Generalized rational first integrals of analytic differential systems

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Outline of the talk

- Background of the problem.
- Statement of the main results.
- Sketch proof of the main results.
- Some results on related problems.
Background of the problem

Given a $C^\omega$ system of differential equations

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n \quad \text{an open set.}$$

It is a classical problem:

- to determine the existence of analytic or generalized rational first integrals in $\Omega$,

or

- to determine the existence of analytic or generalized rational first integrals in a neighborhood of some singularity in $\Omega$. 

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Generalized rational first integrals of analytic systems
Recall that

- an **analytic first integral** is a first integral which is an analytic function.
- a **generalized rational first integral** is a first integral which is a ratio of two analytic functions.

Remark that

- A **generalized rational first integral** with its denominator nonvanishing is an **analytic first integrals**.
- A **generalized rational first integral** with its **numerator** and **denominator** both polynomials is a **rational first integral**.
Remark:

- This problem appears more than one hundred year,
- but the progress is very slow.
The history of the study on the problem:

Equivalent characterization of analytic first integrals (AFI):

- planar nondegenerate centers having AFIs by Poincaré
- planar nondegenerate isochronous centers having AFIs by Poincaré
- planar hyperbolic saddles having AFIs by Moser [Comm. Pure Appl. Math. 1956]
- planar nilpotent centers having AFIs by Chavarriga et al [Ergodic Theory Dynam. Systems 2003], partial results
any dimensional analytic differential systems around nondegenerate singularity which is analytic integrable by Zhang [JDE 2008]

any dimensional analytic differential systems around singularity with non-zero eigenvalues which is analytic integrable by Zhang [preprint]
Rational first integrals (RFI):

- the equivalent characterization, e.g.
  - planar elementary singularities having a generalized rational first integral by Li, Llibre and Zhang [BSM 2001]

- the Darboux integrability
  - the existence of rational first integrals with sufficient number of invariant algebraic surface (curves) by
    - Jouanolou [LNM 1979]
    - Christopher and Llibre [ADE 2000, QTDS 1999]
Necessary conditions on existence of analytic first integrals

Consider analytic differential systems

\[ \dot{x} = Ax + f(x), \quad (1) \]

with \( f(x) = O(|x|^2) \) analytic.

Let

\[ \lambda = (\lambda_1, \ldots, \lambda_n) \]

be the \( n \)--tuple of eigenvalues of \( A \).
Definition: the eigenvalues $\lambda$ satisfy

- $\mathbb{Z}^+-$resonant condition if
  $$\langle \lambda, k \rangle = 0, \quad \text{for some } k \in (\mathbb{Z}^+)^n, \quad k \neq 0,$$
  where $\mathbb{Z}^+$ is the set of nonnegative integers.

- $\mathbb{Z}-$resonant condition if
  $$\langle \lambda, k \rangle = 0, \quad \text{for some } k \in \mathbb{Z}^n, \quad k \neq 0,$$
  where $\mathbb{Z}$ is the set of integers.
Existence of analytic first integrals associated with the eigenvalues of $A$:

Poincaré is the first one studying the relation between the existence of analytic first integrals and resonances:

**Poincaré Theorem**

Assume that

- the eigenvalues $\lambda$ of $A$ do not satisfy any $\mathbb{Z}^+$–resonant conditions.

Then

- system (1) has no analytic first integrals in $(\mathbb{C}^n, 0)$. 

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Generalized rational first integrals of analytic systems
Remark:

- $\lambda$ do not satisfy resonant conditions implies that $A$ has no zero eigenvalues.

The Poincaré’s result was extended by

- Li, Llibre and Zhang [ZAMP 2003]

  to the case that $\lambda$ admits one zero eigenvalue.
Li, Llibre and Zhang’s Theorem [ZAMP, 2003]

Assume that

- $A$ has one zero eigenvalue, and
- the others are not $\mathbb{Z}^+-$resonant.

Then

- in the planar case, system (1) has an analytic first integral if and only if the origin is a non–isolated singularities
- in the higher dimensional case, system (1) has a formal first integral if and only if the origin is a non–isolated singularities
In 2008, the Poincaré’s result was further extended to the case of several first integrals:

**Chen, Yi and Zhang’s Theorem [JDE, 2008]**

The number of functionally independent analytic first integrals of system (1) **does not exceed** the maximal number of linearly independent elements of \( \{ k \in (\mathbb{Z}^+)^n : \langle k, \lambda \rangle = 0, k \neq 0 \} \).
In 2007, the Poincaré’s result was extended to the existence of rational first integrals:

Shi’s Theorem [JMAA, 2007]

- If system (1) has a rational first integral, then the eigenvalues $\lambda$ of $A$ satisfy a $\mathbb{Z}$–resonant condition.

