Bounding the number of zeros of certain Abelian integrals

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Outline of the talk

- Introduction.
- An embedding problem.
- Zeros of Abelian integrals.
Let $\mathcal{E}$ be a finite-dimensional space of analytic functions on an interval $I$. If $\dim(\mathcal{E}) = n$, then it is easy to prove that there exist $g \in \mathcal{E}$ with at least $n - 1$ zeros counting multiplicities.
Let \( \mathcal{E} \) be a finite-dimensional space of analytic functions on an interval \( I \). If \( \dim(\mathcal{E}) = n \), then it is easy to prove that there exist \( g \in \mathcal{E} \) with at least \( n - 1 \) zeros counting multiplicities.

\[
\begin{align*}
\alpha_1 f_1(x_1) + \alpha_2 f_2(x_1) + \ldots + \alpha_n f_n(x_1) &= 0, \\
\alpha_1 f_1(x_2) + \alpha_2 f_2(x_2) + \ldots + \alpha_n f_n(x_2) &= 0, \\
& \vdots \\
\alpha_1 f_1(x_{n-1}) + \alpha_2 f_2(x_{n-1}) + \ldots + \alpha_n f_n(x_{n-1}) &= 0,
\end{align*}
\]

The space \( \mathcal{E} \) is Chebyshev if any \( g \in \mathcal{E} \) has at most \( n - 1 \) zeros counting multiplicities. If this upper bound is greater, say \( n + k - 1 \), then \( \mathcal{E} \) is Chebyshev with accuracy \( k \). We study this property in case that the functions in \( \mathcal{E} \) are Abelian integrals.
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The space $\mathcal{E}$ is **Chebyshev** if any $g \in \mathcal{E}$ has at most $n - 1$ zeros counting multiplicities. If this upper bound is greater, say $n + k - 1$, then $\mathcal{E}$ is **Chebyshev with accuracy $k$**. We study this property in case that the functions in $\mathcal{E}$ are **Abelian integrals**.
Let \( f_0, f_1, \ldots, f_{n-1} \) be analytic functions on an interval \( I \).
An embedding problem

Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an interval $I$.

**Extended complete Chebyshev system**

$(f_0, f_1, \ldots, f_{n-1})$ is an extended complete Chebyshev system (in short, ECT-system) on $I$ if, for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \ldots + \alpha_{k-1} f_{k-1}(x) = 0$$

has at most $k - 1$ isolated zeros on $I$ counted with multiplicities.
An embedding problem

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(“T” stands for Tchebycheff.)
Let $\mathcal{E}$ be a finite-dimensional space of analytic functions on $I$ such that any $f \in \mathcal{E}$ has at most $n$ zeros on $I$ counted with multiplicities.
An embedding problem

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Note that $1 \leq \dim(\mathcal{E}) \leq n + 1$. 
An embedding problem

Let $\mathcal{E}$ be a finite-dimensional space of analytic functions on $I$ such that any $f \in \mathcal{E}$ has at most $n$ zeros on $I$ counted with multiplicities. The problem consists in finding necessary and sufficient conditions for the existence of an ECT-system on $I$ of dimension $n + 1$ whose linear span contains $\mathcal{E}$.

Note that $1 \leq \dim(\mathcal{E}) \leq n + 1$. 
An embedding problem

The result is (almost) true when $\dim(\mathcal{E}) = n + 1$. 

Theorem (Mazure)

The ECT-system exists if $\dim(\mathcal{E}) = n + 1$ and $I$ is closed and bounded.


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**Theorem (Mazure)**

The ECT-system exists if \( \dim(\mathcal{E}) = n + 1 \) and \( I \) is closed and bounded.

The result is true when $\dim(\mathcal{E}) = 1$.

**Lemma 1**

Let $f$ be an analytic function on an open interval $I$. If $f$ has exactly $n$ zeros on $I$ counted with multiplicities, then there exist $g_0, g_1, \ldots, g_{n-1}$ analytic functions on $I$ such that $(g_0, g_1, \ldots, g_{n-1}, f)$ is an ECT-system on $I$. 
The result is true when \( \text{dim}(\mathcal{E}) = 1 \).

**Lemma 1**

Let \( f \) be an analytic function on an open interval \( I \). If \( f \) has exactly \( n \) zeros on \( I \) counted with multiplicities, then there exist \( g_0, g_1, \ldots, g_{n-1} \) analytic functions on \( I \) such that \((g_0, g_1, \ldots, g_{n-1}, f)\) is an ECT-system on \( I \).

