Global dynamics of Planar Quintic Quasi–homogeneous Differential Systems

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Outline

1. Definitions and advances on quasi–homogeneous systems
2. Classification of the quintic quasi–homogeneous systems
3. Global structures of quintic quasi –homogeneous systems
4. Global structures of generic quasi –homogeneous systems

Definitions

Consider a real planar polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$ and the origin $O = (0, 0)$ is a singularity.

System (1) has *degree* n if $n = \max\{\deg P, \deg Q\}$.

System (1) is *coprime* if the polynomials $P(x, y)$ and $Q(x, y)$ have only constant common factors in the ring $\mathbb{R}[x, y]$.

System (1) is called a *homogeneous polynomial differential system* (HS for short) if for an arbitrary $\gamma \in \mathbb{R}^+$ it holds

$$P(\gamma x, \gamma y) = \gamma^n P(x, y) \quad \text{and} \quad Q(\gamma x, \gamma y) = \gamma^n Q(x, y).$$

System (1) is called a *quasi–homogeneous polynomial differential system* (QHS for short) if there exist constants $s_1, s_2, d \in \mathbb{N}$ such that for an arbitrary $\gamma \in \mathbb{R}^+$ it holds

$$P(\gamma^{s_1}x, \gamma^{s_2}y) = \gamma^{s_1+d-1}P(x,y) \quad \text{and} \quad Q(\gamma^{s_1}x, \gamma^{s_2}y) = \gamma^{s_2+d-1}Q(x,y).$$

(s_1, s_2) — *weight exponents*
d — *weight degree* with respect to the weight exponents
$w = (s_1, s_2, d)$ — *weight vector*

$\widetilde{w} = (\widetilde{s}_1, \widetilde{s}_2, \tilde{d})$ is a *minimal weight vector* if any other weight vector (s_1, s_2, d) of system (1) satisfies $\tilde{s}_1 \leq s_1, \tilde{s}_2 \leq s_2$ and $\tilde{d} \leq d$.

When $s_1 = s_2 = 1$, system (1) is a homogeneous one of degree d.

Advances on QHS

- Integrability point of view:
  [Edneral & Romanovski, preprint, 2016]
  [Giné, Grau & Llibre, *Discrete Contin. Dyn. Syst.*, 2013]
  [Algaba, Gamero & García C., *Nonlinearity*, 2009]
  [Goriely, *J. Math. Phys.*, 1996]
- Liouvillian integrable:
  [García, Llibre & Pérez del Río, *J. Diff. Eqns.*, 2013]
  [Li, Llibre, Yang & Zhang, *J. Dyn. Diff. Eqns.*, 2009]
- Polynomial and rational integrability:
  [Algaba, García & Reyes, *Nonlinear Anal.*, 2010]
  [Cairó & Llibre, *J. Math. Anal. Appl.*, 2007]
  [Llibre & Zhang, *Nonlinearity*, 2002]
- Center and limit cycle problems:
  [Algaba, Fuentes & García, *Nonlinear Anal. Real World Appl.*, 2012]
  [Gavrilov, Giné & Grau, *J. Diff. Eqns.*, 2009]

Center classification problem

- Classification of polynomial systems formed by linear plus homogeneous nonlinearities

  Cubic polynomial systems
  [Malkin, *Volz. Mat. Sb. Vyp*, 1964]
  [Vulpe & Sibirskii, *Soviet Math. Dokl.*, 1989]

  Quartic or quintic polynomial systems
  [Chavarriga & Gine, *Publ. Mat.*, 1996, 1997] obtained some partial results. For the systems of degree $k > 3$ the centers are not classified completely.

- Classification of HS

Quadratic HS [Sibirskii & Vulpe, *Differential Equations*, 1977]; [Newton, *SIAM Review*, 1978]; [Date, *J. Diff. Eqns.*, 1979]; [Vdovina, *Diff. Uravn.*, 1984]; [Ye, *Theory of Limit Cycles*, 1986]

Cubic HS
[Cima & Llibre, *J. Math. Anal. Appl.*, 1990]
[Ye, *Qualitative Theory of Polynomial Differential Systems*, 1995]

HS of arbitrary degree
[Cima & Llibre, *J. Math. Anal. Appl.*, 1990]
[Llibre, Pérez del Río & Rodríguez, *J. Diff. Eqns.*, 1996]

These papers have either characterized the phase portraits of HS of degrees 2 and 3, or obtained the algebraic classification of that.

