A polynomial class of Markus-Yamabe counterexamples and examples

Abstract

We study the continuous and discrete versions of the Markus–Yamabe Conjecture for polynomial vector fields in $\mathbb{R}^3$ of the form $X = \lambda I + H$, where $\lambda$ is a real number, $I$ the identity map, and $H$ a map with nilpotent Jacobian matrix $JH$. We distinguish the cases when the rows of $JH$ are linearly dependent over $\mathbb{R}$ and when they are linearly independent over $\mathbb{R}$. In the dependent continuous case, we give a polynomial family of counterexamples to the the Markus–Yamabe conjecture which contains and generalizes that of Cima–Gasull–Mañosas. Furthermore, we construct a new class of polynomial vector fields in $\mathbb{R}^3$ having the origin as a global attractor. We also find non-linearly triangularizable vector fields $X$ for which the origin is a global attractor for both the continuous and the discrete dynamical systems generated by $X$. In the independent continuous case, we present a family of vector fields that have orbits escaping to infinity. Finally, in the independent discrete case, we find a family of vector fields that has a periodic point of period 3.

Introduction

We study both conjectures in the case of a special family of polynomial vector fields in $\mathbb{R}^n$, focusing on $n = 3$. Given a real number $\lambda$ and a positive integer $n$, we denote the set consisting of the polynomial vector fields in $\mathbb{R}^n$ of the form $F = \lambda I + H$, where $I$ is the identity map and $H$ has nilpotent Jacobian matrix at every point, by $\mathcal{N}(\lambda, n)$. Note that for this class of vector fields, the Jacobian matrix at each $x \in \mathbb{R}^n$ has all its eigenvalues equal to $\lambda$. Therefore, a vector field $F = \lambda I + H$ in $\mathcal{N}(\lambda, n)$ satisfies the MYC (resp. the DMYC) hypotheses if and only if $\lambda < 0$ (resp. $|\lambda| < 1$). The counterexamples of [2] are, basically, vector fields $X = \lambda I + H$ in $\mathcal{N}(\lambda, 3)$ where $H$ is a quasi-homogeneous vector field of degree one. We give examples of vector fields in $\mathcal{N}(\lambda, 3)$ which are linearly triangularizable (that is, triangular after a linear change of coordinates) in [5]. For these vector fields, the MYC (resp. the DMYC) is true when $\lambda < 0$ (resp. $|\lambda| < 1$). Further, [5] contains a family of counterexamples to the MYC which generalizes that of Cima–Gasull–Mañosas. The examples and counterexamples $X = \lambda I + H \in \mathcal{N}(\lambda, n)$ of above have one common characteristic, namely the rows of $JH$ are linearly dependent over $\mathbb{R}$. Thus we are led to introducing the sets $\mathcal{N}_{d}(\lambda, n)$ and $\mathcal{N}_{i}(\lambda, n)$. The first is the set consisting of the polynomial vector fields $X = \lambda I + (H_1, \ldots, H_d)$ in $\mathcal{N}(\lambda, n)$ such that $(H_1, \ldots, H_d)$ is linearly dependent over $\mathbb{R}$. The second set is $\mathcal{N}_{d}(\lambda, n) = \mathcal{N}(\lambda, n) - \mathcal{N}_{i}(\lambda, n)$. We denote the set consisting of the vector fields in $\mathcal{N}(\lambda, n)$ which are linearly triangularizable by $\mathcal{N}_{LT}(\lambda, n)$.

Main Results

1.- Let $\lambda \in \mathbb{R}$, and let $m \geq 1$ be an integer. Assume $X_k = \lambda I + H_1 + \cdots + H_k \in \mathcal{N}_{d}(\lambda, 3)$ for $1 \leq k \leq m$, where $H_i$ is a homogeneous polynomial of degree $ii$, with $i = 1, \ldots, m$. Then $X_k \in \mathcal{N}_{LT}(\lambda, 3)$ and

$$X_m(x, y, z) = \xi(x, y, z) +$$

$$+ (0, a_1 x + \cdots + a_m x^m, r_1(x, y) + \cdots + r_m(x, y))$$

where $r_i(x, y)$ is a homogeneous polynomial of degree $i$, with $1 \leq i \leq m$, up to a linear change of coordinates.

2.- For each $a, b, \lambda \in \mathbb{R}$, each $k, l, m \in \mathbb{N}$, and each polynomial map $f : \mathbb{R} \to \mathbb{R}$, the vector field

$$X(x, y, z) = \lambda(x, y, z) + f(az' + byz^m) (-bz^m, az', 0)$$

satisfies the following two properties:

i) For all $\lambda \in \mathbb{R}$, with $\lambda < 0$, the vector field $X \in \mathcal{N}(\lambda, 3)$, and

ii) for all $\lambda \in \mathbb{R}$ with $\lambda < 0$, all $l, m, k$ positive integers with $l - m \neq 0$ and $k \geq 2$, and all $a, b \in \mathbb{R} - \{0\}$ such that either $m + 1$ is odd, or $m + l$ is even and $(m - l) ab_{H_k} < 0$, the differential system $\dot{x} = X(x)$ has unbounded orbits.

3.- Let $X = \lambda I + H \in \mathcal{N}_{d}(\lambda, 3)$ where

$$H(x, y, z) = g(z)(a(z)x + b(z)y)(-b(z), a(z), 0)$$

$$+ (c(z), d(z), 0)$$

with $\lambda < a, b, c, d, g \in \mathbb{R}[z]$ and $X(0) = 0$. Then $X \in \mathcal{N}_{CY}(\lambda, 3)$ and $X \in \mathcal{N}_{DY}(\lambda, 3)$.

When the degree of the polynomial $f(t)$ is greater than one, there are examples of polynomial vector fields in $\mathcal{N}_{d}(\lambda, 3)$ having orbits that escape to infinity.

4.- Let $X = \lambda I + H \in \mathcal{N}_{d}(\lambda, 3)$ where

$$H(x, y, z) = z^{k-1}(x + y z)^{k+1}(-z, 1, 0).$$

Then:

a) If $\lambda < 0$ and $k$ odd, then $X \not\in \mathcal{N}_{CY}(\lambda, 3)$.

b) If $0 < |\lambda| < 1$, then $X \not\in \mathcal{N}_{DY}(\lambda, 3)$.

Now, we consider the linear independent case.
5.- Consider a vector field $X \in \mathcal{N}_{\lambda}(\lambda, 3)$, with $\lambda < 0$, of the form

$$X(x, y, z) = \lambda(x, y, z) + (0, v_1 z, 0) + g(t)(1, -(b_1 + 2\alpha x), \alpha g(t))$$

where $g(t) = A_1 t + A_2 \frac{t^2}{2}$. Then $X$ has orbits that escape to infinity.

6.- For $|\lambda| < 1$, consider

$$X(x, y, z) = \lambda(x, y, z) + (0, v_1 z, 0) + g(t)(1, -(b_1 + 2\alpha x), \alpha g(t))$$

with $t = y + b_1 x + v_1 \alpha x^2$ and $v_1 \alpha \neq 0$, and $g(t) \in \mathbb{R}[t]$ with $g(0) = 0$ and $g'(0) \neq 0$. Then there exists $(x_0, y_0, z_0) \neq (0, 0, 0)$ which is a periodic point of period 3 of $X$.

References


