Local first integrals of three dimensional Lotka-Volterra system
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Introduction:
We investigate the local integrability and linearity of the Lotka-Volterra systems,
\[ x' = x(1 - x^2 + yz), \quad y' = y(x - y + z), \quad z' = z(xy - x - y), \]
where \( y = 0 \).
By integrability, we mean the existence of a change of variables \( x = x(\Phi, \Psi, \Theta), y = y(\Phi, \Psi, \Theta), z = z(\Phi, \Psi, \Theta) \),
(1) to a system orbitally equivalent to its linear system
\[ X = AX, \quad Y = BY, \quad Z = C \cdot Z, \]
where \( A \) is non-singular. The system (2) is linearizable.

Definition 1: We say that a non-constant analytic function \( \Phi(\cdot, \cdot, \cdot) \) of \( \Phi(\cdot, \cdot, \cdot) \) is a first integral of (1) if it is constant on all solutions of (1).

Definition 2: A polynomial \( P(x, y, z) \) of \( r \) variables is called an invariant algebraic surface of the system (1), if the polynomial \( P(x, y, z) \) satisfies the partial differential equation \( \frac{\partial P}{\partial x}(x' + y' + z') = 0 \) for some polynomial \( C(x, y, z) \). Such a polynomial is called the cofactor of the invariant algebraic surface \( \Phi(\cdot, \cdot, \cdot) \).

Definition 3: A function \( M(x, y, z) \) is an inverse Jacobi multiplier for the vector field \( (\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z}) \) if it satisfies the equation
\[ M' = M \left( \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} \right) = 0. \]

Theorem 1: Suppose the analytic vector field \( \Phi(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z)) \) has an analytic first integral \( \Psi(x, y, z) = 0 \), then the vector field \( \Phi(\cdot, \cdot, \cdot) \) has at least one of \( f, g, h \) and a Jacobi multiplier \( M(x, y, z) = \Psi^2(x, y, z) \), and the cross product of \( (f, g, h)(L) \) is bounded away from zero for any integer \( L \cdot K \cdot L \), then the system has a second analytic first integral of the form \( \Psi^2(x, y, z) = 0 \), and hence the integrability of the system is guaranteed.

Case 1 (Darboux Method): If \( \Phi(\cdot, \cdot, \cdot) \) is an invariant algebraic surface \( f(x, y, z) = 0 \) with cofactor \( \Phi(x, y, z) \).
Two independent first integrals are \( \Psi_1 = \Psi^2(x, y, z) \) and \( \Psi_2 = \Psi^2(x, y, z) \).

Case 2 (Linearizable): In this case, we have the system
\[ x' = x(1 - x^2 + yz), \quad y' = y(x - y + z), \quad z' = z(xy - x - y). \]
Writing \( Y = x + y + z \), we obtain \( Y' = Y \).
The resulting system is linearizable. Thus, there exist a change of coordinates \( x = q(x, y, z), y = q(x, y, z), z = q(x, y, z) \) such that \( X = x + y + z \).

Case 3 (Special Transformation to Linearizable): In this case, we have the system
\[ x' = x(a + x + y + z), \quad y' = y(1 - a + y + z), \quad z' = z(a + x + y + z). \]
The transformation \( Y = x + a + y + z \) gives \( Y' = Y(1 + a + b + c) \).
This system again is localizable, and so there exists a change of coordinates \( x = q(x, y, z), y = q(x, y, z), z = q(x, y, z) \) such that \( X = x + y + z \).

Case 4 (Blow-down method): If the system (1) is linearizable, we can find a first integral \( \Phi(x, y, z) \) such that the transformation \( Y = \Phi(x, y, z) \) linearizes the system.

Case 5 (Darboux Method): If \( \Phi(x, y, z) \) is an invariant algebraic surface \( f(x, y, z) = 0 \) with cofactor \( \Phi(x, y, z) \).
Two independent first integrals are \( \Psi_1 = \Psi^2(x, y, z) \) and \( \Psi_2 = \Psi^2(x, y, z) \).

Case 6 (Blow-down method): If the system (1) is linearizable, we can find a first integral \( \Phi(x, y, z) \) such that the transformation \( Y = \Phi(x, y, z) \) linearizes the system.

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References: