Hopf bifurcation of limit cycles of Liénard systems

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1. Introduction and problems

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3. Proof of Theorems 2.1, 2.2 and 2.3
1. Introduction and problems

Consider a Liénard system with vector parameters of the form

\[ \dot{x} = y - F(x, a), \quad \dot{y} = -g(x, b), \]  \hspace{1cm} (1.1)

where \( F \) and \( g \) are \( C^\infty \) functions, and \( a \in D_1, b \in D_2 \) with \( D_i \subset \mathbb{R}^{m_i} \).
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For (1.1) we make the following assumption

\[
g(0) = 0, \quad g_x'(0, b) > 0, \quad F(0, a) = 0, \quad \frac{\partial F}{\partial x}(0, a_0) = 0 \tag{1.2}
\]

for some \( a_0 \in D_1 \). Then the origin is a focus or center of (1.1) for \( a = a_0 \), and (1.1) may have a limit cycle near the origin for \( a \) near \( a_0 \).
Further, for $a$ near $a_0$ the Poincaré return map, denoted by $P(r, a, b)$, can be defined for $|r|$ small and it has a formal expansion of the form

$$P(r, a, b) - r = \sum_{j \geq 1} d_j(a, b) r^j.$$
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\[ P(r, a, b) - r = \sum_{j \geq 1} d_j(a, b) r^j. \]

For fixed $a \in D_1$ and $b \in D_2$, the origin is called a focus of order $k$ if

\[ d_j(a, b) = 0, \quad j = 1, \ldots, 2k, \quad d_{2k+1}(a, b) \neq 0. \]
Further, for a near $a_0$ the Poincaré return map, denoted by $P(r, a, b)$, can be defined for $|r|$ small and it has a formal expansion of the form

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$$d_j(a, b) = 0, \quad j = 1, \ldots, 2k, \quad d_{2k+1}(a, b) \neq 0.$$ 

Let us introduce two numbers $H^*_{D_1, D_2}$ and $\hat{H}_{D_1, D_2}$ for the family (1.1). First, $H^*_{D_1, D_2}$ is defined as follows

$$H^*_{D_1, D_2} = \max_{a \in D_1, b \in D_2} \{\text{the order of the focus at the origin for (1.1)}\}.$$
More precisely, for all \( a \in D_1 \) and \( b \in D_2 \), the origin is a focus of order at most \( H^*_{D_1,D_2} \) unless it is a center, and there exists \( a^* \in D_1 \) and \( b^* \in D_2 \) such that the origin is a focus of order \( H^*_{D_1,D_2} \).
More precisely, for all $a \in D_1$ and $b \in D_2$, the origin is a focus of order at most $H_{D_1,D_2}^*$ unless it is a center, and there exists $a^* \in D_1$ and $b^* \in D_2$ such that the origin is a focus of order $H_{D_1,D_2}^*$.

Thus, $H_{D_1,D_2}^*$ is the maximal order of the origin as a focus of (1.1) for all possible $a \in D_1$ and $b \in D_2$. 
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Thus, $H^*_{D_1,D_2}$ is the maximal order of the origin as a focus of (1.1) for all possible $a \in D_1$ and $b \in D_2$.

Then, we define $\hat{H}_{D_1,D_2}$ by

$$\hat{H}_{D_1,D_2} = \max_{a \in D_1, b \in D_2} \{ \text{the number of limit cycles of (1.1) near the origin} \}.$$
More precisely, for all \( a \in D_1 \) and \( b \in D_2 \), the origin is a focus of order at most \( H^*_{D_1,D_2} \) unless it is a center, and there exists \( a^* \in D_1 \) and \( b^* \in D_2 \) such that the origin is a focus of order \( H^*_{D_1,D_2} \).

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Then, we define \( \hat{H}_{D_1,D_2} \) by

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\hat{H}_{D_1,D_2} = \max_{a \in D_1, b \in D_2} \{ \text{the number of limit cycles of (1.1) near the origin} \}.
\]

In other words (more precisely), there exists a neighborhood \( V \) of the origin such that (1.1) has at most \( \hat{H}_{D_1,D_2} \) limit cycles in \( V \) for all \( a \in D_1 \) and \( b \in D_2 \), and for any neighborhood \( U \subset V \) of the origin, there exists \( a \in D_1 \) and \( b \in D_2 \) such that (1.1) has \( \hat{H}_{D_1,D_2} \) limit cycles in \( U \).
The number $\hat{H}_{D_1,D_2}$ is called the cyclicity of the family (1.1) at the origin.

In the case that $g(x) = g(x, b)$, i.e., $g$ is a function independent of any parameters, we let

$$H^*_{D_1,D_2} = H^*_{D_1}, \quad \hat{H}_{D_1,D_2} = \hat{H}_{D_1}.$$
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From [Han, Tian and Yu, 2011] we know that for

$$\dot{x} = y - (a_2x^3 + a_3x^5), \quad \dot{y} = -x$$

with $a = (a_2, a_3) \in D = R^2$ we have

$$H^*_D = 2 > \hat{H}_D = 1;$$
The number $\hat{H}_{D_1,D_2}$ is called the cyclicity of the family (1.1) at the origin.

In the case that $g(x) = g(x, b)$, i.e., $g$ is a function independent of any parameters, we let

$$H_{D_1,D_2}^* = H_{D_1}^*, \quad \hat{H}_{D_1,D_2} = \hat{H}_{D_1}.$$ 

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with $a = (a_1, a_3) \in D = \mathbb{R}^2$ we have $H_{D}^* = 1 < \hat{H}_{D} = 2$. 
In [Han, 1999] the author gives a way to find $H_{D_1,D_2}^*$ and $\hat{H}_{D_1,D_2}$ for fixed $b \in D_2$, obtaining the following.
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**Theorem 1.1.** Let (1.2) be satisfied. Suppose $g(x, b) = g(x)$, and

$$F(\alpha(x), a) - F(x, a) = \sum_{j \geq 1} B_j(a)x^j,$$

where $\alpha(x) = -x + O(x^2)$ is the solution of the equation $G(x) = G(y)$ on $y < 0 < x$ with $G(x) = \int_0^x g(x)dx$. Then

(1) for all $k \geq 1$ $B_{2k} = O(|B_1, B_3, \ldots, B_{2k-1}|)$, and for fixed $a \in D_1$ the origin is a focus of order $k$ if and only if

$$B_j(a) = 0, \ j = 1, \ldots, 2k, \quad B_{2k+1}(a) \neq 0;$$

in this case, it is stable (unstable) if $B_{2k+1}(a) < 0(>0)$. 


(2) If (i) $B_{2j+1}(a_0) = 0$, $j = 0, \ldots, k$, and

$$\text{rank} \left. \frac{\partial (B_1, B_3, \ldots, B_{2k+1})}{\partial (a_1, a_2, \ldots, a_m)} \right|_{a = a_0} = k + 1,$$

then (1.1) has at least $k$ limit cycles near the origin for some $a$ near $a_0$, each having an odd multiplicity.
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If further (ii) \( F(\alpha(x), a) - F(x, a) \equiv 0 \) as \( B_{2j+1} = 0, \ j = 0, \ldots, k, \) then the cyclicity of (1.1) at the origin is \( k \) for all \( a \) near \( a_0 \).
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If further (ii) \( F(\alpha(x), a) - F(x, a) \equiv 0 \) as \( B_{2j+1} = 0, \ j = 0, \ldots, k, \) then the cyclicity of (1.1) at the origin is \( k \) for all \( a \) near \( a_0 \).

Moreover, when \( F \) is linear in \( a \) then the cyclicity of (1.1) at the origin is \( k \) for all \( a \in D \), and hence we have \( H^*_D = \hat{H}_D = k \) in this case.
Remark 1.1.

If $F(\alpha(x), a) - F(x, a) \equiv 0$ (resp., $F(\alpha(x), a) - F(x, a) \neq 0$) for $0 < x < x_0$ with $x_0$ small then the origin is a center (resp., focus) of (1.1) (Filippov).
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An easy corollary of the above theorem is that $H^*_D = \hat{H}_D = \left[\frac{n-1}{2}\right]$ for the system

$$\dot{x} = y - \sum_{j=1}^{n} a_j x^j, \quad \dot{y} = -x,$$

where $D = R^n = \{(a_1, a_2, \ldots, a_n)| a_j \in R\}$, $n \geq 1$. 
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where $D = \mathbb{R}^n = \{(a_1, a_2, \ldots, a_n)| a_j \in \mathbb{R}\}, \ n \geq 1$.