Our aim in the first part of the talk is to show a (very) partial result for \( \mathcal{E} \) of “arbitrary” dimension.
An embedding problem

The result is true when $\dim(\mathcal{E}) = 1$.

**Lemma 1**

Let $f$ be an analytic function on an open interval $I$. If $f$ has exactly $n$ zeros on $I$ counted with multiplicities, then there exist $g_0, g_1, \ldots, g_{n-1}$ analytic functions on $I$ such that $(g_0, g_1, \ldots, g_{n-1}, f)$ is an ECT-system on $I$.

Our aim in the first part of the talk is to show a (very) partial result for $\mathcal{E}$ of “arbitrary” dimension. First we shall prove Lemma 1.
Lemma 2

Let \((f_0, f_1, \ldots, f_{n-1})\) be an ECT-system on \(I\).

(a) If \(\varphi : L \rightarrow I\) is a diffeomorphism, then 
\((f_0 \circ \varphi, f_1 \circ \varphi, \ldots, f_{n-1} \circ \varphi)\) be an ECT-system on \(L\).

(b) If \(g\) is a non-vanishing function on \(I\), then 
\((gf_0, gf_1, \ldots, gf_{n-1})\) be an ECT-system on \(I\).
Proof of Lemma 1. Let $a_1 \leq a_2 \ldots \leq a_n$ be the zeros of $f$. Define

$$g_i(x) = \frac{f(x)}{\prod_{j=i+1}^{n} (x - a_j)} \text{ for } i = 0, 1, \ldots, n - 1,$$

so that $g_i$ has exactly $i$ zeros on $I$ counted with multiplicities. It is clear that by construction

$$\left( \frac{g_0}{g_0}, \ldots, \frac{g_{n-1}}{g_0}, \frac{f}{g_0} \right) = \left( 1, x - a_1, (x - a_1)(x - a_2), \ldots, \prod_{i=1}^{n} (x - a_i) \right),$$

and this shows, by (b) in Lemma 2, that $(g_0, \ldots, g_{n-1}, f)$ is an ECT-system on $I$. ■
**An embedding problem**

**Wronskian**

\[
W \left[ f_0, f_1, \cdots, f_{k-1} \right] (x) = \det \left( f_j^{(i)}(x) \right)_{0 \leq i, j \leq k-1} \\
= \left| \begin{array}{ccc}
  f_0(x) & \cdots & f_{k-1}(x) \\
  f_1(x) & \cdots & f_{k-1}'(x) \\
  \vdots & & \vdots \\
  f_{k-1}^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \\
\end{array} \right|
\]
Lemma (Howland)

Let $f_0, f_1, \ldots, f_n$ be analytic functions on an open interval $I$ such that $W[f_0, \ldots, f_{n-2}, f_{n-1}]$ does not vanish on $I$. Then

$$
\left( \frac{W[f_0, \ldots, f_{n-2}, f_n]}{W[f_0, \ldots, f_{n-2}, f_{n-1}]} \right)' = \frac{W[f_0, \ldots, f_n] W[f_0, \ldots, f_{n-2}]}{(W[f_0, \ldots, f_{n-2}, f_{n-1}])^2}.
$$
**Problem**: Bound the number of zeros of $g \in \langle f_0, f_1, f_2, f_3 \rangle$. 

**The derivation-division algorithm**

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\[\Downarrow\text{ division (if } f_0 \neq 0\text{)}\]

$$\frac{g}{f_0} = \alpha_0 + \alpha_1 \frac{f_1}{f_0} + \alpha_2 \frac{f_2}{f_0} + \alpha_3 \frac{f_3}{f_0}$$
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\downarrow \text{division (if } f_0 \neq 0) \quad \frac{g}{f_0} = \alpha_0 + \frac{\alpha_1}{f_0} f_1 + \frac{\alpha_2}{f_0} f_2 + \frac{\alpha_3}{f_0} f_3
\]

\[
\downarrow \text{derivation} \quad \left( \frac{g}{f_0} \right)' = \alpha_1 \left( \frac{f_1}{f_0} \right)' + \alpha_2 \left( \frac{f_2}{f_0} \right)' + \alpha_3 \left( \frac{f_3}{f_0} \right)'
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**Problem:** Bound the number of zeros of \( g \in < f_0, f_1, f_2, f_3 > \).