Classifications of QHS with degree ≤ 4

Cubic QHS

[García, Llibre & Pérez del Río, *J. Diff. Eqns.*, 2013]
provided an algorithm for obtaining all QHS with a given degree and characterized QHS of degrees 2 and 3 having a polynomial, rational or global analytical first integral.

[Aziz, Llibre & Pantazi, *Adv. Math.*, 2014]
characterized the centers of the QHS of degree 3. By the averaging theory, at most one limit cycle can bifurcate from the periodic orbits of a center of a cubic HS.

Quartic QHS

[Liang, Huang & Zhao, *Nonlinear Dyn.*, 2014]
proved the non-existence of centers for the QHS of degree 4 and completed classification of global phase portraits.

Forms of quintic QHS

Theorem

[Tang, Wang & Zhang, DCDS, 2015] Every planar real quintic quasi –homogeneous but non–homogeneous coprime polynomial differential system (1) *can be written as one of the following 15 systems.*

$$X_{011}: \quad \dot{x} = a_{05}y^5 + a_{13}xy^3 + a_{21}x^2y, \quad \dot{y} = b_{04}y^4 + b_{12}xy^2 + b_{20}x^2,$$
$$\text{with } a_{05}b_{20} \neq 0 \text{ and the weight vector } \widetilde{w} = (2,1,4),$$
$$X_{012}: \quad \dot{x} = a_{05}y^5 + a_{22}x^2y^2, \quad \dot{y} = b_{13}xy^3 + b_{30}x^3,$$
$$\text{with } a_{05}b_{30} \neq 0 \text{ and the weight vector } \widetilde{w} = (3,2,8),$$
$$X_{014}: \quad \dot{x} = a_{05}y^5 + a_{40}x^4, \quad \dot{y} = b_{31}x^3y,$$
$$\text{with } a_{05}a_{40}b_{31} \neq 0 \text{ and the weight vector } \widetilde{w} = (5,4,16),$$
$$\ldots$$
$$X_{1}: \quad \dot{x} = a_{05}y^5 + a_{10}x, \quad \dot{y} = b_{01}y,$$
$$\text{with } a_{05}a_{10}b_{01} \neq 0, \text{ and the weight vector } \widetilde{w} = (5,1,1).$$

Proof

[García, Llibre & Pérez del Río, *J. Diff. Eqns.*, 2013]
The quasi–homogeneous but non–homogeneous polynomial differential system of degree n with the weight vector (s_1, s_2, d) can be written in

$$X_{ptk} = X_n^p + X_{n-t}^{ptk} + \sum_{\substack{s \in \{1,\ldots,n-p\} \setminus \{t\} \\ k_s t = ks \text{ and} \\ k_s \in \{1,\ldots,n-s-p+1\}}} X_{n-s}^{psk_s},$$

where $p \in \{0, 1, ..., n-1\}$, $t \in \{1, 2, ..., n-p\}$, $k \in \{1, \ldots, n-p-t+1\}$,

$$X_n^p = (a_{p,n-p} x^p y^{n-p}, b_{p-1,n-p+1} x^{p-1} y^{n-p+1}).$$

and

$$X_{n-t}^{ptk} = (a_{p+k,n-t-p-k} x^{p+k} y^{n-t-p-k},\ b_{p+k-1,n-t-p-k+1} x^{p+k-1} y^{n-t-p-k+1}).$$

Center classification of quintic QHS

Theorem

[Tang, Wang & Zhang, DCDS, 2015] The quintic quasi–homogeneous but non–homogeneous coprime polynomial differential system (1) *having a center at the origin, together with possible invertible changes of variables, must be of the form*

$$\dot{x} = axy^2 - y^5, \qquad \dot{y} = by^3 + x, \tag{2}$$

with $a = -3b$ *and* $b^2 < \frac{1}{3}$. *Furthermore, the center is not isochronous and the period of the periodic orbits is a monotonic function.*

Proof

Deleting some vector fields having invariant lines by simple anaysis, there remain three vector fields X_{011}, X_{015} and X_{021} to be studied.

X_{015} is a Hamiltonian system and its origin is a degenerate singularity.

Lemma

The origin O of the Hamiltonian system

$$X_{015}: \ \dot{x} = a_{05}y^5, \quad \dot{y} = b_{40}x^4, \ \textit{with } a_{05}b_{40} \neq 0$$

consists of two hyperbolic sectors.

