Limit cycles in planar polynomial systems

Joan Torregrosa

Universitat Autònoma de Barcelona

http://www.gsd.uab.cat

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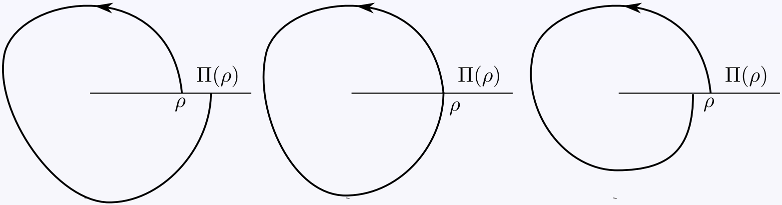
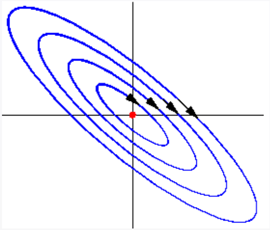
The center-focus problem



How to distinguish if a monodromic point is a repellor, a center, or an attractor?



The center-focus problem for nondegenerate singular points

For differential systems , $X = (P, Q)$, an elementary singular point is of center-focus type if $\operatorname{tr} DX(x_0) = 0$ and $\det DX(x_0) > 0$. Then, after a change of variables and time, the system writes as:

$$\begin{cases} \dot{x} = -y + \sum\limits_{k+\ell=m} p_{k,\ell} x^k y^\ell, \\ \dot{y} = x + \sum\limits_{k+\ell=m} q_{k,\ell} x^k y^\ell, \end{cases}$$

and, in complex coordinates ($z = x + iy$ and $z' = x' + iy'$),

$$z' = i\,z + \sum_{k+\ell=m} r_{k,\ell}\, z^k \bar{z}^\ell,$$

with $m \geq 2$.

Here $p_{k,\ell}$, $q_{k,\ell}$ (resp. $r_{k,\ell}$) are real (resp. complex) parameters.

The Lyapunov constants

Definition

If $V_K \neq 0$ and

$$\Pi(\rho) - \rho = V_K \rho^K + O(\rho^{K+1})$$

for $\rho > 0$ close to zero, then V_K is called the K-th Lyapunov constant.

- $V_{2K} = 0$, consequently the first nonvanishing coefficient of the displacement map corresponds to an odd exponent of ρ.

We will use indistinctly V_{2K+1} or L_K.

The related problems via Lyapunov constants

Problems

- $\Pi(\rho) \equiv \rho$? $\Pi(\rho) > \rho$? $\Pi(\rho) < \rho$?
- V_{2k+1} gets the stability of the origin.
- The characterization of centers is equivalent to solve the system $\{V_3 = 0, V_5 = 0, \ldots, V_{2K+1} = 0, \ldots\}$.
- Maximum order of a weak focus: Highest K in a fixed (perturbation) family such that $V_{2K+1} \neq 0$?
- A limit cycle is a isolated solution of $\Pi(\rho) = \rho$.
- Local cyclicity: Number of limit cycles bifurcating from $\rho = 0$.

In polar coordinates: The Andronov's method

$$\begin{cases} \dot{x} = -y + X_2(x,y) + X_3(x,y) + X_4(x,y) + \cdots, \\ \dot{y} = x + Y_2(x,y) + Y_3(x,y) + Y_4(x,y) + \cdots, \end{cases}$$

where $X_j(x,y)$ and $Y_j(x,y)$ are homogeneous polynomials of degree j depending on the original parameters $p_{k,\ell}$ and $q_{k,\ell}$.
In polar coordinates, $x = r\cos\theta, y = r\sin\theta$, we can write the above system as

$$\frac{dr}{dt} = r^2 P_2(\theta) + r^3 P_3(\theta) + \cdots,$$
$$\frac{d\theta}{dt} = 1 + r\, Q_2(\theta) + r^2 Q_3(\theta) + \cdots,$$

where

$$P_j(\theta) = \cos\theta\, X_j(\cos\theta, \sin\theta) + \sin\theta\, Y_j(\cos\theta, \sin\theta),$$
$$Q_j(\theta) = \cos\theta\, Y_j(\cos\theta, \sin\theta) - \sin\theta\, X_j(\cos\theta, \sin\theta),$$

homogeneous trigonometric polynomials of degree $j+1$.

