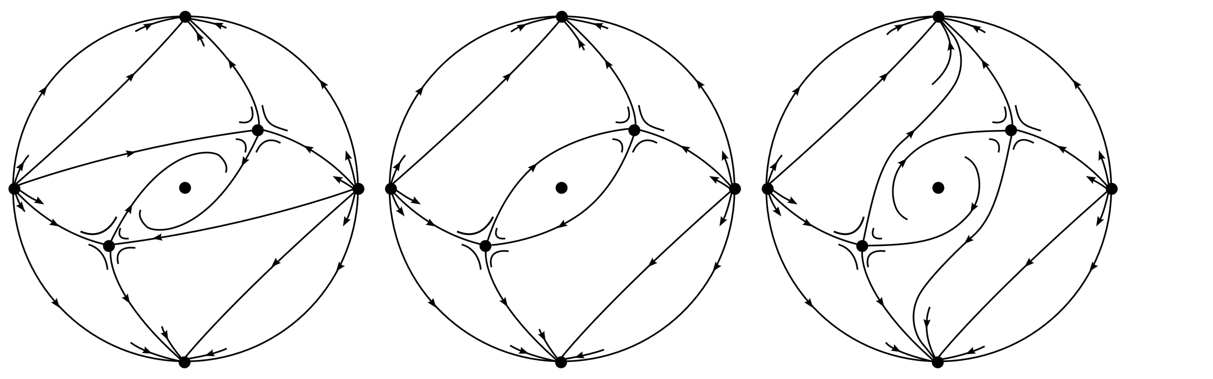
Abstract

The dynamical mechanism underlying the bifurcation of separatrices is revealed for the 1-parameter family $(X_m^k)_{m\in\mathbb{R}}$, where

$$X_m^k \leftrightarrow \dot{x} = y^3 - x^{2k+1}, \dot{y} = -x + my^{4k+1}, \text{ for } m \in \mathbb{R}$$

and $k \geq 1$ an arbitrary but fixed integer. Then the separatrix skeleton for X_m^k for varying $m > 0$ is determined by the figure on the right. Recall that the *separatrix skeleton* is the union of singularities and separatrices. Adding *limit cycles* to it, it is to say the isolated periodic orbits, one obtains the *extended separatrix skeleton.* By the Theorem of Markus, Neumann and Peixoto the extended separatrix skeleton completely determines the topological structure of continuous planar vector fields having only isolated singular points. In this way, proving the absence of limit cycles for sufficiently small and sufficiently big m, in these cases we obtain the global phase portraits of X_m^k for arbitrary $k \geq 1$. Furthermore it allows to answer the nilpotent Center/Focus Problem and Hilbert's 16th Problem for $(X_m^k)_{m\in\mathbb{R}}$. Besides, applying the result on limit cycles of X_m^1 from [3], the bifurcation diagram of global phase portraits is completed for $(X_m^1)_{m\in\mathbb{R}}$.



$m < m_C(k)$ $\quad$ $m = m_C(k)$ $\quad$ $m > m_C(k)$

1. Introduction

- For fixed but arbitrary integer $k \geq 1$ we aim at analytic understanding of
  1. bifurcation of the separatrix skeleton for X_m^k in function of $m > 0$;
  2. rule of separatrix skeleton in bifurcation of limit cycles;
  3. Hilbert's 16th Problem for $(X_m^k)_{m\in\mathbb{R}}$.
- Note: $(X_m^k)_{m>0}$ is not a semi-complete family of rotated vector fields. There are three singularities: a nilpotent monodromic singularity and two symmetric hyperbolic saddles $p_\pm$ that move with $m > 0$.
- Known results from [3] for $k \geq 1$:
  1. for $m < m_S(k)$ the origin is a nilpotent attracting focus and for $m > m_S(k)$ it is a repelling focus, with $m_S(k) \equiv (2k+1)!!/(4k+1)!!!!$.
  2. Global phase portraits of X_m^1 up to 'unicity' of 2-saddle cycle.