In the same paper [Han,1999], the author proved $H^*_D = \hat{H}_D = \left[\frac{2n-1}{3}\right]$ for the system

$$\dot{x} = y - \sum_{j=1}^{n} a_j x^j, \quad \dot{y} = -x(1 + x), \quad n \geq 1.$$
A new proof of this conclusion can be found in [Tian and Han, 2011, JDE].
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By applying Theorem 1.1, [Jiang, Han, Yu and Lynch, 2007] considered a $\mathbb{Z}_2$-equivariant degree $n$ Liénard system of the form

$$\begin{align*}
\dot{x} &= y - \sum_{i=0}^{n} a_i x^{2i+1}, \\
\dot{y} &= -x(x^2 - 1)
\end{align*}$$

which has two singular points of index +1, $A(1, y_0)$ and $B(-1, -y_0)$, with

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$$y_0 = \sum_{i=0}^{n} a_i,$$

and showed that the cyclicity of the system at $A$ and $B$ each is $n$. Hence, the maximal number of small-amplitude limit cycles is $2n$. 
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**Theorem 1.2.** If there exists \( k \geq 1 \) such that for \( j \geq k + 1 \),
\[
B_{2j+1} = O(B_1, B_3, \cdots, B_{2k+1}) \text{ as } |B_1|, |B_3|, \cdots, |B_{2k+1}| \text{ are sufficiently small, then there exists a neighborhood } U \text{ of the origin such that Eq. (1.1) has at most } k \text{ limit cycles in } U \text{ for all } a \in D.
\]
By using Theorems 1.1 and 1.2 it was proved in [Jiang and Han, 2009] that the system

\[
\dot{x} = y - \frac{\sum_{i=1}^{n} a_i x^i}{1 + \sum_{i=1}^{m} b_i x^i}, \quad \dot{y} = -g(x),
\]

where \( g(0) = 0 \), \( g'(0) > 0 \), \( g(-x) = -g(x) \), has the cyclicity \( \left\lfloor \frac{n+m-1}{2} \right\rfloor \) at the origin.
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where \( g(0) = 0, \) \( g'(0) > 0, \) \( g(-x) = -g(x), \) has the cyclicity \( \left\lfloor \frac{n + m - 1}{2} \right\rfloor \) at the origin. The result implies that

\[
H_D^* = \hat{H}_D = \left\lfloor \frac{n + m - 1}{2} \right\rfloor,
\]

with \( D = R^{n+m} = \{(a_1, \cdots, a_n, b_1, \cdots, b_m)\} \) for the above system.
In [Tian and Han, 2011, JDE] we also use Theorems 1.1 and 1.2 to consider the following system

\[
\dot{x} = y - \frac{\sum_{i=1}^{n} a_i x^i}{1 + \sum_{i=1}^{m} b_i x^i}, \quad \dot{y} = -x(x + 1),
\]

proving that an upper bound of the maximum number of limit cycles near the origin is \(\left\lfloor \frac{4n+2m-4}{3} \right\rfloor - \left\lfloor \frac{n-m}{3} \right\rfloor\) as \(n \geq m\) or \(\left\lfloor \frac{4m+2n-4}{3} \right\rfloor - \left\lfloor \frac{m-n}{3} \right\rfloor\) as \(n < m\).
In [Tian and Han, 2011, JDE] we also use Theorems 1.1 and 1.2 to consider the following system

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\dot{x} = y - \sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{m} b_i x^i, \quad \dot{y} = -x(x + 1),
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proving that an upper bound of the maximum number of limit cycles near the origin is \(\left\lfloor \frac{4n+2m-4}{3} \right\rfloor - \left\lfloor \frac{n-m}{3} \right\rfloor\) as \(n \geq m\) or \(\left\lfloor \frac{4m+2n-4}{3} \right\rfloor - \left\lfloor \frac{m-n}{3} \right\rfloor\) as \(n < m\), and that the cyclicity of the system at the origin is \(2n - 2\) in the case \(m = n = 1, 2, 3, 4\).
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proving that an upper bound of the maximum number of limit cycles near the origin is \([\frac{4n+2m-4}{3}] - [\frac{n-m}{3}]\) as \(n \geq m\) or \([\frac{4m+2n-4}{3}] - [\frac{m-n}{3}]\) as \(n < m\), and that the cyclicity of the system at the origin is \(2n - 2\) in the case \(m = n = 1, 2, 3, 4\).

The idea of Theorems 1.1 and 1.2 can be applied to piecewise smooth system of the form (1.1), see [Liu and Han, 2010, IJBC] and [Tian and Han, 2011, JDE].
Consider a polynomial Liénard system of the form

\[ \dot{x} = y, \quad \dot{y} = -g_m(x) - f_n(x)y \] (1.3)

where \( f_n \) and \( g_m \) are polynomials in \( x \) of degrees \( n \) and \( m \) respectively, and \( g_m(0) = 0 \), \( g'_m(0) > 0 \).
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where \( f_n \) and \( g_m \) are polynomials in \( x \) of degrees \( n \) and \( m \) respectively, and \( g_m(0) = 0, \ g'_m(0) > 0 \). Taking all coefficients of \( f_n \) and \( g_m \) as parameters, one can define two numbers \( H^*_{n,m} \) and \( \hat{H}_{n,m} \) for (1.3) as before, see [Christopher and Lynch, 1999] and [Gasull and Torregrosa, 1999].
Consider a polynomial Liénard system of the form

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\end{align*} \tag{1.3}

where $f_n$ and $g_m$ are polynomials in $x$ of degrees $n$ and $m$ respectively, and $g_m(0) = 0$, $g'_m(0) > 0$. Taking all coefficients of $f_n$ and $g_m$ as parameters, one can define two numbers $H_{n,m}^*$ and $\hat{H}_{n,m}$ for (1.3) as before, see [Christopher and Lynch, 1999] and [Gasull and Torregrosa, 1999].

In other words, $H_{n,m}^*$ is the maximal order of the origin as a focus of (1.3), and $\hat{H}_{n,m}$ is the cyclicity of (1.3) at the origin for all possible $f_n$ and $g_m$. 
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\dot{x} &= y, \\
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where \( f_n \) and \( g_m \) are polynomials in \( x \) of degrees \( n \) and \( m \) respectively, and \( g_m(0) = 0, \ g'_m(0) > 0 \). Taking all coefficients of \( f_n \) and \( g_m \) as parameters, one can define two numbers \( H_{n,m}^* \) and \( \hat{H}_{n,m} \) for (1.3) as before, see [Christopher and Lynch, 1999] and [Gasull and Torregrosa, 1999].

In other words, \( H_{n,m}^* \) is the maximal order of the origin as a focus of (1.3), and \( \hat{H}_{n,m} \) is the cyclicity of (1.3) at the origin for all possible \( f_n \) and \( g_m \).

For (1.3) we also define \( H_{n,m} \) to be the maximal number of limit cycles of (1.3) on the plane for all possible \( f_n \) and \( g_m \). Obviously, \( H_{n,m} \geq \hat{H}_{n,m} \).
For
\[ \dot{x} = y, \quad \dot{y} = -\bar{g}_k(x) - \varepsilon g_m(x) - \varepsilon f_n(x)y \quad (1.4) \]
where \( \varepsilon \) is a small number, \( \bar{g}_k \) is a given polynomial of degree \( k \) satisfying \( \bar{g}_k(0) = 0, \quad \bar{g}_k'(0) > 0 \), we define two numbers \( \hat{H}_{n,m}^{(k)} \) and \( \tilde{H}_{n,m}^{(k)} \) as follows:
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where \( \varepsilon \) is a small number, \( \bar{g}_k \) is a given polynomial of degree \( k \) satisfying \( \bar{g}_k(0) = 0, \bar{g}_k'(0) > 0 \), we define two numbers \( \hat{H}^{(k)}_{n,m} \) and \( \tilde{H}^{(k)}_{n,m} \) as follows:
\[ \hat{H}^{(k)}_{n,m} = \text{the cyclicity of (1.4) at the origin for all possible } f_n \text{ and } g_m. \]
\[ \tilde{H}^{(k)}_{n,m} = \text{the maximal number of limit cycles of (1.4) for all possible } f_n \text{ and } g_m. \]
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\( \hat{H}^{(k)}_{n,m} =: \) the cyclicity of (1.4) at the origin for all possible \( f_n \) and \( g_m \).
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Finally, for
\[ \dot{x} = y, \quad \dot{y} = -g_m(x) - \varepsilon f_n(x)y \] (1.5)
we introduce \( \tilde{H}_{n,m} \) to be the maximal number of limit cycles of (1.5) on the plane for all possible \( f_n \) and \( g_m \).
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\( \hat{H}_{n,m}^{(k)} \): the cyclicity of (1.4) at the origin for all possible \( f_n \) and \( g_m \).
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Finally, for
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we introduce \( \tilde{H}_{n,m} \) to be the maximal number of limit cycles of (1.5) on the plane for all possible \( f_n \) and \( g_m \).

It is obvious that \( H_{n,m} \geq \tilde{H}_{n,m} \).
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1. For (1.3), Is it true that $H^*_n,m = \hat{H}_n,m$? Does it hold $H^*_n,m \geq \hat{H}_n,m$ or $H^*_n,m \leq \hat{H}_n,m$?
Then we have the following problems:

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2. For (1.3), Find $H_{n,m}^*$, $\hat{H}_{n,m}$.
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1. For (1.3), Is it true that $H_{n,m}^* = \hat{H}_{n,m}$? Does it hold
   $H_{n,m}^* \geq \hat{H}_{n,m}$ or $H_{n,m}^* \leq \hat{H}_{n,m}$?

2. For (1.3), Find $H_{n,m}^*$, $\hat{H}_{n,m}$.

3. For (1.3) and (1.5), Find a sharp lower bound of $H_{n,m}$ and
   $\tilde{H}_{n,m}$. 
Then we have the following problems:

1. For (1.3), Is it true that \( H_{n,m}^* = \hat{H}_{n,m} \)? Does it hold \( H_{n,m}^* \geq \hat{H}_{n,m} \) or \( H_{n,m}^* \leq \hat{H}_{n,m} \)?

2. For (1.3), Find \( H_{n,m}^*, \hat{H}_{n,m} \).

3. For (1.3) and (1.5), Find a sharp lower bound of \( H_{n,m} \) and \( \tilde{H}_{n,m} \).

4. For (1.4), Find \( \hat{H}_{n,m}^{(k)} \) and \( \tilde{H}_{n,m}^{(k)} \).
From the discussion after Theorem 1.1 we have

\[ H_{n,1}^* = \hat{H}_{n,1} = \left[ \frac{n}{2} \right], \quad H_{n,2}^* = \hat{H}_{n,2} = \left[ \frac{2n + 1}{3} \right]. \]
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From the discussion in [Christopher and Lynch, 1999], one can see that

\[ H_{n,3}^* = 2\left[ \frac{3n + 2}{8} \right] \leq \hat{H}_{n,3} \text{ for } 1 \leq n \leq 50, \]

\[ H_{3,m}^* = 2\left[ \frac{3m + 2}{8} \right] \leq \hat{H}_{3,m} \text{ for } 1 \leq m \leq 50. \]
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One can find the values of \( H_{n,m}^* \) and \( \hat{H}_{n,m} \) for many concrete \( n \) and \( m \) from the works of

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[Gasull and Torregrosa, 1999].
[Jiang, Han, Yu and Lynch, 2007],
[Lloyd and Lynch, 1988],
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[Lloyd and Lynch, 1988],
[Lynch and Christopher, 1999], [Yu and Han, 2006] and many more works
[Jiang, Han, Yu and Lynch, 2007], [Lloyd and Lynch, 1988], [Lynch and Christoper, 1999], [Yu and Han, 2006] and many more works which suggest $H_{n,m}^* = \hat{H}_{n,m}$. However, it must be nontrivial to prove either $H_{n,m}^* \leq \hat{H}_{n,m}$ or $H_{n,m}^* \geq \hat{H}_{n,m}$ for general system (1.3).
From [Han, 2002], we know that for the system

\[ \dot{x} = y, \quad \dot{y} = -x(1 - x) - \varepsilon f_n(x)y \]

we have \( \tilde{H}_{n,2} = \left[ \frac{2n+1}{3} \right] \).
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In [Llibre, Mereu and Teixeira, 2010], the authors consider a system of the form

\[ \dot{x} = y, \quad \dot{y} = -x - \varepsilon g_m(x) - \varepsilon f_n(x)y. \quad (1.6) \]
By using the averaging theory of order 3, the authors obtained for (1.6)

\[ \tilde{H}^{(1)}_{n,m} \geq \left\lfloor \frac{n + m - 1}{2} \right\rfloor. \]

The proof of the above inequality is very technical and complicated.
By using the averaging theory of order 3, the authors obtained for (1.6)

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There are many studies on global bifurcations of limit cycles for (1.3), by using the first order Melnikov functions,
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The proof of the above inequality is very technical and complicated.

There are many studies on global bifurcations of limit cycles for (1.3), by using the first order Melnikov functions, related to homoclinic bifurcation or heteroclinic bifurcation, or bifurcation from a periodic annular or from infinity. We will not list them.
2. Some new studies on Hopf bifurcation

As was shown in [Han, 1999], [Jiang and Han, 2009], [Tian and Han, 2011], if the function $F$ or $g$ in (1.1) has a particular form, then one can use Theorems 1.1 and 1.2 to find a sharp estimate of the number of limit cycles in Hopf bifurcations.
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However, if both of the functions are polynomials of arbitrary degrees, the theorems are very hard to be used to find a sharp estimate.
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To overcome this difficulty, recently, [Han, Tian and Yu, 2011] introduce a small parameter in (1.1) and establish a general theorem, and then apply it to give a new lower bound of the maximal number of limit cycles for arbitrary $m$ and $n$. 
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In this section we introduce the main results of the paper.
Consider a system of the form with small parameter $\lambda$

$$\dot{x} = y, \quad \dot{y} = -g_0(x) - \lambda g_1(x) - \lambda^2 g_2(x) - [f_0(x) + \lambda f_1(x) + \lambda^2 f_2(x)]y,$$

(2.1)

where $f_j$ and $g_j$ are $C^\infty$ functions with

$$g_j(0) = 0, \quad j = 0, 1, 2, \quad g'_0(0) > 0.$$

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We will discuss the number of limit cycles in Hopf bifurcation.

Let

$$F(x, \lambda) = F_0(x) + \lambda F_1(x) + \lambda^2 F_2(x),$$

$$G(x, \lambda) = G_0(x) + \lambda G_1(x) + \lambda^2 G_2(x),$$

where

$$F_j(x) = \int_0^x f_j(x) \, dx, \quad G_j(x) = \int_0^x g_j(x) \, dx, \ j = 0, 1, 2.$$
The main results of [Han, Tian and Yu, 2011] can be stated as follows.

**Theorem 2.1.** Let

\[ \Phi(x, \lambda) = F(\alpha(x, \lambda), \lambda) - F(x, \lambda) \]

where \( \alpha(x, \lambda) = -x + O(x^2) \) satisfies \( G(\alpha(x, \lambda), \lambda) = G(x, \lambda) \) for \((x, \lambda)\) near \((0, 0)\). Then

\[ \alpha(x, \lambda) = \alpha_0(x) + \lambda \alpha_1(x) + \lambda^2 \alpha_2(x) + \cdots, \]

and

\[ \Phi(x, \lambda) = \Phi_0(x) + \lambda \Phi_1(x) + \lambda^2 \Phi_2(x) + \cdots, \]

where

\[ G_0(\alpha_0(x)) = G_0(x), \quad \alpha_1(x) = \frac{G_1(x) - G_1(\alpha_0(x))}{g_0(\alpha_0(x))}, \]
\[ \alpha_2(x) = \frac{1}{g_0(\alpha_0(x))} \left[ G_2(x) - G_2(\alpha_0(x)) - g_1(\alpha_0(x))\alpha_1(x) \right. \\
\left. \quad - \frac{1}{2} g_0'(\alpha_0(x))\alpha_1^2(x) \right], \]

\Phi_0(x) = F_0(\alpha_0(x)) - F_0(x), \quad \Phi_1(x) = F_1(\alpha_0(x)) - F_1(x) + f_0(\alpha_0(x))\alpha_1(x),

and

\Phi_2(x) = F_2(\alpha_0(x)) - F_2(x) + f_0(\alpha_0(x))\alpha_2(x) + f_1(\alpha_0(x))\alpha_1(x) + \frac{1}{2} f_0'(\alpha_0(x))\alpha_1^2(x).

