Bifurcation values for a one-parameter family of polynomial quintic systems

Johanna Denise García Saldaña

(Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Spain, Laboratory of Mathematical and Theoretical Physics, Université de Tours, France) 

Introduction

We consider the real polynomial differential system

\[ \dot{x} = y, \quad \dot{y} = -x + (a - a^2)(y + y^3), \]

where \( a \) is an arbitrary parameter.

The global bifurcation problem respecting to the parameter \( a \) of system (1) was studied in [3] and [4]. Their main results can be stated as follows.

(i) For \( a \leq 0 \) the origin of (1) is globally asymptotically stable.

(ii) For \( a > 0 \) there exist \( a^* \in (0, a) \), with \( a = \sqrt{\frac{2}{3}} / 2 \approx 0.7746 \) such that system (1) has:

- a unique hyperbolic limit cycle for \( 0 < a < a^* \),
- an internally stable heteroclinic loop for infinity for \( a = a^* \), and
- no limit cycle for \( a > a^* \).

From this result the authors conclude that system (1) has two bifurcation values \( a = 0 \) and \( a = a^* \), and four different global phase portraits on the Poincaré disc.

Our main goal is to improve the interval \((0, 0.7746)\) that contains the bifurcation value \( a^* \) and to show that two phase portraits were missed in [4]. In addition, we will provide an alternative proof, by using a Dulac function, for the uniqueness of the limit cycle when it exists.

The result

Theorem 1. System (1) has the following properties.

(a) For \( a \leq 0 \) the origin is globally asymptotically stable.

(b) If it has a limit cycle then it is unique, hyperbolic and stable.

(c) It has a limit cycle if and only if \( 0 < a < a^* \).

(d) It has exactly three bifurcation values and exactly six global phase portraits on the Poincaré disc which are shown in the figure below.

(\( a \leq 0 \)) \( (0 < a < a^* \)) \( (a = a^* \)) \( (a > a^* \))

The bifurcation values are \( a = 0 \), \( a = a^* \) and \( a > a^* \). Moreover \( a \) and \( a^* \) satisfy that \( 0.6241 < a < a^* < 0.6675 \).

Main ideas of the proof

Statement (a) The function \( V = x^2 + y^2 \) satisfies that \( V \leq 0 \) for \( a \leq 0 \), and by applying Liouvall’s principle the assertion follows.

Statement (b) We consider the function

\[ V(x, y) = \frac{\lambda}{\lambda - 1} V(x, y) - V(x, y), \]

with \( \lambda > 1 \) and \( \lambda = 1 \) respectively. Then \( V(x, y) \geq 0 \) is satisfied if \( a < a^* \) and \( a = a^* \), which implies that \( S_k = (V(x,y) - V(x,y))^2(X) \), where \( X \) is the vector field associated to (1), does not vanish in the region where the limit cycle can live. For proving this fact, which is the most difficult part of the proof, we use some results about the double discriminant of \( S_k \). Hence by using that \((V_0 = 0)\) is a simple closed curve for \( a = 0.6675 \) and by applying a generalization of the Bendixson-Dulac criterion [1][2] we prove that (1) has at most one limit cycle, which is hyperbolic and stable if it exists. To complete the proof, in statement (a) we will prove that (1) has no limit cycles for \( a \geq 0.6675 \).

Statement (c) By using the Hartman-Grobman theorem is easy to see that the origin, the unique limit critical point of (1), is a sink for \( 0 < a \leq a^* \) and a source for \( a > a^* \). Then at \( a = 0.6675 \) occurs a Hopf bifurcation. Therefore (1) has a limit cycle for \( 0 < a < a^* \), bifurcating from the origin, which increases with \( a \). Thus we need to prove that this limit cycle persists for \( 0 < a < a^* \) and that (1) has no limit cycles for \( a^* \leq a \). Hence from now on we assume \( a > 0.6675 \).

System (1) has two critical points at infinity. As usual, for studying the phase portrait at these critical points we use the Poincaré sphere. This is, we use the transformations \((x, y) = (1/z, y/z)\) and \((x, y) = (x/2, y/2)\), with a suitable change of time to transform (1) into two new systems, both having an unique nilpotent critical point at the origin. By using the blow-up technique we prove that their local phase portraits are the ones showed in next figures.

The separatrices \( S_1, S_2, S_3, S_4, S_5 \) and \( S_6 \) have the following configuration on the Poincaré disc.

By using that (1) is invariant by \((x, y) \mapsto (-x, -y)\) and that it is a family of generalized rotated vector fields, we prove that all the possible relative configurations of the separatrices are the following.

Statement (d) It is follows statement (a), with the figures (i), (ii), (iii), (iv) and (v), and the precedent paragraph.

Statement (e) The first assertion is clear from statement (d). The proof of the second assertion will be divided in two parts. In the first part we will prove that for \( a = 0.6675 \) the system (1) has a limit cycle and that the \( \omega \)-limit of \( S_k \) is the limit cycle. In the second part we will prove that (1) has no limit cycles for \( a \geq 0.6675 \). To prove these facts we will use the following property of the separatrices in the plane (\( x, y \)).

- \( S_1 \) is contained in the curve \((y - \phi_1(x) = 0)\) where \( \phi_1 = \phi_1(x) \) and \( \phi \) is the analytic function \( \phi(x) = \phi_1(x) - \sqrt{x(x^2 - y^2)} \), with \( \phi_1 \) and \( \phi_\) are the approximations of order \( i \) of (2) and (3) respectively, to know the exact distribution of the separatrices for each value of \( a \). That is, we want to use approximations of \( S_1, S_2 \) and \( S_3 \). Then by choosing \( \phi_1(x) \) and \( \phi_\) we will prove that for \( a \geq 0.6675 \) the distribution of the separatrices is the one showed figure (i). In fact the goal is to study the direction of the vector field along the graphic of \( \phi_1(x) \). This direction is controlled by the function \( \phi_\) and \( \phi_\) intersect a unique point \((x_0, y_0)\) with \( x_0 > 0 \) and \( y_0 = 0 \). Step 3. The \( \phi_\) and \( \phi_\) intersect a critical point \((x_0, y_0)\). Hence from now on we assume \( x > 0.6675 \).

Further we use the Poincaré algebra \( P^\) which satisfies that \( \phi_\) is not contained in \( \phi_\) and we use \((x_0, y_0)\) with \( \phi_\) and \( \phi_\) for \( a = 0.6675 \) then \( \phi_\) and \( \phi_\) are contained in the curve \((y - \phi_\) = 0\) and \( \phi_\) is the analytic function \( \phi(x) = \phi_1(x) - \sqrt{x(x^2 - y^2)} \), with \( \phi_1 \) and \( \phi_\) are the approximations of order \( i \) of (2) and (3) respectively, to know the exact distribution of the separatrices for each value of \( a \). That is, we want to use approximations of \( S_1, S_2 \) and \( S_3 \). Then by choosing \( \phi_1(x) \) and \( \phi_\) we will prove that for \( a \geq 0.6675 \) the distribution of the separatrices is the one showed figure (i). In fact the goal is to study the direction of the vector field along the graphic of \( \phi_1(x) \). This direction is controlled by the function \( \phi_\). This is the most difficult step of the proof. In addition, the graphic of \( \phi_1(x) \) close to \( \sqrt{a} \) define a hyperbolic sector, \( S_1 \) cannot be asymptotic to the line \( \phi_\), thus we obtain \( \phi_\) (vi). Finally as the origin is a source and from the symmetry of (1) we have figure (vii). Hence it is clear that for \( a = 0.6675 \) the system (1) has a limit cycle which is the \( \omega \)-limit of \( S_1 \).

Part 2. \( S_2 \) and \( S_3 \) have the following configuration on the Poincaré disc.

References


Remark

We believe that the tools used in this paper will be useful for study the bifurcation values of other systems on the plane.