The Andronov's method

Let $r(\theta,\rho)$ be the solution of the initial value problem

$$\begin{aligned} r' = \frac{dr}{d\theta} =& R_2(\theta)\, r^2 + R_3(\theta)\, r^3 + \cdots , \\ r(0,\rho) =& \rho, \end{aligned}$$

that, in series in ρ, writes as

$$r(\theta,\rho) = \rho + u_2(\theta)\,\rho^2 + u_3(\theta)\,\rho^3 + \cdots .$$

From $r(0,\rho)=\rho$, it is clear that $u_k(0)=0$ for all k.

Consequently, the return map can be computed evaluating $r(\theta,\rho)$ at $\theta = 2\pi$.

The Andronov's method

$$u_2(2\pi) = \widetilde{R_2}(2\pi) = \int_0^{2\pi} R_2(\psi)\, d\psi = \int_0^{2\pi} P_2(\psi)\, d\psi = 0,$$

because $P_2(\theta)$ is a homogeneous polynomial of degree 3 in $\sin\theta$ and $\cos\theta$.

$$u_3(2\pi) = \left(\widetilde{R_2}(2\pi)\right)^2 + \widetilde{R_3}(2\pi) = \int_0^{2\pi} R_3(\psi)\, d\psi.$$

In this case $\left(\widetilde{R_2}(2\pi)\right)^2 = 0$.

Finally

$$V_3 = \int_0^{2\pi} \left(P_3(\psi) - P_2(\psi)\, Q_2(\psi)\right)\, d\psi.$$

In general, as P_3 has degree 4 and $P_2 Q_2$ has degree 6, the above integral is non zero.

In Cartesian coordinates: The Poincaré–Lyapunov Method

The differential equation, in complex coordinates writes, as

$$z' = iz + Z(z,\bar{z}) = iz + Z_2(z,\bar{z}) + Z_3(z,\bar{z}) + \cdots .$$

The keypoint of this method is the study of the existence of a "Lyapunov function" of the form

$$F = F_2 + F_3 + F_4 + \cdots ,$$

where F_k are homogeneous polynomials of degree k, starting with terms of degree two because the linear term corresponds to $z' = iz$.
If $\dot{F}$ is nonzero then we will have a "Lyapunov function" in a neighborhood of the origin.

$$\dot{F} = F_z\,\dot{z} + F_{\bar{z}}\,\dot{\bar{z}} = F_z\,(iz + Z(z,\bar{z})) + F_{\bar{z}}\,(-i\bar{z} + \overline{Z(z,\bar{z})}) = \sum_{k\geq 1} V_{2k+1}\,(z\bar{z})^{k+1}.$$

Note that $x^2 + y^2 = z\bar{z}$. We have used the same name for the Lyapunov constants than in the previous method. But they are not exactly the same. They coincide modulus a multiplicative constant.

The Poincaré–Lyapunov Method

When all V_k are zero, $\dot{F} = 0$, then F is a first integral and the origin is a center.
Otherwise, when some (the first) V_{2k+1} is different from zero the origin will be stable ($V_{2k+1} < 0$) or unstable ($V_{2k+1} > 0$).
In this case ($z' = iz + \cdots$) when the system has a center, the first integral converges in a neighborhood of the origin.

The terms F_k can be found recursively.

$$
\begin{aligned}
&(F_{2z} + F_{3z} + F_{4z} + \cdots)\,(iz + Z_2 + Z_3 + Z_4 + \cdots) + \\
&+ (F_{2\bar{z}} + F_{3\bar{z}} + F_{4\bar{z}} + \cdots)\,(-i\bar{z} + \overline{Z_2} + \overline{Z_3} + \overline{Z_4} + \cdots) = \\
&= V_3\,(z\bar{z})^2 + V_5\,(z\bar{z})^3 + V_7\,(z\bar{z})^4 + \cdots .
\end{aligned}
$$

The Poincaré–Lyapunov Method

When p is odd, $V_{\frac{p}{2}-1} = 0$, and the system writes:

$$\begin{pmatrix} -p & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & -p+2 & 0 & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & & 0 & -1 & 0 & & & 0 \\ 0 & & & 0 & 1 & 0 & & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & & 0 & p-2 & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & p \end{pmatrix} \begin{pmatrix} h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p+1}{2},\frac{p-1}{2}} \\ h_{\frac{p-1}{2},\frac{p+1}{2}} \\ \vdots \\ h_{1,p-1} \\ h_{0,p} \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_{\frac{p-1}{2}} \\ \tilde{\phi}_{\frac{p+1}{2}} \\ \vdots \\ \tilde{\phi}_{p-1} \\ \tilde{\phi}_p \end{pmatrix},$$

where the components $\tilde{\phi}_j$ are the coefficients of $-i\sum_{k=2}^{p-1}\phi_{p-k+1,k}$ and all are known in this step p. The system is well defined because the determinant is non zero. Then $h_{p-j,j}$ is uniquely determined $h_{p-j,j} = \frac{\tilde{\phi}_j}{2j-p}$.

The Poincaré–Lyapunov Method

When p is even, then

$$\begin{pmatrix} -p & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & -p+2 & 0 & & & & & & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & & 0 & -2 & 0 & & & & 0 \\ 0 & & & 0 & 0 & 0 & & & 0 \\ 0 & & & & 0 & 2 & 0 & & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & & & & & & 0 & p-2 & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & p \end{pmatrix} \begin{pmatrix} h_{p,0} \\ h_{p-1,1} \\ \vdots \\ h_{\frac{p}{2}+1,\frac{p}{2}-1} \\ h_{\frac{p}{2},\frac{p}{2}} \\ h_{\frac{p}{2}-1,\frac{p}{2}+1} \\ \vdots \\ h_{1,p-1} \\ h_{0,p} \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_0 \\ \tilde{\phi}_1 \\ \vdots \\ \tilde{\phi}_{\frac{p}{2}-1} \\ \tilde{\phi}_{\frac{p}{2}} + i\,V_{p-1} \\ \tilde{\phi}_{\frac{p}{2}+1} \\ \vdots \\ \tilde{\phi}_{p-1} \\ \tilde{\phi}_p \end{pmatrix},$$

where the components $\tilde{\phi}_j$ are the coefficients of $-i\sum_{k=2}^{p-1}\phi_{p-k+1,k}$ in the equation of degree p, that are known. As in the odd case, the coefficients $h_{p-j,j}$ for $j \neq \frac{p}{2}$ can be determined.

The Lyapunov constants using "words"

Let $R \in \mathcal{P}$ be a polynomial of degree n. For all integer number $k \geq 2$, we define the operators $\mathcal{F}_k$ and $\mathcal{H}_k$ as

$$\begin{array}{lccc} \mathcal{G}: & \mathcal{P}_1 & \longrightarrow & \mathcal{P}_1 \\ & R = \sum\limits_{k \neq \ell} r_{k\ell} z^k \bar{z}^\ell & \longmapsto & \sum\limits_{k \neq \ell} \frac{2}{k-\ell} r_{k\ell} z^k \bar{z}^\ell, \end{array}$$

$$\begin{array}{lccc} \mathcal{F}: & \mathcal{P}_2 & \longrightarrow & \mathcal{P}_1 \\ & R & \longmapsto & -\operatorname{Im}\left(\mathcal{G}\left(\frac{\partial R(z,\bar{z})}{\partial z}\right)\right), \\ \mathcal{F}_k: & \mathcal{P} & \longrightarrow & \mathcal{P} \\ & R & \longmapsto & \mathcal{F}(R_k R), \\ \mathcal{H}_k: & \mathcal{P} & \longrightarrow & \mathbb{R} \\ & R & \longmapsto & -\frac{1}{(2\rho)^{\frac{n+1+k}{2}}} \int_{H=\rho} \operatorname{Im}(R_k R d\bar{z}), \end{array}$$

$\mathcal{P}_0$ is the subset formed by the polynomials vanishing at zero; $\mathcal{P}_1 \subset \mathcal{P}_0$ is the subset of all polynomials without monomials of the form $z^k \bar{z}^k$; $\mathcal{P}_2 \subset \mathcal{P}_0$ is the subset of all polynomials without monomials of the form $z^{k+1} \bar{z}^k$.