2. Separatrix skeleton for $k \geq 1$

Let $k \geq 1$. The *separatrix skeleton* of X_m^k is the union of the singularities and separatrices of X_m^k.

Theorem 1. *[1] For $m \leq 0$ the origin is a global attractor of X_m^k. For increasing $m > 0$, the separatrix skeleton of X_m^k undergoes a separatrix bifurcation passing through a unique parameter value $m_C(k) > 0$, giving rise to three separatrix skeletons as drawn in the right upper corn.*

For $m = m_C(k)$ the phase portrait of X_m^k exhibits a 2-saddle cycle, that is broken for $m \neq m_C(k)$.

Proof. Techniques based on Poincaré-Bendixson Theorem, Bendixson-Dulac Theorem (see [3]), rotated vector fields (see Section 3 and [4]), strip flows, continuous dependence on parameter and initial value, compactification, desingularisation, Poincaré return map (see [2]). □

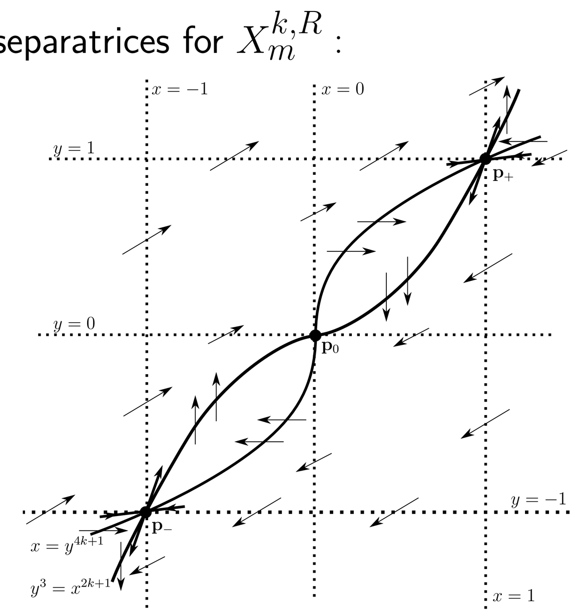
3. Indefinite rotated vector fields

Lemma. *For $m > 0$ the vector field X_m^k is topologically equivalent to*

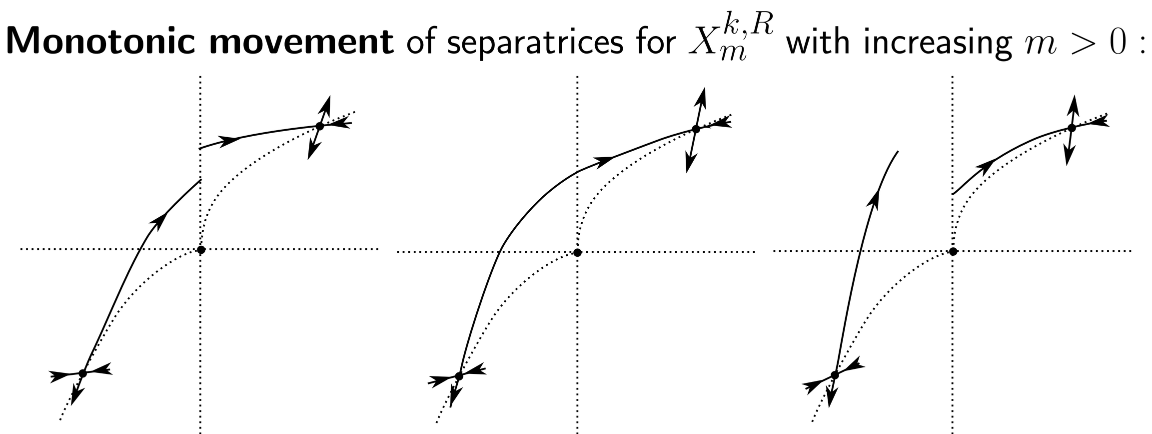
$$X_m^{k,R} \leftrightarrow \dot{x} = y^3 - x^{2k+1}, \dot{y} = m^{\frac{1}{k+1}}(-x + y^{4k+1}).$$

The family $(X_m^{k,R})_{m>0}$ is a semi-complete family of indefinitely rotated vector fields, that is positively rotated in $(y^3 - x^{2k+1})(y^{4k+1} - x) \geq 0$.