- In other words, if $\lambda$ do not satisfy any $\mathbb{Z}$–resonant condition, then system (1) has no rational first integrals in $(\mathbb{C}^n, 0)$. 
Statement of our new results

**Theorem 1**

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be the eigenvalues of \( A \). Then

- the number of functionally independent generalized rational first integrals of system (1) in \((\mathbb{C}^n, 0)\) is at most the dimension of the minimal vector subspace of \( \mathbb{R}^n \) containing the set \( \{k \in \mathbb{Z}^n : \langle k, \lambda \rangle = 0, k \neq 0\} \).

This result was obtained by Cong, Llibre and Zhang in [JDE 2011]
Remark:

This result has improved

- the Poincaré’s one
- the Chen, Yi and Zhang’s one, and
- the Shi’s one

by studying the existence of more than one functionally independent rational first integrals.
Remark:

- The **methods** used in the proof
  - of the Poincaré’s result,
  - of the Chen, Yi and Zhang’s one, and
  - of Shi’s one

are not enough to study the existence of more than one functionally independent generalized rational first integrals.

- We will **use a different approach** to prove Theorem 1.
Let

- \( \mathbb{C}(x) \) be the field of rational functions in the variables \( x \),
- \( \mathbb{C}[x] \) be the ring of polynomials in \( x \).

**Definition**

- functions \( F_1(x), \ldots, F_k(x) \in \mathbb{C}(x) \) are *algebraically dependent* if there exists a complex polynomial \( P \) of \( k \) variables such that \( P(F_1(x), \ldots, F_k(x)) \equiv 0 \).
First step: algebraically and functionally independent

Lemma 1
The functions $F_1(x), \ldots, F_k(x) \in \mathbb{C}(x)$ are algebraically independent if and only if they are functionally independent.

Remark
Lemma 1 has a relation in some sense with the result of Bruns in 1887, which stated that

- if a polynomial differential system of dimension $n$ has $l (1 \leq l \leq n - 1)$ independent algebraic first integrals, then it has $l$ independent rational first integrals.
The main idea of the proof of Lemma 1 follows from that of Lemma 9.1 of Ito [Comment. Math. Helvetici, 1989].
Sufficiency By contradiction, if \( F_1(x), \ldots, F_k(x) \) are algebraically dependent,

\[
\exists \text{ a complex polynomial } P(z_1, \ldots, z_k) \text{ of minimal degree such that } P(F_1(x), \ldots, F_k(x)) \equiv 0.
\]

Minimal means that for any polynomial \( Q(z_1, \ldots, z_k) \) of degree less than \( \deg P \) we have that \( Q(F_1(x), \ldots, F_k(x)) \neq 0 \).

\[
\frac{\partial (F_1(x), \ldots, F_k(x))}{\partial (x_1, \ldots, x_n)} \left( \frac{\partial P}{\partial z_1} (F_1, \ldots, F_k), \ldots, \frac{\partial P}{\partial z_1} (F_1, \ldots, F_k) \right)^T \equiv 0.
\]

\[
\frac{\partial (F_1(x), \ldots, F_k(x))}{\partial (x_1, \ldots, x_n)} \left( \frac{\partial P}{\partial z_1} (F_1, \ldots, F_k), \ldots, \frac{\partial P}{\partial z_1} (F_1, \ldots, F_k) \right)^T \equiv 0.
\]

\[
F_1(x), \ldots, F_k(x) \text{ are functionally dependent}
\]

contradiction with the assumption
Necessary

Using the field extension and the expression of the derivatives on $\mathbb{C}(x)$.