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\begin{align*}
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\downarrow \text{derivation} \\
\frac{W[f_0, g]}{f_0^2} &= \alpha_1 \frac{W[f_0, f_1]}{f_0^2} + \alpha_2 \frac{W[f_0, f_2]}{f_0^2} + \alpha_3 \frac{W[f_0, f_3]}{f_0^2}
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\frac{W[f_0, g]}{f_0^2} = \alpha_1 \frac{W[f_0, f_1]}{f_0^2} + \alpha_2 \frac{W[f_0, f_2]}{f_0^2} + \alpha_3 \frac{W[f_0, f_3]}{f_0^2}
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$\Downarrow$ division (if $f_0 \neq 0$)

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$$W[f_0, g] = \alpha_1 W[f_0, f_1] + \alpha_2 W[f_0, f_2] + \alpha_3 W[f_0, f_3]$$

Then, $\# \{ \text{zeros of } g \} \leq \# \{ \text{zeros of } W[f_0, g] \} + 1$. 
The derivation-division algorithm

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\[ \left( \frac{W[f_0, g]}{W[f_0, f_1]} \right)' = \alpha_2 \left( \frac{W[f_0, f_2]}{W[f_0, f_1]} \right)' + \alpha_3 \left( \frac{W[f_0, f_3]}{W[f_0, f_1]} \right)' \]
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\[ \frac{W[f_0, f_1, g] f_0}{W[f_0, f_1]^2} = \alpha_2 \frac{W[f_0, f_1, f_2] f_0}{W[f_0, f_1]^2} + \alpha_3 \frac{W[f_0, f_1, f_3] f_0}{W[f_0, f_1]^2} \]
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\[ W[f_0, f_1, g] = \alpha_2 W[f_0, f_1, f_2] + \alpha_3 W[f_0, f_1, f_3] \]

Then, \( \# \{ \text{zeros of } g \} \leq \# \{ \text{zeros of } W[f_0, f_1, g] \} + 2 \).
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W[f_0, f_1, g] &= \alpha_2 W[f_0, f_1, f_2] + \alpha_3 W[f_0, f_1, f_3] \\
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\[ \left( \frac{W[f_0, f_1, g]}{W[f_0, f_1, f_2]} \right)' = \alpha_3 \left( \frac{W[f_0, f_1, f_3]}{W[f_0, f_1, f_2]} \right)' \]
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\[ \Downarrow \text{derivation} \]

\[ \frac{W[f_0, f_1, f_2, g]W[f_0, f_1]}{W[f_0, f_1, f_2]^2} = \alpha_3 \frac{W[f_0, f_1, f_2, f_3]W[f_0, f_1]}{W[f_0, f_1, f_2]^2} \]
The derivation-division algorithm

\[ W[f_0, f_1, g] = \alpha_2 W[f_0, f_1, f_2] + \alpha_3 W[f_0, f_1, f_3] \]

\[ \downarrow \quad \text{division (if } W[f_0, f_1, f_2] \neq 0) \]

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\[ \Downarrow \text{ derivation} \]

\[ W[f_0, f_1, f_2, g] = \alpha_3 W[f_0, f_1, f_2, f_3] \]

Then, \#\{zeros of } g\} \leq \#\{zeros of } W[f_0, f_1, f_2, f_3]\} + 3. \]
The derivation-division algorithm

In short,

\[ g = \alpha_0 f_0 + \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \]
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\[ W[f_0, f_1, f_2, g] = \alpha_3 W[f_0, f_1, f_2, f_3] \]
Lemma (Karlin-Studden)

\((f_0, f_1, \ldots, f_{n-1})\) is an ECT-system on \(L\) if and only if, for each \(k = 1, 2, \ldots, n\), it holds

\[ W[f_0, f_1, \ldots, f_{k-1}](x) \neq 0 \text{ for all } x \in L. \]
An embedding problem

The following is a (very) partial result on the embedding problem.

Theorem A

Let \( f_0, f_1, \ldots, f_{n-1} \) and \( h \) be analytic functions on \( I \) such that:

(a) \( W[f_0, f_1, \ldots, f_k] \) is non-vanishing on \( I \) for \( k = 0, 1, \ldots, n-1 \) (i.e., such that \( (f_0, f_1, \ldots, f_{n-1}) \) is an ECT-system on \( I \)), and

(b) \( W[f_0, \ldots, f_{n-1}, h] \) has \( \ell \) zeros on \( I \) counted with multiplicities.