Localization of separatrices for $X_m^{k,R}$:

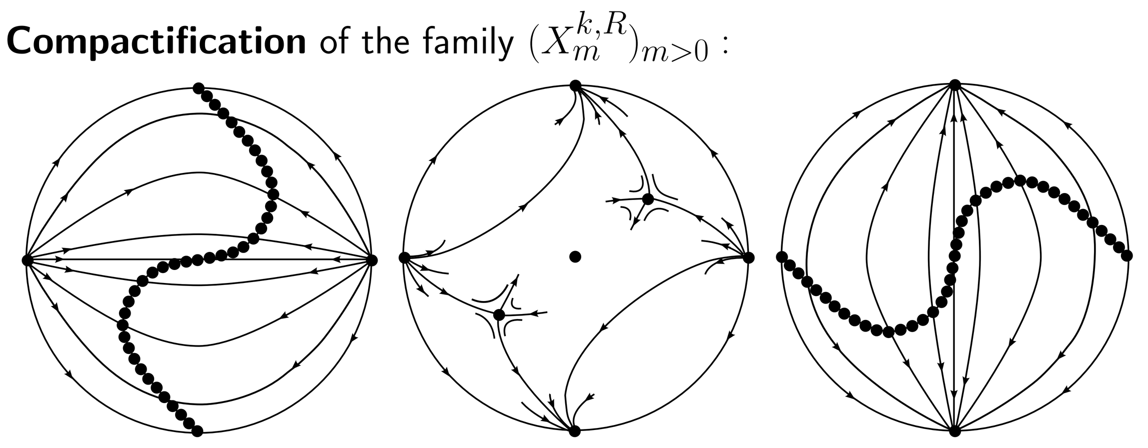


Monotonic movement of separatrices for $X_m^{k,R}$ with increasing $m > 0$:



Proposition. *Let $k \geq 1, m > 0$. Polycycles and limit cycles of $X_m^{k,R}$ are contained in the cube $C \equiv [-1,-1] \times [-1,1]$.*

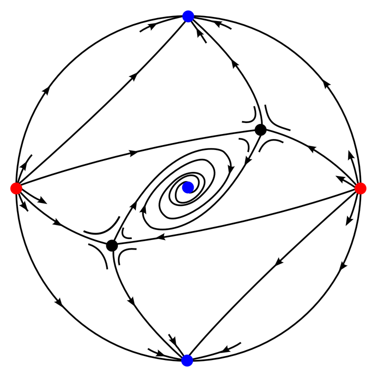
Compactification of the family $(X_m^{k,R})_{m>0}$:



$m = 0$ $\quad$ $m > 0$ $\quad$ $m = \infty$

4. Small m

Theorem 2. *[1] There exists $m_0(k) > 0$ such that X_m^k does not have limit cycles nor polycycles for $m < m_0(k)$. Furthermore, for $0 < m < m_0(k)$, next figure determines the global phase portrait of X_m^k uniquely up to topological equivalence.*



Proof. Define $V_m(x,y) = 2m^{1/(k+1)}x^2 + y^4$ and

$$M(x,y,m) = \langle X_m^{k,R}(x,y), \nabla V_m(x,y)\rangle - \frac{2}{2k+1}V_m(x,y)\triangle X_m^{k,R}(x,y).$$