1. $F_1, \ldots, F_k$ are algebraically independent,

$\Downarrow$

$\mathbb{C}(F_1, \ldots, F_k)$ is a separably generated and finitely generated field extension of $\mathbb{C}$ of transcendence degree $k$

$\Downarrow$

there exist $k$ derivations $D_r (r = 1, \ldots, k)$ on $\mathbb{C}(F_1, \ldots, F_k)$ satisfying

$$D_r F_s = \delta_{rs},$$
2. Since $\mathbb{C}(x)$ is a finitely generated field extension of $\mathbb{C}(F_1, \ldots, F_k)$ of transcendence degree $n - k$

\[ \downarrow \]

there exist $n$ derivations $\tilde{D}_1, \ldots, \tilde{D}_n$ on $\mathbb{C}(x)$ satisfying

\[ \tilde{D}_j = D_j \quad \text{on } \mathbb{C}(F_1, \ldots, F_k) \text{ for } j = 1, \ldots, k. \]

3. all derivations on $\mathbb{C}(x)$ form an $n$–dimensional vector space over $\mathbb{C}(x)$ with base $\{ \frac{\partial}{\partial x_j} : j = 1, \ldots, n \}$

\[ \downarrow \]

\[ \tilde{D}_s = \sum_{j=1}^{n} d_{sj} \frac{\partial}{\partial x_j}, \]

where $d_{sj} \in \mathbb{C}(x)$. 
4. The derivations $\tilde{D}_s$ acting on $\mathbb{C}(F_1, \ldots, F_k)$ satisfy

$$\delta_{sr} = D_s F_r = \tilde{D}_s F_r = \sum_{j=1}^{n} d_{sj} \frac{\partial F_r}{\partial x_j}, \quad r, s \in \{1, \ldots, k\}.$$

↓

the gradients $\nabla_x F_1, \ldots, \nabla_x F_k$ have the rank $k$,

↓

$F_1, \ldots, F_k$ are functionally independent.
Second step: independent in lowest degree terms

Notations

- For an analytic or a polynomial function $F(x)$ in $(\mathbb{C}^n, 0)$, denote by $F^0(x)$ its lowest degree homogeneous term.
- For a rational or a generalized rational function $F(x) = G(x)/H(x)$ in $(\mathbb{C}^n, 0)$, denote by $F^0(x)$ the rational function $G^0(x)/H^0(x)$. 
For analytic functions $G(x)$ and $H(x)$, expanding

$$F(x) = \frac{G(x)}{H(x)} = \frac{G^0(x)}{H^0(x)} + \sum_{i=1}^{\infty} \frac{A^i(x)}{B^i(x)},$$

where $A^i(x)$ and $B^i(x)$ are homogeneous polynomials, and

$$\text{deg} G^0(x) - \text{deg} H^0(x) < \text{deg} A^i(x) - \text{deg} B^i(x) \quad \text{for all } i \geq 1.$$

**Definitions**

- $\text{deg} A^i(x) - \text{deg} B^i(x)$ is called the degree of $A^i(x)/B^i(x)$,
- $G^0(x)/H^0(x)$ is called the lowest degree term of $F(x)$
- $d(F) = \text{deg} G^0(x) - \text{deg} H^0(x)$ is called the lowest degree of $F$. 

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Lemma 2

Assume that

\[ F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)}, \]

are functionally independent generalized rational functions in \((\mathbb{C}^n, 0)\).

Then there exist polynomials \(P_i(z_1, \ldots, z_m)\) for \(i = 2, \ldots, m\) such that

- \(F_1(x), \tilde{F}_2(x) = P_2(F_1(x), \ldots, F_m(x)), \ldots, \tilde{F}_m(x) = P_m(F_1(x), \ldots, F_m(x))\) are functionally independent generalized rational functions,
- and \(F_1^0(x), \tilde{F}_2^0(x), \ldots, \tilde{F}_m^0(x)\) are functionally independent rational functions.
Remark

Lemma 2 was first

- proved by Ziglin [Functional Anal. Appl. 1983],
- and then proved by Baider et al [Fields Institute Communications 7, 1996].

In our paper we also provide a proof using Lemma 1. The idea follows from the proof of Lemma 2.1 of Ito [Comment. Math. Helvetici 1989].
Third step:  
characterization of rational first integrals

**Definition**

- A *rational monomial* is the ratio of two monomials, i.e. of the form $x^k/x^l$ with $k, l \in (\mathbb{Z}^+)^n$.
- The rational monomial $x^k/x^l$ is *resonant* if $\langle \lambda, k - l \rangle = 0$.
- A rational function is *homogeneous* if its denominator and numerator are both homogeneous polynomials.
- A rational homogeneous function is *resonant* if the ratio of any two elements in the set of all its monomials in both denominator and numerator is a resonant rational monomial.
The vector field associated to (1) is written in

\[ \mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h := \langle Ax, \partial_x \rangle + \langle f(x), \partial_x \rangle. \]

**Lemma 3**

If

\[ F(x) = \frac{G(x)}{H(x)} \]

is a generalized rational first integral of the vector field \( \mathcal{X} \) defined by (1), then

\[ F^0(x) = \frac{G^0(x)}{H^0(x)} \]

is a **resonant rational homogeneous first integral** of the linear vector field \( \mathcal{X}_1 \).
Remark:
The proof of Lemma 3 needs the spectrum of linear operators

Define

\[ L_c(h)(x) = \langle \partial_x h(x), Ax \rangle - c h(x), \quad h(x) \in \mathcal{H}_n^m, \]

where \( \mathcal{H}_n^m \) the linear space of complex coefficient homogeneous polynomials of degree \( m \) in \( n \) variables.