The Lyapunov constants using "words"

The first Lyapunov constants expressed as words are:

$$\begin{aligned}
V_3 &= \mathcal{H}_3(1) + \mathcal{H}_2(\mathcal{F}_2(1)), \\
V_5 &= \mathcal{H}_5(1) \\
&\quad + \mathcal{H}_4(\mathcal{F}_2(1)) + \mathcal{H}_3(\mathcal{F}_3(1)) + \mathcal{H}_2(\mathcal{F}_4(1)) \\
&\quad + \mathcal{H}_3(\mathcal{F}_2(\mathcal{F}_2(1))) + \mathcal{H}_2(\mathcal{F}_3(\mathcal{F}_2(1))) + \mathcal{H}_2(\mathcal{F}_2(\mathcal{F}_3(1))) \\
&\quad + \mathcal{H}_2(\mathcal{F}_2(\mathcal{F}_2(\mathcal{F}_2(1)))).
\end{aligned}$$

Notice that the expression of $\mathcal{F}_2$ obtained in the computation of V_3 is used in several places to compute V_5.

Homogeneous nonlinearities using “words”

Then “nonvanishing” Lyapunov constants are

- $n = 2, 4, 6, \ldots$.

$$\begin{aligned} L_1 =& \mathcal{H}_n(\mathcal{F}_n(1)), \\ L_2 =& \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(1)))), \\ L_3 =& \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(1)))))), \\ L_4 =& \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(1)))))))). \end{aligned}$$

- $n = 3, 5, 7, \ldots$.

$$\begin{aligned} L_1 =& \mathcal{H}_n(1), \\ L_2 =& \mathcal{H}_n(\mathcal{F}_n(1)), \\ L_3 =& \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(1))), \\ L_4 =& \mathcal{H}_n(\mathcal{F}_n(\mathcal{F}_n(\mathcal{F}_n(1)))). \end{aligned}$$

The first two Lyapunov constants

We can clarify the meaning of the above properties with the explicit expressions of the first two Lyapunov constants in the variables $r_{k,\ell}$:

$$V_3 = 2\pi\left(\mathrm{Re}(r_{2,1}) + \mathrm{Im}(-r_{2,0}r_{1,1})\right),$$
$$V_5 = \frac{2}{3}\pi(\mathrm{Re}(V_{5,1}) + \mathrm{Im}(V_{5,2}) + \mathrm{Re}(V_{5,3}) + \mathrm{Im}(V_{5,4})),$$

where

$$\begin{aligned}
V_{5,1} =& 3r_{32},\\
V_{5,2} =& -4r_{02}r_{40} - 6r_{31}r_{11} - 3r_{30}r_{12} - r_{02}\bar{r}_{13} - 3\bar{r}_{22}r_{11} - 3r_{31}\bar{r}_{20},\\
V_{5,3} =& -6r_{30}r_{11}^2 - 3r_{30}r_{11}\bar{r}_{20} - 2r_{30}r_{02}r_{20} + 5r_{30}r_{02}\bar{r}_{11}\\
&+ 3r_{12}r_{20}\bar{r}_{11} + 2r_{12}\bar{r}_{02}\bar{r}_{20} + 3r_{12}\bar{r}_{02}r_{11} - 30r_{21}\bar{r}_{20}r_{20}\\
&- 24r_{21}r_{20}r_{11} - 21r_{21}\bar{r}_{20}\bar{r}_{11} - 15r_{21}\bar{r}_{11}r_{11} + r_{03}\bar{r}_{02}\bar{r}_{11} + 2r_{03}\bar{r}_{02}r_{20},\\
V_{5,4} =& 24r_{11}^2r_{20}^2 - 2\bar{r}_{02}r_{11}^3 + 30\bar{r}_{20}r_{11}r_{20}^2 + 15r_{11}^2\bar{r}_{11}r_{20} + 3r_{02}\bar{r}_{11}^2r_{20}\\
&- 2\bar{r}_{11}r_{02}r_{20}^2 - 4r_{11}\bar{r}_{02}r_{02}r_{20}.
\end{aligned}$$