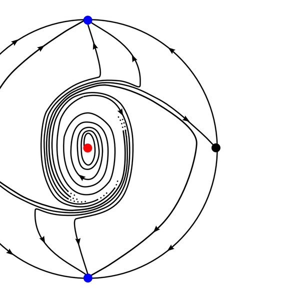
For m small enough $M(x,y,m) \geq 0$. Besides, the origin is the only maximal invariant set contained in $M(x,y,m) = 0$. Then by a generalization of Bendixson-Dulac criterion there exists at most one limit cycle or polycycle, and both cannot coexist. Then stability analysis of the origin/polycycle leads to the absence of limit cycles for m small enough. □

5. Large m

Let $k \geq 1$. For $m > 0$ the vector field X_m^k is topologically equivalent to

$$Y_\eta^{k,S} \leftrightarrow \dot{x} = y^3 - \eta x^{2k+1}, \ \dot{y} = -x + y^{4k+1}, \text{ where } m\eta = 1.$$

Lemma. *$(Y_\eta^{k,S})_{0<\eta\leq\eta_0}$ can analytically be extended to a compact analytic family $(\hat{Y}_\eta^{k,S})_{0\leq\eta\leq\eta_0}$ on the Poincaré disc. The global phase portrait of $\hat{Y}_0^{k,S}$ exhibits a global repeller:*



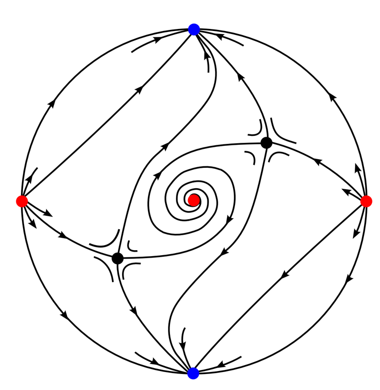
Proof. By compactification on the Poincaré disc and Lypaunov stability's theorem considering the sign of derivative of $V(x,y) = 2x^2 + y^4$ along orbits of $Y_0^{k,S}$. □

Proposition 3. *No large nor medium amplitude limit cycles for $\eta \downarrow 0$. For every open ball $\mathcal{B}_0$ centered at the origin of $\mathbb{R}^2$, there exists $\eta_0 > 0$ such that for $0 \leq \eta \leq \eta_0$ there are no limit cycles of $\hat{Y}_\eta^{k,S}$ outside $\mathcal{B}_0$.*

Proposition 4. *No small amplitude limit cycles for $\eta \downarrow 0$. There exists an open ball $\mathcal{B}_0$ centered at the origin of $\mathbb{R}^2$ and there exists $\eta_1 > 0$ such that for $0 \leq \eta < \eta_1$ there are no limit cycles of $\hat{Y}_\eta^{k,S}$ starting in $\mathcal{B}_0$.*

Proof. For η_0 sufficiently small the only limit periodic set in the family $(Y_\eta^{k,S})_{|\eta|\leq\eta_0}$ is the nilpotent singularity in the origin. To study the cyclicity at the origin, one considers the Poincaré map of first return using coordinates near the origin from a quasi-homogenous blow up.

Theorem 5. *[1]There exists $m_\infty(k) > m_0(k)$ such that X_m^k has no limit cycles nor polycycles for $m > m_\infty(k)$. Furthermore, for $m > m_\infty(k)$, next figure determines the global phase portrait of X_m^k uniquely up to topological equivalence.*



Proof of Theorem 5. By Propositions 3 and 4 using Roussarie compactification-localization technique from [5]. □

6. Center/Focus Problem

A singularity is called a *center* if it has a punctured neighborhood full of concentric non-isolated periodic orbits.
The *Center/Focus Problem* aims at deciding whether a singularity is a center or a focus.

Theorem 6. *[1] Let $k \geq 1$ and $m = m_S(k)$. The nilpotent singularity of $X^k_{m_S(k)}$ at the origin is a focus and not a center.*

Proof. By hyperbolicity of 2-saddle cycle for $m = m_S(k)$. □

The stability of the origin for $m = m_S(k)$ and the bifurcation of small amplitude limit cycles for $m \to m_S(k)$ for $k \geq 2$ is matter of a work in progress (in collaboration with Ilker Çolak).