Then there exist \( l_1, l_2, \ldots, l_\ell \) analytic functions on \( I \) such that \( (f_0, \ldots, f_{n-1}, l_1, \ldots, l_\ell, h) \) is an ECT-system on \( I \).

Note that (a) and (b) imply that any function in \( \langle f_0, f_1, \ldots, f_{n-1}, h \rangle \) has at most \( n+\ell \) zeros on \( I \) counted with multiplicities.
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Note that (a) and (b) imply that any function in $< f_0, f_1, \ldots, f_{n-1}, h >$ has at most $n + \ell$ zeros on $I$ counted with multiplicities.
An embedding problem

*Idea of the proof.* Set $n = \ell = 2$ for simplicity. Thus, given $f_0$, $f_1$ and $h$ such that $f_0$ and $W[f_0, f_1]$ do not vanish, and that $W[f_0, f_1, h]$ has 2 zeros on $I$ counting multiplicities, we must find $l_1$ and $l_2$ such that $(f_0, f_1, l_1, l_2, h)$ is an ECT-system.
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$$g = \alpha_0 f_0 + \alpha_1 f_1 + \beta_1 l_1 + \beta_2 l_2 + \gamma h$$
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\[\downarrow \text{division-derivation}\]
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$$W[f_0, g] = \alpha_1 W[f_0, f_1] + \beta_1 W[f_0, l_1] + \beta_2 W[f_0, l_2] + \gamma W[f_0, h]$$
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\[\downarrow\] division-derivation

$$W[f_0, f_1, g] = \beta_1 W[f_0, f_1, l_1] + \beta_2 W[f_0, f_1, l_2] + \gamma W[f_0, f_1, h]$$
An embedding problem

At this stage, by convenience we divide by \( W[f_0, f_1] \), so that

\[
\frac{W[f_0, f_1, g]}{W[f_0, f_1]} = \beta_1 \frac{W[f_0, f_1, l_1]}{W[f_0, f_1]} + \beta_2 \frac{W[f_0, f_1, l_2]}{W[f_0, f_1]} + \gamma \frac{W[f_0, f_1, h]}{W[f_0, f_1]} \tilde{h}
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\[
\tilde{h}
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Then, since $\tilde{h} := \frac{W[f_0, f_1, h]}{W[f_0, f_1]}$ has 2 zeros on $I$ counting multiplicities, by Lemma 1 there exist $\tilde{l}_1$ and $\tilde{l}_2$ such that $(\tilde{l}_1, \tilde{l}_2, \tilde{h})$ is an ECT-system on $I$. 
An embedding problem

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Then, since \( \tilde{h} := \frac{W[f_0, f_1, h]}{W[f_0, f_1]} \) has 2 zeros on \( I \) counting multiplicities, by Lemma 1 there exist \( \tilde{l}_1 \) and \( \tilde{l}_2 \) such that \( (\tilde{l}_1, \tilde{l}_2, \tilde{h}) \) is an ECT-system on \( I \). (Hence \( \tilde{l}_1, W[\tilde{l}_1, \tilde{l}_2] \) and \( W[\tilde{l}_1, \tilde{l}_2, \tilde{h}] \) do not vanish on \( I \).)
An embedding problem

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Then, since $\tilde{h} := \frac{W[f_0, f_1, h]}{W[f_0, f_1]}$ has 2 zeros on $I$ counting multiplicities, by Lemma 1 there exist $\tilde{l}_1$ and $\tilde{l}_2$ such that $(\tilde{l}_1, \tilde{l}_2, \tilde{h})$ is an ECT-system on $I$. (Hence $\tilde{l}_1$, $W[\tilde{l}_1, \tilde{l}_2]$ and $W[\tilde{l}_1, \tilde{l}_2, \tilde{h}]$ do not vanish on $I$.) We choose $l_1$ and $l_2$ verifying the second order linear differential equations

$$
\frac{W[f_0, f_1, l_1]}{W[f_0, f_1]} = \tilde{l}_1 \quad \text{and} \quad \frac{W[f_0, f_1, l_2]}{W[f_0, f_1]} = \tilde{l}_2.
$$
We can then continue the derivation-division algorithm

\[ W[f_0, f_1, g] = \beta_1 W[f_0, f_1, l_1] + \beta_2 W[f_0, f_1, l_2] + \gamma W[f_0, f_1, h] \]
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\[ \downarrow \text{division-derivation} \quad (\tilde{l}_1 \neq 0) \]
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\[ W[f_0, f_1, l_1, l_2, g] = \gamma W[f_0, f_1, l_1, l_2, h] \neq 0 \]

\[ (W[\tilde{l}_1, \tilde{l}_2, \tilde{h}] \neq 0) \]
**Theorem A**

Let $f_0, f_1, \ldots, f_{n-1}$ and $h$ be analytic functions on $I$ such that:

(a) $W[f_0, f_1, \ldots, f_k]$ is non-vanishing on $I$ for $k = 0, 1, \ldots, n - 1$,

(b) $W[f_0, \ldots, f_{n-1}, h]$ has $\ell$ zeros on $I$ counted with multiplicities.

Then there exist $l_1, l_2, \ldots, l_\ell$ analytic functions on $I$ such that $(f_0, \ldots, f_{n-1}, l_1, \ldots, l_\ell, h)$ is an ECT-system on $I$.

Setting $\tilde{h} := \frac{W[f_0, \ldots, f_{n-1}, h]}{W[f_0, \ldots, f_{n-1}]}$, we take $\tilde{l}_1, \tilde{l}_2, \ldots, \tilde{l}_\ell$ such that $(\tilde{l}_1, \ldots, \tilde{l}_\ell, \tilde{h})$ is an ECT-system on $I$. Then, for $k = 1, 2, \ldots, \ell$, we choose $l_k$ being a solution of the $n$-th order linear differential equation

$$\frac{W[f_0, \ldots, f_{n-1}, y]}{W[f_0, \ldots, f_{n-1}]} = \tilde{l}_k.$$
Corollary

Consider the $n$-th order linear differential equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = b(x), \quad (1)$$

where $a_i$ and $b$ are analytic on $I$. Assume that the homogeneous equation has a fundamental set of solutions $(\varphi_0, \varphi_1, \ldots, \varphi_{n-1})$ being an ECT-system on $I$. Then, if $b$ has $k$ zeros on $I$ counted with multiplicities, any solution of (1) has at most $n + k$ zeros on $I$ counted with multiplicities.
Corollary

Consider the \( n \)-th order linear differential equation

\[
y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = b(x),
\]

(1)

where \( a_i \) and \( b \) are analytic on \( I \). Assume that the homogeneous equation has a fundamental set of solutions \((\varphi_0, \varphi_1, \ldots, \varphi_{n-1})\) being an ECT-system on \( I \). Then, if \( b \) has \( k \) zeros on \( I \) counted with multiplicities, any solution of (1) has at most \( n + k \) zeros on \( I \) counted with multiplicities.

This result is used in [L. Gavrilov, I. Iliev, Quadratic perturbations of quadratic codimension-four centers, J. Math. Anal. Appl. 357 (2009) 69-76] for \( n = 2 \).
**Proof.** Note that if \( \{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\} \) is a fundamental set of solutions of \( y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0 \), then

\[
y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = \frac{W[\varphi_0, \ldots, \varphi_{n-1}, y]}{W[\varphi_0, \ldots, \varphi_{n-1}]}(x).
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Therefore the linear differential equation writes as

\[
\frac{W[\varphi_0, \ldots, \varphi_{n-1}, y]}{W[\varphi_0, \ldots, \varphi_{n-1}]}(x) = b(x).
\]
An embedding problem

**Proof.** Note that if \( \{\varphi_0, \varphi_1, \ldots, \varphi_{n-1}\} \) is a fundamental set of solutions of \( y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = 0 \), then

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Therefore the linear differential equation writes as

\[
\frac{W[\varphi_0, \ldots, \varphi_{n-1}, y]}{W[\varphi_0, \ldots, \varphi_{n-1}]}(x) = b(x).
\]

Accordingly, if \( h \) is a solution, then \( W[\varphi_0, \ldots, \varphi_{n-1}, h] \) has \( k \) zeros on \( I \) counted with multiplicities. Then, by Theorem A, \( h \) belongs to an ECT-system on \( I \) of dimension \( n + k + 1 \).
Let $H(x, y) = A(x) + B(x)y^2$ be an analytic function in the plane with a local minimum at the origin. Then there exists a period annulus $\mathcal{P}$ foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$ for $h \in (0, h_0)$. The projection of $\mathcal{P}$ on the $x$-axis is an interval $(x_\ell, x_r)$ with $x_\ell < 0 < x_r$. 