Order of a weak focus

Order of a weak focus

The order of a weak focus is the smallest value of K such that

$$\Pi(\rho) - \rho = V_{2K+1}\rho^{2K+1} + O(\rho^{2K+2}).$$

Maximum order problem

For a given family of polynomial vector fields, which is the highest value for the order of a weak focus?

Known higher order weak foci

General n

- $n^2 - 1$ for n even [QiuYan2010,LliRab2012]
- $(n^2 - 1)/2$ for n odd [QiuYan2010,LliRab2012]

Concrete n

- 3 for $n = 2$ [Bau1952]
- 11 for $n = 3$ [Zol1995,Chr2006]
- 21 for $n = 4$ [Gin2012]
- 33 for $n = 5$ [Gin2012]
- $n^2 + n - 2$ for $n = 6, 8, \ldots, 18, 20, \ldots, 32$ [QiuYan2010,LiaTor2015]

The systems providing the best lower bound for the order of a weak-focus are nonexplicit. They are "perturbing" results from centers.

For even "low" degrees

We extend the work of QiuYan2010 up to degree 32, using the "words" algorithm for homogeneous nonlinearities.

Proposition (LiaTor2015)

For n (even) $\in \{20, 22, \ldots, 32\}$ consider the system of degree n

$$z' = i\,z - \frac{n}{n-2}\,z^n + z\bar{z}^{n-1} + iC_n\,\bar{z}^n.$$

Then there exists a number C_n such that the above system has a weak focus at the origin of order $n^2 + n - 2$.

Highest order weak foci for odd degree

Theorem (LiaTor2015)

For $n \leq 100$, the origin of equation

$$z' = i\,z + \bar{z}^{n-1} + z^n$$

is a weak focus of order $(n-1)^2$.

Highest order weak foci for odd degree

Theorem (LiaTor2015)

For $n \leq 100$, the origin of equation

$$z' = i\,z + \bar{z}^{n-1} + z^n$$

is a weak focus of order $(n-1)^2$.

1. With the classical algorithm solving linear equations recursively we can comput, only, up to $n = 12$.
2. We can adapt de algorithm to prove up to $n = 100$. The necessary memory to prove the case $n = 100$ is around $384\,Gb$ or RAM.

The adapted algorithm (with GP-Pari)

```
firstliapunov(N)=
{
  local(last,H,L);
  last=2*(N-1)^2+2;      H=matrix(last+1,last+1);     L=vector(last+1);
  H[N+1,1]=1;  H[1,N+1]=1;  H[N+1,2]=1;  H[2,N+1]=1;
  for(i=3,last,
    for(j=0,floor((i+1)/2),
      if (j-N+1>=0,
         H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j+1+1,j-N+1+1]*(i-j+1)/(i-2*j+N)/I+H[i-j+1,j+1-N+1]*(j+1-N)/(i-2*j+N-1)/I;
      );
      if(i-j-N+1>=0,
        if(i-2*j-N !=0,
          H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j-N+1+1,j+1+1]*(j+1)/(i-2*j-N)/I;
         );
      if(i-2*j-N+1 !=0,
          H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j-N+1+1,j+1]*(i-j-N+1)/(i-2*j-N+1)/I;
        );
      );
      if(i-2*j==0,
        L[j+1]=H[i-j+1,j+1];
        if(L[j+1]!=0,
          print("N=",N,", j=",j);       print(L[j+1]);
         );
      );
    );
   for(j=floor((i+1)/2)+1,i,H[i-j+1,j+1]=conj(H[j+1,i-j+1]););
  );
}
```

Order of a weak focus and its cyclicity

Cyclicity of a weak-focus

For a given family of polynomial vector fields, which is the maximum number of limit cycles that bifurcate from an elementary weak focus?