Apply the *Bendixson's formula* that

$$\mathcal{I}(O) = 1 + \frac{\widehat{e} - \widehat{h}}{2}.$$

$\mathcal{I}(O)$ — Poincaré index of the singularity O
$\hat{e}$ — number of elliptic sectors
$\hat{h}$ — number of hyperbolic sectors adjacent to the singularity O

By [Zhang, Ding, Huang and Dong, *Qualitative Theory of Differential Equations*, 1992], $\mathcal{I}(O) = 0$ because the sum of degrees of two components of the vector field X_{015} is odd. Since $\widehat{e} = 0$, it follows that $\hat{h} = 2$.

This lemma shows that the origin of the vector field X_{015} is not a center.

Actually, if we only want to prove that the origin of the vector field X_{015} is not a center, the proof can be simplified.

It follows from the second equation $y'(t) = b_{40}x^4$ of X_{015} that $y(t)$ is increasing if $b_{40} > 0$ and decreasing if $b_{40} < 0$ for $t \in (-\infty, +\infty)$. Therefore, $y(t)$ is not a periodic function, which yields that X_{015} has no periodic orbits. It is obvious that the origin is not center if $b_{40} = 0$.

Lemma

For systems

$$X_{021}^{\pm}: \ \dot{x} = axy^2 \pm y^5, \ \ \dot{y} = x + by^3,$$

the following statements hold.

(a) *The origin O of system X_{021}^{+} is not a center.*

(b) *System X_{021}^{-} has a center at the origin O if and only if $a = -3b, \ b^2 < \frac{1}{3}$.*

$$\frac{\partial P_\pm}{\partial x} + \frac{\partial Q_\pm}{\partial y} = (a+3b)y^2.$$

By Bendixson's Criteria, system $X_{021}^{\pm}$ has no periodic orbit if $a+3b \neq 0$.

Apply the theory of nilpotent center in [Dumortier, Llibre and Artés, *Qualitative Theory of Planar Differential Systems*, 2006], we have

(a) O of system X_{021}^{+} is not a center provided $a=-3b$.

(b) O of system X_{021}^{-} is monodromy iff $-1+3b^2<0$ in the case $a=-3b$. The polynomial first integral $H^+(x,y)=\frac{x^2}{2}+bxy^3+\frac{y^6}{6}$ forces that the origin O must be a center.

Lemma

System

$$X_{011}: \dot{x} = a_{05}y^5 + a_{13}xy^3 + a_{21}x^2y, \quad \dot{y} = b_{04}y^4 + b_{12}xy^2 + b_{20}x^2$$

has an invariant curve passing through the origin O, where $a_{05}b_{20} \neq 0$.

We can check thay X_{011} has the invariant curve $x - \lambda_1 y^2 = 0$, where λ_1 is a real zero of the cubic polynomial

$$\eta(1, \lambda) = a_{05} + (a_{13} - 2b_{04})\lambda + (a_{21} - 2b_{12})\lambda^2 - 2b_{20}\lambda^3.$$

This lemma shows that the origin of the vector field X_{011} is not a center.

X_{021}: Center at the origin

X_{011}: No centers

X_{015}: No centers

Center of X_{021} is NOT isochronous, since the center is not elementary by [Mardesic, Rousseau & Toni, *J. Diff. Eqns.*,1995].

The period function

$$T(h) = \frac{1}{3\sqrt{2}} \left(\frac{6}{1 - 3b^2} \right)^{\frac{1}{6}} h^{-\frac{2}{3}} \int_0^{2\pi} (\sin s)^{-\frac{2}{3}} ds.$$

Clearly the period of closed orbits inside the period annulus of the center is monotonic in h. We completed the proof of this theorem.

Global center of X_{021}

Theorem

[Tang, Wang & Zhang, DCDS, 2015] The center of system X_{021} is global if it exists.

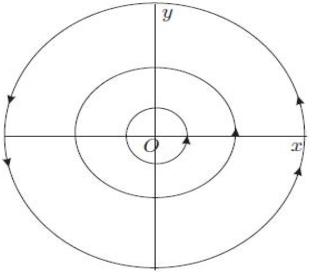

Figure: Global phase portrait of system X_{021}.

Proof

First integral is NOT enough to get the global phase portrait. We need to know the properties of orbits at infinity.