In particular, if \( \Phi_0(x) \equiv 0 \), then

\[ \Phi_1(x) = \frac{1}{g_0(x)} \left\{ g_0(x)[F_1(\alpha_0(x)) - F_1(x)] - f_0(x)[G_1(\alpha_0(x)) - G_1(x)] \right\}. \tag{2.2} \]
Now suppose the functions $f_j$ and $g_j$ in (2.1) depend on a vector parameter $\delta \in \mathbb{R}^m$. Then the function $\Phi = \Phi(x, \lambda, \delta)$ is a function of $(x, \lambda, \delta)$. In this case we have

**Theorem 2.2.** Let for $k = 1$ or $k = 2$

$$\Phi(x, \lambda, \delta) = \lambda^k \varphi_0(x, \delta) \tilde{\Phi}_k(x, \delta) + O(\lambda^{k+1}),$$

$$\tilde{\Phi}_k(x, \delta) = \sum_{j \geq 1} B^*_j(\delta) x^j, \quad \varphi_0(0, \delta) \neq 0.$$

Suppose there exists $\delta_0 \in \mathbb{R}^m$ such that

$$B^*_{2j+1}(\delta_0) = 0, \quad j = 0, 1, \ldots, m - 1,$$

and

$$\det \frac{\partial (B^*_1, B^*_3, \ldots, B^*_{2m-1})}{\partial (\delta_1, \delta_2, \ldots, \delta_m)}(\delta_0) \neq 0.$$
Then

(i) If $B_{2m+1}(\delta_0) \neq 0$, Eq.(2.1) has at least $m$ limit cycles, each having an odd multiplicity, in an arbitrary neighborhood of the origin for some $(\lambda, \delta)$ sufficiently closed to $(0, \delta_0)$.
Then

(i) If \( B_{2m+1}(\delta_0) \neq 0 \), Eq.(2.1) has at least \( m \) limit cycles, each having an odd multiplicity, in an arbitrary neighborhood of the origin for some \((\lambda, \delta)\) sufficiently closed to \((0, \delta_0)\).

(ii) If \( \Phi(x, \lambda, \delta) \equiv 0 \) as \( B_{2j+1}^* = 0 \), \( j = 0, 1, \ldots, m - 1 \), then there exist a constant \( \varepsilon > 0 \) and a neighborhood \( U \) of the origin such that for all \( |\lambda| < \varepsilon, |\delta - \delta_0| < \varepsilon \) Eq.(2.1) has at most \( m - 1 \) limit cycles in \( U \). Moreover, \( m - 1 \) limit cycles can appear in an arbitrary neighborhood of the origin for some \((\lambda, \delta)\) near \((0, \delta_0)\).
We remark that if $\delta \in \mathbb{R}^{m+1}$ and

$$B_{2j+1}^*(\delta_0) = 0, \ j = 0, 1, \cdots, m,$$

$$\det \frac{\partial (B_1^*, B_3^*, \cdots, B_{2m+1}^*)}{\partial (\delta_1, \delta_2, \cdots, \delta_{m+1})}(\delta_0) \neq 0$$

for some $\delta_0$, then by the first conclusion of the above theorem it is obvious that Eq.(2.1) has at least $m$ limit cycles in an arbitrary neighborhood of the origin for some $(\lambda, \delta)$ sufficiently closed to $(0, \delta_0)$. 

To show that the above theorem is a nice development of Theorem 1.1, we apply it to the polynomial system (1.4)

$$\dot{x} = y,$$

$$\dot{y} = -\bar{g}_k(x) - \epsilon g_m(x) - \epsilon f_n(x)y,$$

where $\epsilon$ is small and $\bar{g}_k$ is a polynomial of degree $k$ satisfying $\bar{g}_k(0) = 0$ and $\bar{g}_k'(0) > 0$. 
We remark that if $\delta \in \mathbb{R}^{m+1}$ and

$$B_{2j+1}^*(\delta_0) = 0, \ j = 0, 1, \cdots, m,$$

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We can obtain

**Theorem 2.3.** Let $\hat{H}^{(k)}_{n,m}$ denote the maximal number of limit cycles near the origin of (1.4) for all possible $f_n$ and $g_m$. Then for $m, n \geq 1$

$$\hat{H}^{(1)}_{n,m} \geq \left\lceil \frac{n + m - 1}{2} \right\rceil,$$

$$\hat{H}^{(2)}_{n,m} \geq \max\{\left\lceil \frac{m - 2}{3} \right\rceil + \left\lceil \frac{2n + 1}{3} \right\rceil, \left\lceil \frac{n - 2}{3} \right\rceil + \left\lceil \frac{2m + 1}{3} \right\rceil\}.$$
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**Corollary 2.1.** For (1.3) we have

$$\hat{H}_{n,m} \geq \max\{\left\lfloor \frac{m - 2}{3} \right\rfloor + \left\lfloor \frac{2n + 1}{3} \right\rfloor, \left\lfloor \frac{n - 2}{3} \right\rfloor + \left\lfloor \frac{2m + 1}{3} \right\rfloor\},$$

where $m \geq 2$, $n \geq 1$. 
It is not hard to prove that
\[ h_{m,n} \equiv \max\{\left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{2n+1}{3} \right\rfloor, \left\lfloor \frac{n-2}{3} \right\rfloor + \left\lfloor \frac{2m+1}{3} \right\rfloor \} \geq \left\lceil \frac{n+m-1}{2} \right\rceil. \]
It is not hard to prove that

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For example, in the case of \( m = 3n + 1, 3n + 2 \) we have

\[ h_{m,n} = \left\lceil \frac{n-2}{3} \right\rceil + 2n + 1 > 2n = \left\lceil \frac{(n + m - 1)}{2} \right\rceil. \]
3. Proof of Theorems 2.1, 2.2 and 2.3

Consider an equivalent form of (2.1) as follows

\[ \dot{x} = y - F(x, \lambda), \quad \dot{y} = -g_0(x) - \lambda g_1(x) - \lambda^2 g_2(x) \equiv -g(x, \lambda). \] (1)
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\]

**Lemma 3.1.** There exists a unique function \( y = \alpha(x, \lambda) \) with \( \alpha(0, 0) = 0 \) and \( \frac{\partial \alpha}{\partial x}(0, \lambda) = -1 \) such that \( G(\alpha(x, \lambda), \lambda) = G(x, \lambda) \). Moreover,

\[
\alpha(x, \lambda) = \alpha_0(x) + \lambda \alpha_1(x) + \lambda^2 \alpha_2(x) + \cdots,
\]

where

\[
G_0(\alpha_0(x)) = G_0(x), \quad \alpha_1(x) = \frac{G_1(x) - G_1(\alpha_0(x))}{g_0(\alpha_0(x))},
\]

\[
\alpha_2(x) = \frac{1}{g_0(\alpha_0(x))}[G_2(x) - G_2(\alpha_0(x)) - g_1(\alpha_0(x))\alpha_1(x) - \frac{1}{2}g'_0(\alpha_0(x))\alpha^2_1(x)]
\]
Proof.

Let

\[ H(x, y, \lambda) = \begin{cases} \frac{G(y, \lambda) - G(x, \lambda)}{y - x}, & y \neq x \\ g(x, \lambda), & y = x. \end{cases} \]

Then \( H \) is of \( C^\infty \) and satisfies

\[ H(x, y, \lambda) = \sum_{j=0}^{2} \frac{G_j(y) - G_j(x)}{y - x} \lambda^j. \]
Proof.

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\[ H(x, y, \lambda) = \begin{cases} \frac{G(y, \lambda) - G(x, \lambda)}{y - x}, & y \neq x \\ g(x, \lambda), & y = x. \end{cases} \]

Then \( H \) is of \( C^\infty \) and satisfies

\[ H(x, y, \lambda) = \sum_{j=0}^{2} \frac{G_j(y) - G_j(x)}{y - x} \lambda^j. \]

Since \( G_j(x) = \frac{1}{2} g_j'(0)x^2 + O(|x|^3) \) we have

\[ G_j(y) - G_j(x) = (y - x)\left[\frac{1}{2} g_j'(0)(y + x) + O(|x, y|^2)\right]. \]
Proof.