Problem

Fixed the degree or the family, does the number of limit cycles coincide with the order of the weak focus?

Theorem

For a general system, the number of limit cycles that bifurcate from a weak focus of order K ($V_{2K+1} \neq 0$) is K.

Simple systems with weak-foci of high order $(n^2 + n - 2)$

Proposition

The system of degree 4

$$z' = i\,z - 2\,z^4 + z\bar{z}^3 + i\sqrt{\frac{52278}{20723}}\,\bar{z}^4$$

has a weak-focus at the origin of order 18. Moreover there exist polynomial perturbations of degree 4 such that from the origin bifurcate 18 limit cycles.

Proof.

The linear parts of the first 17 Lyapunov constants, with the restriction trace=0, are linearly independent. Consequently there are 17 limit cycles emerging from the origin with the assumption trace=0. The proof follows adding the trace and bifurcating the last one. □

Simple systems with weak-foci of high order (n^2+n-2)

Proposition

The system of degree 6

$$z' = i\,z - \frac{3}{2}\,z^6 + z\bar{z}^5 + i\sqrt{\frac{963010778697180}{95872134236688 1}}\,\bar{z}^6$$

has a weak-focus at the origin of order 40. Moreover there exist polynomial perturbations of degree 4 such that from the origin bifurcate 39 limit cycles.

Proof.

The linear parts of the first 37 Lyapunov constants, with the restriction trace=0, are linearly independent. Consequently there are 37 limit cycles emerging from the origin with the assumption trace=0. With the terms of degree two and the trace we can add 2 extra limit cycles. □

Simple systems with weak-foci of high order $(n^2 + n - 2)$

n	order	cyclicity up to order 1
4	18	18
6	40	37
8	70	63

Which are the number of limit cycles using higher order terms?

The cyclicity of holomorphic centers

Theorem

The cyclicity of the holomorphic center

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^{n-1} + z^n$$

is $n^2 + n - 2$ *for* $4 \leq n \leq 13$ *and at least* 9 *and no more than* 10 *for* $n = 3$, *under general polynomial perturbations of degree* n.

The proof follows computing the first Lyapunov quantities up to order one in the perturbed parameters.

Linear term of Lyapunov constants: Parallelization

Theorem (LiaTor2015)

Let $p(z,\bar{z})$ be a polynomial starting with terms of degree 2. Let $Q_i(z,\bar{z},\lambda)$ be analytic functions such that $Q_i(0,0,\lambda)\equiv 0$ and $Q_i(z,\bar{z},0)\equiv 0$, for $i=1,\ldots,s$. Let $a_1,\ldots,a_s$ be any s fixed constants. Suppose that $V_k^{Q_i}$ are the k-Lyapunov constants of equations

$$\dot{z}=iz+p(z,\bar{z})+Q_i(z,\bar{z},\lambda),\ \lambda\in\mathbb{C}^m,\quad \text{for } i=1,\ldots,s.$$

Then the linear part of $a_1V_k^{Q_1}+\cdots+a_sV_k^{Q_s}$ is the linear part of the k-Lyapunov constant of equation

$$\dot{z}=iz+p(z,\bar{z})+a_1Q_1(z,\bar{z},\lambda)+\cdots+a_sQ_s(z,\bar{z},\lambda),$$

with respect to the parameters λ.

Computing in parallel for holomorphic center with order 1

Computation time for the perturbation of the holomorphic family:

n	4	5	6	7	8	9	10	11	12	13
	1.7m	12.2m	1.2h	5.8h	1.4d	4.9d	1.8w	1.1M	3.3M	1y
P64	7s	0.5m	2m	8m	1.1h	3.1h	6.3h	0.9d	2.5d	8d

Computing in parallel allow us to get new results for $M(n)$

Best lower bounds for $M(n)$

The number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus in the class of polynomial vector fields of degree n is

- $M(n) \geq n^2 + 3n - 7$ for $n = 2, 3, 4.$
  [Bau1952,Zol1995,Chr2006,BouSad2008,Gin2012]
- $M(n) \geq n^2 + n - 2$ for $n = 5, 6, \ldots, 13.$
  [LiaTor2015]

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