Jordi Villadelprat

Bounding the number of zeros of certain Abelian integrals
Let $H(x, y) = A(x) + B(x)y^2$ be an analytic function in the plane with a local minimum at the origin. Then there exists a period annulus $\mathcal{P}$ foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$ for $h \in (0, h_0)$. The projection of $\mathcal{P}$ on the $x$-axis is an interval $(x_\ell, x_r)$ with $x_\ell < 0 < x_r$.

Clearly, $A$ has a local minimum at $x = 0$, and so there exists an analytic involution $\sigma$ such that $A(x) = A(\sigma(x))$ for all $x \in (x_\ell, x_r)$. (Recall that an involution is a mapping $\sigma \neq Id$ verifying that $\sigma^2 = Id$.)
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Given a function $\kappa$ defined on $I \setminus \{0\}$, we define its balance as

$$\mathcal{B}_\sigma(\kappa)(x) = \frac{\kappa(x) - \kappa(\sigma(x))}{2}.$$
Let $H(x, y) = A(x) + B(x)y^2$ be an analytic function in the plane with a local minimum at the origin. Then there exists a period annulus $\mathcal{P}$ foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$ for $h \in (0, h_0)$. The projection of $\mathcal{P}$ on the $x$-axis is an interval $(x_\ell, x_r)$ with $x_\ell < 0 < x_r$.

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$$B_\sigma(\kappa)(x) = \frac{\kappa(x) - \kappa(\sigma(x))}{2}.$$ 

For example, if $\sigma = -Id$, then the balance of a function is its odd part.
Zeros of Abelian integrals

Let $H(x, y) = A(x) + B(x)y^2$ be an analytic function in the plane with a local minimum at the origin. Then there exists a period annulus $\mathcal{P}$ foliated by ovals $\gamma_h \subset \{H(x, y) = h\}$ for $h \in (0, h_0)$. The projection of $\mathcal{P}$ on the $x$-axis is an interval $(x_\ell, x_r)$ with $x_\ell < 0 < x_r$.

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We say that a function $g$ is $\sigma$-odd (respectively, $\sigma$-even) if $f \circ \sigma = -f$ (respectively, $f \circ \sigma = f$).
Zeros of Abelian integrals

Let \( H(x, y) = A(x) + B(x)y^2 \) be an analytic function in the plane with a local minimum at the origin. Then there exists a period annulus \( \mathcal{P} \) foliated by ovals \( \gamma_h \subset \{ H(x, y) = h \} \) for \( h \in (0, h_0) \). The projection of \( \mathcal{P} \) on the \( x \)-axis is an interval \( (x_\ell, x_r) \) with \( x_\ell < 0 < x_r \).

Clearly, \( A \) has a local minimum at \( x = 0 \), and so there exists an analytic involution \( \sigma \) such that \( A(x) = A(\sigma(x)) \) for all \( x \in (x_\ell, x_r) \). (Recall that an involution is a mapping \( \sigma \neq \text{Id} \) verifying that \( \sigma^2 = \text{Id} \).)

Given a function \( \kappa \) defined on \( I \setminus \{0\} \), we define its balance as

\[
\mathcal{B}_\sigma(\kappa)(x) = \kappa(x) - \kappa(\sigma(x)) \frac{2}{2}.
\]

We say that a function \( g \) is \( \sigma \)-odd (respectively, \( \sigma \)-even) if \( f \circ \sigma = -f \) (respectively, \( f \circ \sigma = f \)). Thus, the balance of any function is \( \sigma \)-odd.
Zeros of Abelian integrals

**Theorem (Grau-Mañosas-Villadelprat)**

Let $I_i(h) := \int_{\gamma_n} f_i(x) y^{2s-1} dx$, $i = 0, 1, \ldots, n - 1$, with $s \geq n - 1$. 

**Jordi Villadelprat**

Bounding the number of zeros of certain Abelian integrals
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Define $\ell_i = \mathcal{B}\sigma\left(\frac{f_i}{A'B^{s-\frac{1}{2}}}\right)$. 

The idea is simple. If $W[\ell_0, \ell_1, \ldots, \ell_{i-1}]$ does not vanish on $(0, x_r)$ for all $i = 1, 2, \ldots, n$, then $(I_0, I_1, \ldots, I_{n-1})$ is an ECT-system on $(0, x_0)$. Then we define $J_i(h) := \int_{\gamma_n} g_i(x)y^{2s-1}dx$ and use the above result to show that $(I_0, \ldots, I_{n-2}, J_1, \ldots, J_k, I_{n-1})$ is an ECT-system on $(0, x_0)$. 