7. Hilbert's 16th Problem

Hilbert's 16th Problem asks, if it exists, for the maximal number of limit cycles of a planar polynomial vector field

$$\dot{x} = P_n(x,y), \dot{y} = Q_n(x,y),$$

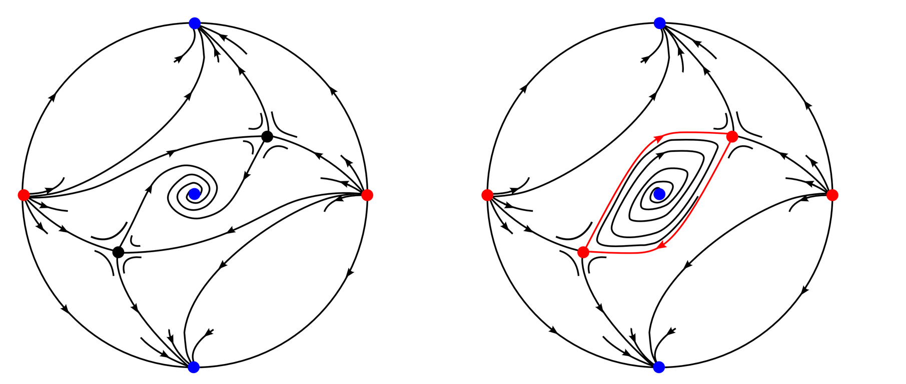
only depending on the degree n of the polynomials P_n, Q_n. This problem still is open, even for $n = 2$.

Theorem 7. *[1] For all $k \geq 1$ there exists $\mathcal{H}(k) < \infty$ such that for all $m \in \mathbb{R}$ the number of limit cycles of X_m^k in the global plane is bounded by $\mathcal{H}(k)$. Furthermore, it is necessary that $\mathcal{H}(k) \geq 1$.*

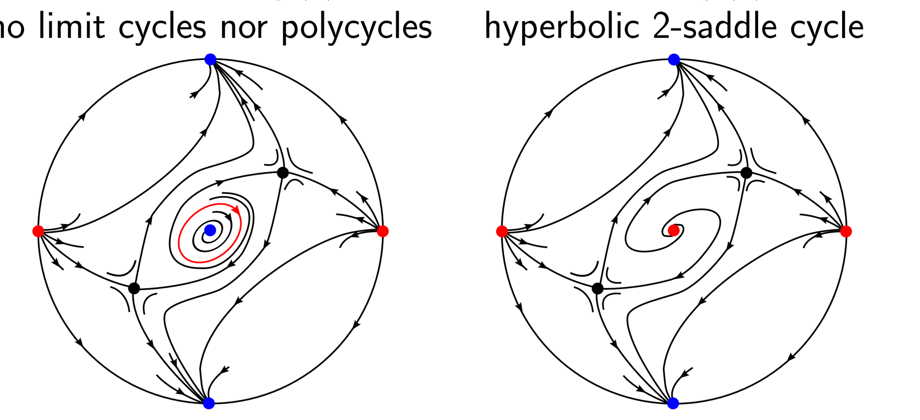
Proof. By Roussarie compactification-localization technique (see [5]) and Theorems 2, 5 and 6. □

8. Particular case $k = 1$

Theorem 8. *[1,3] There exists a unique $547/1000 < m_C(1) < 3/5$ such that the bifurcation diagram of global phase portraits of X_m^1 in function of m is given in next figure.*



$0 < m < m_C(1)$ — no limit cycles nor polycycles $\quad$ $m = m_C(1)$ — hyperbolic 2-saddle cycle



$m_C(1) < m < 3/5$ — 1 limit cycle, no polycycles $\quad$ $m \geq 3/5$ — no limit cycles nor polycycles

Proof. Existence of $m_C(1)$ in [3] and its uniqueness by Theorem 1. Bifurcation diagram is completed by Poincaré-Bendixson Theorem. □

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