Then the spectrum of \( L_c \) is

\[ \{ \langle k, \lambda \rangle - c : k \in (\mathbb{Z}^+)^n, |k| = k_1 + \ldots + k_n = m \}, \]

where \( \lambda \) are the eigenvalues of \( A \).
Proof of Theorem 1

Let
\[ F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)}, \]
be the \( m \) functionally independent generalized rational first integrals of \( \mathcal{X} \).

\[ \downarrow \text{ by Lemma 2} \]
we can assume that

\[ F^0_1(x) = \frac{G^0_1(x)}{H^0_1(x)}, \ldots, F^0_m(x) = \frac{G^0_m(x)}{H^0_m(x)}, \]
are functionally independent.

\[ \downarrow \text{ by Lemma 3} \]
\( F^0_1(x), \ldots, F^0_m(x) \) are resonant rational homogeneous first integrals of the linear vector field \( \mathcal{X}_1 \),
By the linear algebra, the matrix $A$ in $\mathbb{C}$ has a unique representation in the form

$$A = A_s + A_n$$

with

- $A_s$ semi-simple, $A_n$ nilpotent, $A_sA_n = A_nA_s$
- $A_s$ is similar to a diagonal matrix, and assume $A_s$ diagonal

Define

- $\mathcal{X}_s = \langle A_s x, \partial_x \rangle$ and
- $\mathcal{X}_n = \langle A_n x, \partial_x \rangle$.
- Separate $\mathcal{X}_1 = \mathcal{X}_s + \mathcal{X}_n$.  

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Generalized rational first integrals of analytic systems
Direct calculations show that

- any resonant rational monomial is a first integral of $\mathcal{X}_s$, for example,
  - let $x^m$ be a resonant rational monomial, i.e. it satisfies $\langle \lambda, m \rangle = 0$.
  - Then $\mathcal{X}_s(x^m) = \langle \lambda, m \rangle x^m = 0$.

- So $F_1^0(x), \ldots, F_m^0(x)$ are also first integrals of $X_s$.

- This means that $m$ is less than or equal to the number of functionally independent resonant rational monomials.

- In addition, the number of functionally independent resonant rational monomials is equal to the maximum number of linearly independent vectors in $\mathbb{R}^n$ of the set $\{k \in \mathbb{Z}^n : \langle k, \lambda \rangle = 0\}$.

We complete the proof of the Theorem
Related problem 1: semi-quasi-homogeneous systems

Remark

- If $A$ of (1) has all its eigenvalues zero, Theorem 1 is trivial.
- In this case, we consider semi–quasi–homogeneous systems.
System

\[ \dot{x} = f(x) = (f_1(x), \ldots, f_n(x)), \quad (2) \]

- is quasi–homogeneous of degree \( q \in \mathbb{N} \setminus \{1\} \) with exponents \( s = (s_1, \ldots, s_n) \in \mathbb{Z}^n \setminus \{0\} \) if for all \( \rho > 0 \)

\[ f_i(\rho^{s_1}x_1, \ldots, \rho^{s_n}x_n) = \rho^{q+s_i-1}f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n. \]

- is semi–quasi–homogeneous of degree \( q \) with the weight exponent \( s \) if

\[ f(x) = f_q(x) + f_h(x), \]

with

\[ \dot{x} = f_q(x), \]

quasi-homogeneous and \( f_h(x) \) consisting of higher order terms.
Set
\[ W = \text{diag}(s_1/(q-1), \ldots, s_n/(q-1)). \]

- Any solution \( c = (c_1, \ldots, c_n) \) of
  \[ f_q(c) + Wc = 0, \]
  is called a *balance*.

- For each balance \( c \), the eigenvalues of
  \[ K = Df_q(c) + W \]
  are called *Kowalevskaya exponents*, denoted by \( \lambda_c \).

- Let \( d_c \) be the dimension of the minimal vector subspace of \( \mathbb{R}^n \) containing the set
  \[ \left\{ k \in \mathbb{Z}^n : \langle k, \lambda_c \rangle = 0, k \neq 0 \right\}. \]
Theorem 2

Assume that

- system (2) is semi–quasi–homogeneous of weight degree $q$ with weight exponent $s$.

Then

- the number of functionally independent generalized rational first integrals of (2) is at most $d = \min_{c \in \mathcal{B}} d_c$, where $\mathcal{B}$ is the set of balances.
Remark

- Theorem 2 is an extension
  - of Theorem 1 of Furta [ZAMP 1996]
  - of Corollary 3.7 of Goriely [JMP 1996] and
  - of Theorem 2 of Shi [JMAA 2007].

- In some sense Theorem 2 is also an extension
  - of the main results of Yoshida [Celestial Mech 1983],

where he proved that if a quasi–homogenous differential system is algebraically integrable, then every Kowalevskaya exponent should be a rational number.
Remark:

Theorems 1 and 2 studied the existence of functionally independent generalized rational first integrals in a neighborhood of a singularity.

Next we investigate the existence of generalized rational first integrals of system (2) in a neighborhood of a periodic orbit.
Definition

- The *multipliers* of a periodic orbit are the eigenvalues of the linear part of the Poincaré map at the fixed point corresponding to the periodic orbit.

Recall that

- Associated to Poincaré map of a periodic orbit, its linear part has the eigenvalue 1 along the direction tangent to the periodic orbit.
Theorem 3

Assume that

- the analytic differential system (2) has a periodic orbit with multipliers \( \mu = (\mu_1, \ldots, \mu_{n-1}) \).

Then

- the number of functionally independent generalized rational first integrals of system (2) in a neighborhood of the periodic orbit is at most the maximum number of linearly independent vectors in \( \mathbb{R}^n \) of the set

\[
\{ k \in \mathbb{Z}^{n-1} : \mu^k = 1, k \neq 0 \}.
\]
Remark:

Previously, we studied the existence of functionally independent generalized rational first integrals of autonomous differential systems.

Finally we consider the periodic differential systems

\[ \dot{x} = f(t, x), \quad (t, x) \in S^1 \times (\mathbb{C}^n, 0), \]  

(3)

where

- \( S^1 = \mathbb{R}/(2\pi \mathbb{N}) \), and
- \( f(t, x) \) is analytic in its variables and \( 2\pi \) periodic in \( t \).
Definition

A non–constant function $F(t, x)$ is a **generalized rational first integral** of system (3) if

- $F(t, x) = G(t, x)/H(t, x)$ with $G(t, x)$ and $H(t, x)$ analytic in their variables and $2\pi$ periodic in $t$,
- and it satisfies

$$\frac{\partial F(t, x)}{\partial t} + \langle \partial_x F(t, x), f(t, x) \rangle \equiv 0 \quad \text{in } S^1 \times (\mathbb{C}^n, 0).$$

functions $F_1(t, x), \ldots, F_m(t, x)$ are **functionally independent** in $S^1 \times (\mathbb{C}^n, 0)$ if

- $\partial_x F_1(t, x), \ldots, \partial_x F_m(t, x)$ have the rank $m$ in a full Lebesgue measure subset of $S^1 \times (\mathbb{C}^n, 0)$. 

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Generalized rational first integrals of analytic systems
Assume that

- $x = 0$ is a constant solution of (3), i.e. $f(t, 0) = 0$.

then system (3) can be written in

$$\dot{x} = A(t)x + g(t, x),$$

(4)

where $A(t)$ and $g(t, x) = O(x^2)$ are $2\pi$ periodic in $t$.

Let

- $\mathcal{L}$ be the *monodromy operator* associated with the linear periodic equation

$$\dot{x} = A(t)x.$$
Theorem 4

Assume that

- the monodromy operator $\mathcal{L}$ has the eigenvalues $\mu = (\mu_1, \ldots, \mu_n)$.

Then

- the number of functionally independent generalized rational first integrals of system (4) in a neighborhood of the constant solution $x = 0$ is at most the maximum number of linearly independent vectors in $\mathbb{R}^n$ of the set

$$\Xi := \left\{ k \in \mathbb{Z}^n : \mu^k = 1, k \neq 0 \right\} \subset \mathbb{Z}^n.$$
Expresiones

致谢

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Gracias!

Thanks for your attention!