**Zeros of Abelian integrals**
Theorem (Grau-Mañosas-Villadelprat)

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We shall make an slight improvement of this result for the case that the last Wronskian fails to be non-vanishing.
Theorem (Grau-Mañosas-Villadelprat)

Let $I_i(h) := \int_{\gamma_h} f_i(x)y^{2s-1}dx$, $i = 0, 1, \ldots, n - 1$, with $s \geq n - 1$.

Define $\ell_i = B_{\sigma} \left( \frac{f_i}{A'B^{s-\frac{1}{2}}} \right)$. If $W[\ell_0, \ell_1, \ldots, \ell_{i-1}]$ does not vanish on $(0, x_r)$ for all $i = 1, 2, \ldots, n$, then $(I_0, I_1, \ldots, I_{n-1})$ is an ECT-system on $(0, h_0)$.

The idea is simple. If $W[\ell_0, \ell_1, \ldots, \ell_{n-1}]$ has $k$ zeros, we apply Theorem A to obtain $g_1, g_2, \ldots, g_k$ such that $(\ell_0, \ldots, \ell_{n-2}, g_1, \ldots, g_k, \ell_{n-1})$ is an ECT-system on $(0, x_r)$. 
Zeros of Abelian integrals

**Theorem (Grau-Mañosas-Villadelprat)**

Let \( I_i(h) := \int_{\gamma h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \ldots, n-1, \) with \( s \geq n-1. \)

Define \( \ell_i = B_\sigma \left( \frac{f_i}{A'B^{s-\frac{1}{2}}} \right). \) If \( W[\ell_0, \ell_1, \ldots, \ell_{i-1}] \) does not vanish on \((0, x_r)\) for all \( i = 1, 2, \ldots, n, \) then \((I_0, I_1, \ldots, I_{n-1})\) is an ECT-system on \((0, h_0)\).

The idea is simple. If \( W[\ell_0, \ell_1, \ldots, \ell_{n-1}] \) has \( k \) zeros, we apply Theorem A to obtain \( g_1, g_2, \ldots, g_k \) such that \((\ell_0, \ldots, \ell_{n-2}, g_1, \ldots, g_k, \ell_{n-1})\) is an ECT-system on \((0, x_r)\). Then we define \( J_i(h) := \int_{\gamma h} g_i(x) y^{2s-1} dx \) and use the above result to show that \((I_0, \ldots, I_{n-2}, J_1, \ldots, J_k, I_{n-1})\) is an ECT-system on \((0, h_0)\).
Following this idea we can prove the following result:

**Theorem B**

Let \( I_i(h) := \int_{\gamma_h} f_i(x) y^{2s-1} dx, \ i = 0, 1, \ldots, n - 1. \) Define \( \ell_i = \mathcal{B}_\sigma \left( \frac{f_i}{A'B^{s-\frac{1}{2}}} \right). \) If the following conditions are verified:

\((a)\) \( W[\ell_0, \ldots, \ell_i] \) is non-vanishing on \((0, x_0)\) for \( i = 0, 1, \ldots, n-2, \)

\((b)\) \( W[\ell_0, \ldots, \ell_{n-1}] \) has \( k \) zeros on \((0, x_0)\) counting multiplicities,

\((c)\) \( s \geq n + k - 1, \)

then any nontrivial linear combination of \( I_0, I_1, \ldots, I_{n-1} \) has at most \( n + k - 1 \) zeros on \((0, h_0)\) counted with multiplicities.
Erasmus de la Prat

Zeros of Abelian integrals

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If the following conditions are verified:

(a) $W[\ell_0, \ldots, \ell_i]$ is non-vanishing on $(0, x_r)$ for $i = 0, 1, \ldots, n - 2$,
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then any nontrivial linear combination of $I_0, I_1, \ldots, I_{n-1}$ has at most $n + k - 1$ zeros on $(0, h_0)$ counted with multiplicities.
Since $A$ vanishes at $x = 0$ with even multiplicity, say $2m$, the functions

$$
\ell_i = \mathcal{B}_\sigma \left( \frac{f_i}{A'B^s - \frac{1}{2}} \right), \quad i = 0, 1, \ldots, n - 1,
$$

have a pole at $x = 0$ of order $\leq 2m - 1$ Moreover, they are $\sigma$-odd.
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have a pole at $x = 0$ of order $\leq 2m - 1$. Moreover, they are $\sigma$-odd. Hence the functions $g_1, g_2, \ldots, g_k$ such that $(\ell_0, \ldots, \ell_{n-2}, g_1, \ldots, g_k, \ell_{n-1})$ is an ECT-system on $(0, x_r)$ must also be $\sigma$-odd and have a pole of order at most $2m - 1$. 

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The key point to solve this problem is the following result, which enables us to suppose without loss of generality that $\sigma = -Id$. 

Lemma 3

Consider $I = (a, b)$ with $a < 0 < b$ and let $\sigma$ be an analytic involution on $I$ such that $\sigma(0) = 0$. Define $\phi(x) = x - \sigma(x)$, i.e., $\phi = B_{\sigma}(Id)$. Then $\phi$ is a diffeomorphism from $I$ to $(a - b^2, b - a^2)$. Moreover an analytic function $f$ on $I$ is $\sigma$-odd if, and only if, $f \circ \phi^{-1}$ is odd.
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**Lemma 3**

Consider $I = (a, b)$ with $a < 0 < b$ and let $\sigma$ be an analytic involution on $I$ such that $\sigma(0) = 0$. \[ \phi(x) = x - \sigma(x)^2, \] i.e., $\phi = B_{\sigma}(Id)$. Then $\phi$ is a diffeomorphism from $I$ to $(a - b^2, b - a^2)$. Moreover an analytic function $f$ on $I$ is $\sigma$-odd if, and only if, $f \circ \phi^{-1}$ is odd.
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Then $\varphi$ is a diffeomorphism from $I$ to $(\frac{a-b}{2}, \frac{b-a}{2})$. 
Zeros of Abelian integrals

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Proof of Lemma 3. That $\varphi$ is a diffeomorphism follows from the fact that an involution is monotonous decreasing. Due to $\sigma^2 = Id$, note that $\varphi(\sigma(x)) = -\varphi(x)$, so that $\varphi(\sigma(\varphi^{-1}(x))) = -x$. 


Proof of Lemma 3. That \( \varphi \) is a diffeomorphism follows from the fact that an involution is monotonous decreasing. Due to \( \sigma^2 = \text{Id} \), note that \( \varphi(\sigma(x)) = -\varphi(x) \), so that \( \varphi(\sigma(\varphi^{-1}(x))) = -x \). Thus, \( \sigma(\varphi^{-1}(x)) = \varphi^{-1}(-x) \).
Proof of Lemma 3. That \( \varphi \) is a diffeomorphism follows from the fact that an involution is monotonous decreasing. Due to \( \sigma^2 = Id \), note that \( \varphi(\sigma(x)) = -\varphi(x) \), so that \( \varphi(\sigma(\varphi^{-1}(x))) = -x \). Thus, \( \sigma(\varphi^{-1}(x)) = \varphi^{-1}(-x) \). Hence, if \( f \) is \( \sigma \)-odd, then

\[
(f \circ \varphi^{-1})(-x) = f(\varphi^{-1}(-x)) = f(\sigma(\varphi^{-1}(x))) = -(f \circ \varphi^{-1})(x)
\]

and this shows that \( f \circ \varphi^{-1} \) is an odd function.
**Proof of Lemma 3.** That \( \varphi \) is a diffeomorphism follows from the fact that an involution is monotonous decreasing. Due to \( \sigma^2 = I_d \), note that \( \varphi(\sigma(x)) = -\varphi(x) \), so that \( \varphi(\sigma(\varphi^{-1}(x))) = -x \). Thus, \( \sigma(\varphi^{-1}(x)) = \varphi^{-1}(-x) \). Hence, if \( f \) is \( \sigma \)-odd, then

\[
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\]

and this shows that \( f \circ \varphi^{-1} \) is an odd function. Reciprocally, if \( f \circ \varphi^{-1} \) is odd, then

\[
\mathcal{B}_\sigma(f)(x) = \frac{f(x) - f(\sigma(x))}{2} = \frac{f(x) - f(\varphi^{-1}(-\varphi(x)))}{2} = f(x),
\]

where in the second equality we use that \( \sigma(x) = \varphi^{-1}(-\varphi(x)) \) and in the third one that \( f \circ \varphi^{-1} \) is odd. This proves that \( f \) is \( \sigma \)-odd. \( \blacksquare \)