Let

\[ H(x, y, \lambda) = \begin{cases} 
\frac{G(y, \lambda) - G(x, \lambda)}{y - x}, & y \neq x \\
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Since \( G_j(x) = \frac{1}{2} g_j'(0)x^2 + O(|x|^3) \) we have

\[ G_j(y) - G_j(x) = (y - x)[\frac{1}{2} g_j'(0)(y + x) + O(|x, y|^2)]. \]

Hence, the first part follows from the implicit function theorem.
Then twice differentiating both sides of the quality \( G(\alpha(x, \lambda), \lambda) = G(x, \lambda) \) in \( \lambda \) yields

\[
G_x(\alpha, \lambda)\alpha_\lambda + G_\lambda(\alpha, \lambda) = G_\lambda(x, \lambda),
\]

and

\[
G_{xx}(\alpha, \lambda)(\alpha_\lambda)^2 + G_x(\alpha, \lambda)\alpha_{\lambda\lambda} + 2G_{\lambda x}(\alpha, \lambda)\alpha_\lambda + G_{\lambda\lambda}(\alpha, \lambda) = G_{\lambda\lambda}(x, \lambda).
\]
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G_{xx}(\alpha, \lambda)(\alpha_\lambda)^2 + G_x(\alpha, \lambda)\alpha_{\lambda\lambda} + 2G_{\lambda x}(\alpha, \lambda)\alpha_\lambda + G_{\lambda\lambda}(\alpha, \lambda) = G_{\lambda\lambda}(x, \lambda).
\]

Thus, taking \( \lambda = 0 \) in the above two formulas gives the formula of \( \alpha_1 \) and \( \alpha_2 \). This finishes the proof.
Lemma 3.2.

Let
\[ \Phi(x, \lambda) = F(\alpha(x, \lambda), \lambda) - F(x, \lambda) = \Phi_0(x) + \lambda \Phi_1(x) + \lambda^2 \Phi_2(x) + \cdots. \]

Then
\[ \Phi_0(x) = F_0(\alpha_0(x)) - F_0(x), \quad \Phi_1(x) = F_1(\alpha_0(x)) - F_1(x) + f_0(\alpha_0(x)) \]
\[ \Phi_2(x) = F_2(\alpha_0(x)) - F_2(x) + f_0(\alpha_0(x)) \alpha_2(x) + f_1(\alpha_0(x)) \alpha_1(x) + \frac{1}{2} f'_0(\alpha_0(x)). \]

In particular, if \( \Phi_0(x) \equiv 0 \), then (2.2) holds.
Proof.

We have

\[ F(\alpha, \lambda) = F_0(\alpha) + \lambda F_1(\alpha) + \lambda^2 F_2(\alpha), \]

\[ F_0(\alpha) = F_0(\alpha_0) + \lambda f_0(\alpha_0) \alpha_1 + \lambda^2 [f_0(\alpha_0) \alpha_2 + \frac{1}{2} f_0'(\alpha_0) \alpha_1^2] + O(\lambda^3), \]

and

\[ F_1(\alpha) = F_1(\alpha_0) + \lambda f_1(\alpha_0) \alpha_1 + O(\lambda^2). \]
Proof.

We have
\[ F(\alpha, \lambda) = F_0(\alpha) + \lambda F_1(\alpha) + \lambda^2 F_2(\alpha), \]
\[ F_0(\alpha) = F_0(\alpha_0) + \lambda f_0(\alpha_0)\alpha_1 + \lambda^2 [f_0(\alpha_0)\alpha_2 + \frac{1}{2} f'_0(\alpha_0)\alpha_1^2] + O(\lambda^3), \]
and
\[ F_1(\alpha) = F_1(\alpha_0) + \lambda f_1(\alpha_0)\alpha_1 + O(\lambda^2). \]

Then the formula of \( \Phi_0, \Phi_1 \) and \( \Phi_2 \) follows immediately.
When $\Phi_0(x) \equiv 0$, we have

$$f_0(\alpha_0)\alpha'_0 = f_0(x), \quad g_0(\alpha_0)\alpha'_0 = g_0(x)$$

which implies

$$\frac{f_0(\alpha_0)}{g_0(\alpha_0)} = \frac{f_0(x)}{g_0(x)}.$$

When $\Phi_0(x) \equiv 0$, we have

$$f_0(\alpha_0)\alpha'_0 = f_0(x), \quad g_0(\alpha_0)\alpha'_0 = g_0(x)$$

which implies

$$\frac{f_0(\alpha_0)}{g_0(\alpha_0)} = \frac{f_0(x)}{g_0(x)}.$$

Hence, (2.2) follows from the formula of $\alpha_1$ and $\Phi_1$. The proof is completed.
Let \((x(t), y(t))\) be the solution of (1) with the initial value \(x(0) = 0, \ y(0) = r_0\). Then there exists a first return time \(\tau_0(r_0, \lambda) \in C^\infty\) such that \(x(\tau_0) = 0\).
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\[x(\tau_0) = 0.\]

Define 
\[d(r_0, \lambda) = y(\tau_0) - r_0.\] It is obvious that the function \(d\) is
\(C^\infty\) for \((r_0, \lambda)\) near \((0, 0)\) and system (1) has a periodic orbit near
the origin if and only if the function has two zeros in \(r_0\) (with one
positive and the other negative) near \(r_0 = 0.\)
Let \((x(t), y(t))\) be the solution of (1) with the initial value \(x(0) = 0, \ y(0) = r_0\). Then there exits a first return time \(\tau_0(r_0, \lambda) \in C^\infty\) such that \(x(\tau_0) = 0\).

Define \(d(r_0, \lambda) = y(\tau_0) - r_0\). It is obvious that the function \(d\) is \(C^\infty\) for \((r_0, \lambda)\) near \((0, 0)\) and system (1) has a periodic orbit near the origin if and only if the function has two zeros in \(r_0\) (with one positive and the other negative) near \(r_0 = 0\).

The function \(d\) is called a displacement function or a bifurcation function.
From [Han, 1999] we have the following lemma.

**Lemma 3.3.** Suppose formally

\[ F(\alpha(x, \lambda), \lambda) - F(x, \lambda) = \sum_{i \geq 1} B_i(\lambda)x^i, \tag{2} \]

and

\[ d(r_0, \lambda) = \sum_{i \geq 1} d_i(\lambda)r_0^i. \]

Then

\[ d_1 = B_1 N_0^*(B_1), \]
\[ d_{2j} = O(|B_1, B_3, \cdots, B_{2j-1}|), \]
\[ d_{2j+1} = B_{2j+1} N_j^*(B_1) + O(|B_1, B_3, \cdots, B_{2j-1}|), \]

with \( N_j^* \in C^\infty \) and \( N_j^*(0) > 0 \) for \( j \geq 0 \).
Moreover, we have

\[ F(\alpha(x, \lambda), \lambda) - F(x, \lambda) = \sum_{i \geq 0} A_{2i+1}(a)u^{2i+1}, \quad (3) \]

where \( u = (\text{sgn} \, x) \sqrt{G(x, \lambda)} \),

\[ A_1 = \left(\frac{2}{g_x(0, \lambda)}\right)^{\frac{1}{2}} B_1, \]

\[ A_{2k+1} = \left(\frac{2}{g_x(0, \lambda)}\right)^{k+\frac{1}{2}} B_{2k+1} + O(|B_1, B_3, \cdots, B_{2k-1}|), \quad k \geq 1. \]
Using the above lemma it is easy to prove the following

**Lemma 3.4.** Suppose formally

\[
F(\alpha(x, \lambda), \lambda) - F(x, \lambda) = \lambda^k \varphi_0(x) \sum_{i \geq 1} \tilde{B}_i(\lambda)x^i,
\]

where \( \varphi_0 \in C^\infty, \varphi_0(0) \neq 0 \). Then

\[
d(r_0, \lambda) = \lambda^k \sum_{i \geq 1} \tilde{d}_i(\lambda)r_0^i,
\]

where
Using the above lemma it is easy to prove the following

**Lemma 3.4.** Suppose formally

\[ F(\alpha(x, \lambda), \lambda) - F(x, \lambda) = \lambda^k \varphi_0(x) \sum_{i \geq 1} \tilde{B}_i(\lambda)x^i, \tag{4} \]

where \( \varphi_0 \in C^\infty, \varphi_0(0) \neq 0 \). Then

\[ d(r_0, \lambda) = \lambda^k \sum_{i \geq 1} \tilde{d}_i(\lambda)r_0^i, \]

where

\[ \tilde{d}_1 = \tilde{B}_1(\beta_0 + O(\lambda^k)), \]
\[ \tilde{d}_{2j} = O(\|\tilde{B}_1, \tilde{B}_3, \cdots, \tilde{B}_{2j-1}\|), \]
\[ \tilde{d}_{2j+1} = \tilde{B}_{2j+1}(\beta_j + O(\lambda^k)) + O(\|\tilde{B}_1, \tilde{B}_3, \cdots, \tilde{B}_{2j-1}\|), \]

with \( \beta_0, \beta_1, \cdots \) being all nonzero constants.
Now we are in a position to prove Theorems 2.1 and 2.2. It is clear that Theorem 2.1 is direct from Lemmas 3.1 and 3.2.
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Then let us prove Theorem 2.2. Suppose all conditions of Theorem 2.2 are satisfied. Then

$$\Phi(x, \lambda, \delta) = \lambda^k \varphi_0(x, \delta) \sum_{i \geq 1} \tilde{B}_i(\lambda, \delta)x^i,$$

where

$$\tilde{B}_i(\lambda, \delta) = B_i^*(\delta) + O(\lambda), \ j \geq 1.$$
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where

\[
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\]

Consider the change of parameters

\[
b_j = \tilde{B}_{2j-1}(\lambda, \delta), \quad j = 1, 2, \cdots, m.
\]
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Consider the change of parameters

\[ b_j = \tilde{B}_{2j-1}(\lambda, \delta), \quad j = 1, 2, \cdots, m. \]

By our assumptions, we can solve from the above equations

\[ \delta = \psi(\lambda, b) = \delta_0 + O(|\lambda, b|), \]

where \( b = (b_1, b_2, \cdots, b_m) \).
Then by Lemma 3.4 the succession function \( d \) has the form

\[
d(r_0, \lambda, \delta) = \lambda^k \sum_{i \geq 1} \tilde{d}_i(\lambda, \delta) r_0^i \equiv \bar{d}(r_0, \lambda, b),
\]

where
Then by Lemma 3.4 the succession function $d$ has the form

$$d(r_0, \lambda, \delta) = \lambda^k \sum_{i \geq 1} \tilde{d}_i(\lambda, \delta) r_0^i \equiv \overline{d}(r_0, \lambda, b),$$

where

$$\tilde{d}_{2j-1} = b_j(\beta_{j-1} + O(|\lambda^k, b|)) + O(|b_1, b_2, \cdots, b_{j-1}|),$$

$$\tilde{d}_{2j} = O(|b_1, b_2, \cdots, b_j|), \quad j = 1, 2, \cdots, m.$$
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$$\tilde{d}_{2j} = O(|b_1, b_2, \cdots, b_j|), \quad j = 1, 2, \cdots, m.$$

Therefore, we can rewrite $\bar{d}$ as follows

$$\bar{d}(r_0, \lambda, b) = \lambda^k [\sum_{j=1}^{m} r_0^{2j-1} b_j P_j(r_0, \lambda, b) + r_0^{2m+1} P_{m+1}(r_0, \lambda, b)], \quad (5)$$
Then by Lemma 3.4 the succession function $d$ has the form

$$d(r_0, \lambda, \delta) = \lambda^k \sum_{i \geq 1} \tilde{d}_i(\lambda, \delta) r_0^i \equiv \bar{d}(r_0, \lambda, b),$$

where

$$\tilde{d}_{2j-1} = b_j(\beta_{j-1} + O(\lambda^k, b)) + O(|b_1, b_2, \cdots, b_{j-1}|),$$

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Therefore, we can rewrite $\bar{d}$ as follows

$$\bar{d}(r_0, \lambda, b) = \lambda^k \left[ \sum_{j=1}^{m} r_0^{2j-1} b_j P_j(r_0, \lambda, b) + \lambda^{2m+1} P_{m+1}(r_0, \lambda, b) \right], \quad (5)$$

where $P_1, \cdots, P_m$ are polynomials in $r_0$ of degree at most $2m-1$, $P_{m+1} \in C^\infty$, $P_j(0,0,0) = \beta_{j-1} \neq 0$ for $j = 1, 2, \cdots, m$, and $P_{m+1}(0,0,0) = B_{2m+1}^*(\delta_0) \beta_m$. 
For definiteness, we may suppose $\varphi_0(0, \delta_0) > 0$, which yields $\beta_j > 0$ for all $j \geq 1$. 
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If $B_{2m+1}^*(\delta_0) \neq 0$, we can change $b_m, b_{m-1}, \cdots, b_1$ in turn such that

$$0 \ll |b_j| \ll |b_{j+1}| \ll 1, \ b_j b_{j+1} < 0, \ b_m B_{2m+1}^*(\delta_0) < 0.$$  

Then by the form of (5), the function $\tilde{d}$ has at least $m$ positive zeros in $r_0$ near $r_0 = 0$ in this case.
For definiteness, we may suppose $\varphi_0(0, \delta_0) > 0$, which yields
$\beta_j > 0$ for all $j \geq 1$.

If $B^*_{2m+1}(\delta_0) \neq 0$, we can change $b_m, b_{m-1}, \cdots, b_1$ in turn such
that

$$0 \ll |b_j| \ll |b_{j+1}| \ll 1, \quad b_j b_{j+1} < 0, \quad b_m B^*_{2m+1}(\delta_0) < 0.$$ 

Then by the form of (5), the function $\tilde{d}$ has at least $m$ positive
zeros in $r_0$ near $r_0 = 0$ in this case.

If $\Phi(x, \lambda, \delta) \equiv 0$ as $B^*_{2j+1} = 0$, $j = 0, 1, \cdots, m - 1$, then

$$\tilde{B}_{2j-1}(\lambda, \delta) = B^*_{2j-1}(\delta) + O(\lambda) = O(B^*_{2j-1}(\delta)), \quad j = 1, 2, \cdots, m,$$

which implies

$$B^*_{2j-1}(\delta) = 0, \quad j = 1, 2, \cdots, m \Leftrightarrow b = 0.$$
Thus, by Lemma 3.4, in this case we have $\bar{d}(r_0, \lambda, 0) = 0$. Hence, the function $P_{m+1}$ in (5) satisfies $P_{m+1}(r_0, \lambda, 0) = 0$. It follows that $\bar{d}(r_0, \lambda, b) = \lambda k \sum_{j=1}^{m} r_j^2 - 1 b \bar{P}_j(r_0, \lambda, b)$, where $\bar{P}_j = P_j + O(r^{2(m-j)+2})$. Then using the above and following the idea from Bautin, we can prove that $\bar{d}$ has at most $m-1$ positive zeros in $r_0$ near $r_0 = 0$ for all small $|b|$, and $m-1$ positive zeros can appear for some $b$. Then the conclusion of Theorem 2.2 follows.
Thus, by Lemma 3.4, in this case we have \( \bar{d}(r_0, \lambda, 0) = 0 \). Hence, the function \( P_{m+1} \) in (5) satisfies \( P_{m+1}(r_0, \lambda, 0) = 0 \).

It follows that

\[
\bar{d}(r_0, \lambda, b) = \lambda^k \sum_{j=1}^{m} r_0^{2j-1} b_j \bar{P}_j(r_0, \lambda, b),
\]
Thus, by Lemma 3.4, in this case we have $\bar{d}(r_0, \lambda, 0) = 0$. Hence, the function $P_{m+1}$ in (5) satisfies $P_{m+1}(r_0, \lambda, 0) = 0$.

It follows that

$$\bar{d}(r_0, \lambda, b) = \lambda^k \sum_{j=1}^{m} r_0^{2j-1} b_j \bar{P}_j(r_0, \lambda, b),$$

where $\bar{P}_j = P_j + O(r_0^{2(m-j)+2})$. Then using the above and following the idea from Bautin, we can prove that $\bar{d}$ has at most $m - 1$ positive zeros in $r_0$ near $r_0 = 0$ for all small $|b|$, and $m - 1$ positive zeros can appear for some $b$.

Then the conclusion of Theorem 2.2 follows.
Now we consider a system of Liénard type with the form

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -g_m(x, \lambda) - f_n(x, \lambda)y,
\end{align*}
\]

(6)

where \( g_m \) and \( f_n \) are polynomials in \( x \) of degree \( m \) and \( n \) respectively, and have the form

\[
\begin{align*}
g_m(x, \lambda) &= g_0(x) + \lambda g_1(x), \\
f_n(x, \lambda) &= f_0(x) + \lambda f_1(x).
\end{align*}
\]

(7)
Now we consider a system of Liénard type with the form

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where \( g_m \) and \( f_n \) are polynomials in \( x \) of degree \( m \) and \( n \) respectively, and have the form

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\begin{align*}
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\end{align*}
\]

(7)

First, we take

\[
\begin{align*}
g_0(x) &= \sum_{j=0}^{m_1} a_{0j}x^{2j+1}, \quad 2m_1 + 1 \leq m, \\
g_1(x) &= \sum_{j=1}^{m} a_{1j}x^j, \\
f_0(x) &= \sum_{j=0}^{n_1} b_{0j}x^{2j+1}, \quad 2n_1 + 1 \leq n, \\
f_1(x) &= \sum_{j=0}^{n} b_{1j}x^j,
\end{align*}
\]

(8)

where \( a_{00} > 0 \).
In this case, we have by (2.2)

\[ \Phi_1(x) = \frac{1}{g_0^*(x)} \tilde{\Phi}_1(x), \]
In this case, we have by (2.2)

$$\Phi_1(x) = \frac{1}{g_0^*(x)} \tilde{\Phi}_1(x),$$

where

$$\tilde{\Phi}_1(x) = g_0^*(x)[F_1(-x) - F_1(x)] - f_0^*(x)[G_1(-x) - G_1(x)],$$

$$g_0^*(x) = \frac{g_0(x)}{x} = \sum_{j=0}^{m_1} a_{0j} x^{2j}, \quad f_0^*(x) = \frac{f_0(x)}{x} = \sum_{j=0}^{n_1} b_{0j} x^{2j},$$

$$F_1(-x) - F_1(x) = \sum_{j=0}^{n_2} b_j x^{2j+1}, \quad n_2 = \lfloor n/2 \rfloor, \quad b_j = -\frac{b_{1,2j}}{2j + 1},$$

and

$$G_1(-x) - G_1(x) = \sum_{j=1}^{m_2} \bar{a}_j x^{2j+1}, \quad m_2 = \lceil m/2 \rceil, \quad \bar{a}_j = -\frac{a_{1,2j}}{2j + 1}.$$
Letting $r = x^2$, then it is easy to see that

$$\tilde{\Phi}_1(x) = x\left[ \sum_{j=0}^{m_1} a_{0j} r^j \sum_{j=0}^{n_2} b_{j} r^j - \sum_{j=0}^{n_1} b_{0j} r^j \sum_{j=1}^{m_2} \bar{a}_j r^j \right] \equiv x S_{\bar{M}}(r),$$

where
Letting $r = x^2$, then it is easy to see that

$$
\tilde{\Phi}_1(x) = x \left[ \sum_{j=0}^{m_1} a_0 j^r \sum_{j=0}^{n_2} b_j r^j - \sum_{j=0}^{n_1} b_0 j^r \sum_{j=1}^{m_2} \bar{a}_j r^j \right] \equiv x S_{\bar{M}}(r),
$$

where

$$
S_{\bar{M}}(r) = \sum_{k=0}^{\bar{M}} B_{k+1} r^k, \quad \bar{M} = \max\{m_1 + n_2, \ n_1 + m_2\},
$$

$$
B_{k+1} = \sum_{i+j=k, \ 0 \leq i \leq m_1} a_0 i \bar{b}_j - \sum_{i+j=k, \ 0 \leq i \leq n_1} b_0 i \bar{a}_j.
$$
Let first $m$ be even. In this case, we take $m_1 = 0$, $n_1 = \left[\frac{n-1}{2}\right]$ so that

$$m_1 + n_2 \leq \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right] \leq \left[\frac{m-1}{2}\right] + \left[\frac{m}{2}\right] = \bar{M}.$$  

We further take $a_{00} = 1$, $b_{0n_1} \neq 0$, $b_{0i} = 0$ for $0 \leq i \leq n_1 - 1$, and set

$$\mu = b_{0n_1}, \quad \delta = (\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n_2}, \bar{a}_{n_2-n_1+1}, \ldots, \bar{a}_{m_2}) \in \mathbb{R}^{\bar{M}+1}.$$  

Then

$$B_{k+1} = \begin{cases} 
\bar{b}_k, & 0 \leq k \leq n_1 \\
\bar{b}_k - b_{0n_1} \bar{a}_{n_2-n_1}, & k = n_2 \text{ (for } n \text{ even)} \\
b_{0n_1} \bar{a}_{k-n_1}, & n_2 + 1 \leq k \leq \bar{M}.
\end{cases}$$

Evidently

$$B_{k+1} = 0, \quad k = 0, 1, \ldots, \bar{M} \iff \delta = 0;$$

$$\det \frac{\partial (B_1, \ldots, B_{\bar{M}+1})}{\partial \delta}|_{\delta=0} = (-b_{0n_1})^{\bar{M}-n_2}.$$
Now let $m$ be odd. Then take $n_1 = 0$, $m_1 = \lceil \frac{m-1}{2} \rceil$ so that  

$$n_1 + m_2 \leq \left\lceil \frac{n-1}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor \leq \left\lceil \frac{m-1}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor = \bar{M}.$$

Further take $a_{00} = 1$, $a_{0m_1} \neq 0$, $b_{00} \neq 0$, $a_{0i} = 0$ for $0 \leq i \leq m_1 - 1$, and set  

$$\mu = (a_{0m_1}, b_{00}), \quad \delta = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{m_2}, \bar{b}_0, \ldots, \bar{b}_{n_2}) \in \mathbb{R}^{\bar{M}+1}.$$

Then noting $m_1 = m_2$, we have  

$$B_{k+1} = \begin{cases} 
\bar{b}_0, & k = 0 \\
\bar{b}_k - b_{00} \bar{a}_k, & 1 \leq k \leq m_2 - 1 \\
\bar{b}_k + a_{0m_1} \bar{b}_0 - b_{00} \bar{a}_k, & k = m_2 \\
\bar{b}_k + a_{0m_1} \bar{b}_{k-m_1}, & m_2 + 1 \leq k \leq \bar{M}.
\end{cases}$$
As above

\[ B_{k+1} = 0, \ k = 0, 1, \cdots, \tilde{M} \iff \delta = 0; \]
\[ \det \frac{\partial (B_1, \cdots, B_{\tilde{M}+1})}{\partial \delta} \bigg|_{\delta=0} \neq 0. \]

Then by Theorem 2.2(i) under (8) for all \( 0 < |\lambda| \ll |\mu| \) and some \( \delta \) near 0, the system (6) can have at least \( \tilde{M} \) limit cycles near the origin.
As above

\[ B_{k+1} = 0, \ k = 0, 1, \cdots, \tilde{M} \Leftrightarrow \delta = 0; \]
\[ \det \frac{\partial (B_1, \cdots, B_{\tilde{M}+1})}{\partial \delta} |_{\delta=0} \neq 0. \]

Then by Theorem 2.2(i) under (8) for all \( 0 < |\lambda| \ll |\mu| \) and some \( \delta \) near 0, the system (6) can have at least \( \tilde{M} \) limit cycles near the origin.

Note that \( \tilde{M} = \left\lfloor \frac{m+n-1}{2} \right\rfloor \) and that \( |\mu| \) can be taken very small. We have immediately
Lemma 3.5. For any neighborhood $U$ of the origin there exist $\varepsilon_0 > 0$ and functions $g_0$, $g_1$, $f_0$ and $f_1$ of the form (8) where $a_{00} > 0$, $|a_{0j}| < \varepsilon_0$ for $1 \leq j \leq m_1$, $|b_{0j}| < \varepsilon_0$ for $0 \leq j \leq n_1$, $|a_{1j}| < \varepsilon_0$ for $1 \leq j \leq m$ and $|b_{1j}| < \varepsilon_0$ for $0 \leq j \leq n$, such that for all $0 < |\lambda| < \varepsilon_0$, Eq.(6) has at least $\left\lceil \frac{m+n-1}{2} \right\rceil$ limit cycles in $U$.

Next, we take $g_0$, $g_1$, $f_0$ and $f_1$ in (7) to have the form

$$
\begin{align*}
g_0(x) &= \bar{g}(x) \sum_{j=0}^{m_1} a_{0j}[\bar{G}(x)]^j, \quad g_1(x) = \sum_{j=1}^{m} a_{1j}x^j, \\
f_0(x) &= \bar{g}(x) \sum_{j=0}^{n_1} b_{0j}[\bar{G}(x)]^j, \quad f_1(x) = \sum_{j=0}^{n} b_{1j}x^j,
\end{align*}
$$

(9)
where

\[ a_{00} > 0, \quad \bar{g}(x) = x(1-x), \quad \bar{G}(x) = x^2/2 - x^3/3, \quad 3m_1 + 2 \leq m, \quad 3n_1 + 2 \leq n. \]

Obviously,

\[ G_0(x) = \int_0^x g_0(x) \, dx = \sum_{j=0}^{m_1} \frac{a_{0j}}{j+1} [\bar{G}(x)]^{j+1}, \]

\[ F_0(x) = \int_0^x f_0(x) \, dx = \sum_{j=0}^{n_1} \frac{b_{0j}}{j+1} [\bar{G}(x)]^{j+1} \]

which satisfy

\[ G_0(\alpha_0(x)) = G_0(x), \quad F_0(\alpha_0(x)) = F_0(x), \]

where \( \alpha_0(x) = -x + O(x^2) \) is defined by \( \bar{G}(\alpha_0(x)) = \bar{G}(x) \).
By (2.2) we have

\[ \Phi_1(x) = \frac{1}{g_0^*(x)} \tilde{\Phi}_1(x), \]

where

\[ \tilde{\Phi}_1(x) = g_0^*(x) [F_1(\alpha_0) - F_1(x)] - f_0^*(x) [G_1(\alpha_0) - G_1(x)], \]

\[ g_0^*(x) = \frac{g_0(x)}{\bar{g}(x)} = \sum_{j=0}^{m_1} a_{0j} \bar{G}^j, \quad f_0^*(x) = \frac{f_0(x)}{\bar{g}(x)} = \sum_{j=0}^{n_1} b_{0j} \bar{G}^j. \]
By (2.2) we have

$$\Phi_1(x) = \frac{1}{g_0^*(x)} \tilde{\Phi}_1(x),$$

where

$$\tilde{\Phi}_1(x) = g_0^*(x) [F_1(\alpha_0) - F_1(x)] - f_0^*(x) [G_1(\alpha_0) - G_1(x)],$$

$$g_0^*(x) = \frac{g_0(x)}{\bar{g}(x)} = \sum_{j=0}^{m_1} a_{0j} \bar{G}^j, \quad f_0^*(x) = \frac{f_0(x)}{\bar{g}(x)} = \sum_{j=0}^{n_1} b_{0j} \bar{G}^j.$$

By (3), we have

$$F_1(\alpha_0) - F_1(x) = \sum_{j \geq 0} \alpha_j u^{2j+1},$$

$$G_1(\alpha_0) - G_1(x) = \sum_{j \geq 1} \beta_j u^{2j+1},$$
where $u = (\text{sgn} x) \sqrt{G(x)}$. By Theorem 3 and the proof of Theorem 5 in [Han, 1999] and Lemma 3.3 (or Lemma 1 in [Han, 1999]), we know that $\alpha_0, \alpha_1, \cdots, \alpha_{n_3}$ and $\beta_1, \beta_2, \cdots, \beta_{m_3}$ can be taken as free parameters, and

$$\alpha_j = O(|\alpha_0, \alpha_1, \cdots, \alpha_{n_3}|), \quad j \geq n_3 + 1,$$

$$\beta_j = O(|\beta_1, \beta_2, \cdots, \beta_{m_3}|), \quad j \geq m_3 + 1,$$

where $m_3 = \left\lfloor \frac{2m+1}{3} \right\rfloor$, $n_3 = \left\lfloor \frac{2n+1}{3} \right\rfloor$. 
Therefore,

$$\tilde{\Phi}_1(x) = u\left[\sum_{j=0}^{m_1} a_{0j} u^{2j} \sum_{j \geq 0} \alpha_j u^{2j} - \sum_{j=0}^{n_1} b_{0j} u^{2j} \sum_{j \geq 1} \beta_j u^{2j}\right] \equiv uS(u),$$

where
Therefore,

\[
\tilde{\Phi}_1(x) = u \left[ \sum_{j=0}^{m_1} a_{0j}u^{2j} \sum_{j \geq 0} \alpha_j u^{2j} - \sum_{j=0}^{n_1} b_{0j}u^{2j} \sum_{j \geq 1} \beta_j u^{2j} \right] \equiv uS(u),
\]

where

\[
S(u) = \sum_{k \geq 0} B_{k+1} u^{2k},
\]

\[
B_{k+1} = \sum_{i + j = k \atop 0 \leq i \leq m_1 \atop 0 \leq j} a_{0i} \alpha_j - \sum_{i + j = k \atop 0 \leq i \leq n_1 \atop 1 \leq j} b_{0i} \beta_j.
\]
Let $M = \max\{m_1 + n_3, \; n_1 + m_3\}$, where $m_1 = \left\lfloor \frac{m-2}{3} \right\rfloor$, $n_1 = \left\lfloor \frac{n-2}{3} \right\rfloor$. Then for $m_1 \geq 0$ or $n_1 \geq 0$ it is easy to see, as before, that $B_0, B_1, \ldots, B_M$ can be taken as free parameters, which implies the following
Let $M = \max\{m_1 + n_3, \ n_1 + m_3\}$, where $m_1 = \left\lfloor \frac{m-2}{3} \right\rfloor$, $n_1 = \left\lfloor \frac{n-2}{3} \right\rfloor$. Then for $m_1 \geq 0$ or $n_1 \geq 0$ it is easy to see, as before, that $B_0, B_1, \cdots, B_M$ can be taken as free parameters, which implies the following

**Lemma 3.6.** Let $m, n \geq 1$ and $\max\{m, n\} \geq 2$. Then for any neighborhood $U$ of the origin there exist $\varepsilon_0 > 0$ and functions $g_0$, $g_1$, $f_0$ and $f_1$ of the form (9) such that for all $0 < |\lambda| < \varepsilon_0$, Eq.(6) has at least $M$ limit cycles in $U$. It is clear that Theorem 2.3 follows from Lemmas 3.5 and 3.6.
Let $M = \max\{m_1 + n_3, n_1 + m_3\}$, where $m_1 = \left\lfloor \frac{m-2}{3} \right\rfloor$, $n_1 = \left\lfloor \frac{n-2}{3} \right\rfloor$.

Then for $m_1 \geq 0$ or $n_1 \geq 0$ it is easy to see, as before, that $B_0, B_1, \cdots, B_M$ can be taken as free parameters, which implies the following

**Lemma 3.6.** Let $m, n \geq 1$ and $\max\{m, n\} \geq 2$. Then for any neighborhood $U$ of the origin there exist $\varepsilon_0 > 0$ and functions $g_0, g_1, f_0$ and $f_1$ of the form (9) such that for all $0 < |\lambda| < \varepsilon_0$, Eq.(6) has at least $M$ limit cycles in $U$.

It is clear that Theorem 2.3 follows from Lemmas 3.5 and 3.6.
Remark

In the above application to Theorem 2.2, we use $\tilde{\Phi}_k(x, \delta)$ with $k = 1$ in the expansion

$$\Phi(x, \lambda, \delta) = \lambda^k \varphi_0(x, \delta) \tilde{\Phi}_k(x, \delta) + O(\lambda^{k+1}).$$
Remark

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I believe more limit cycles can be obtained if using $\tilde{\Phi}_2(x, \delta)$. 

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Thank You!