Poincaré compactification $\to$ Poincaré sphere:

$$\begin{aligned} \dot{u} &= u^6 + (b-a)u^3z^2 + z^4 := P_1(u,z), \\ \dot{z} &= u^2z(u^3 - az^2) := Q_1(u,z). \end{aligned}$$

$E = (0,0) \leftrightarrow \infty$ on the x-axis, which is the unique singularity at infinity of X_{021}.

We can prove $E = (0,0)$ is NOT monodromy by the method of generalized normal sectors [Tang & Zhang, *Nonlinearity*, 2004].

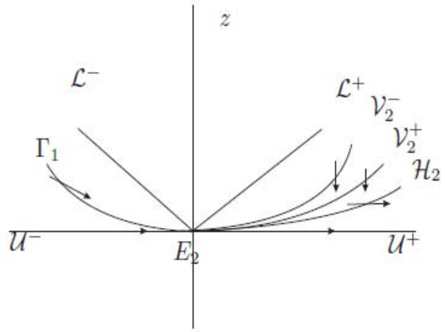

Figure: Directions of vector field for system X_{021}.

Global structures of quintic QHS

Among all quintic QHS for the global structures, the most difficult case is to discuss that of

$$X_{111}: \dot{x} = a_{14}xy^4 + a_{22}x^2y^2 + a_{30}x^3, \quad \dot{y} = b_{05}y^5 + b_{13}xy^3 + b_{21}x^2y,$$

where $a_{14}^2 + b_{05}^2 \neq 0$. We will mainly introduce the results of X_{111}.

Theorem

[Tang& Zhang, preprint, 2016] The global phase portrait of system X_{111} is topologically equivalent to one of 52 ones without taking into account the direction of the time.

Proof.

Step 1. Simplification of quintic QHS

The quintic quasi–homogeneous system X_{111} can be transformed into homogeneous system of degree 3

$$\mathcal{H}: \begin{cases} \dot{x} = x(c_{12}y^2 + c_{21}xy + c_{30}x^2) := P_3(x,y), \\ \dot{y} = y(y^2 + d_{12}xy + d_{21}x^2) := Q_3(x,y), \end{cases}$$

by using the change

$$\tilde{x} = x, \qquad \tilde{y} = y^2,$$

together with a time scaling, where $c_{30} \neq 0$ and we keep the notations of parameters c_{ij}, d_{ij} and variables x, y for simplicity..

Then, for studying topological phase portraits of X_{111}, we need the knowledge on homogeneous systems of degree 3.

Based on the classification of fourth–order binary forms,
[Cima & Llibre, *J. Math. Anal. Appl.*, 1990]
obtained the algebraic characteristics of cubic HS and further they researched all phase portraits of such canonical cubic HS.

However, it is NOT easy to change a cubic homogeneous system to its canonical form since one needs to solve four quartic polynomial equations.

We will apply the idea in [Cima & Llibre, 1990] to obtain the global dynamics of system $\mathcal{H}$ and consequently those of X_{111}.

Step 2. Blow-up along a line

For vector field $\mathcal{H}$ of degree 3, its origin is a highly degenerate singularity. For studying its local dynamics around the origin, the blow–up technique is useful. Commonly, we can blow up a degenerate singularity into several less degenerate singularities either on a cycle or on a line. Here, we choose the latter, which can be applied to the singularities both in the finite plane and at the infinity.

The change of variables

$$x = x, \quad y = ux,$$

transforms system $\mathcal{H}$ into

$$\hat{\mathcal{H}}: \quad \begin{cases} \dot{x} = x\widehat{P}_3(u) := x(c_{12}u^2 + c_{21}u + c_{30}), \\ \dot{u} = \widehat{G}_3(u) := u((1 - c_{12})u^2 + (d_{12} - c_{21})u + d_{21} - c_{30}). \end{cases}$$

The singularity $E_0 = (0, u_0)$ of system $\hat{\mathcal{H}}$ is a saddle if either $\widehat{P}_3(u_0)\widehat{G}_3'(u_0) < 0$, or $\widehat{G}_3'(u_0) = \widehat{G}_3''(u_0) = 0$ and $\widehat{P}_3(u_0)\widehat{G}_3'''(u_0) < 0$.

E_0 is a node if either $\widehat{P}_3(u_0)\widehat{G}_3'(u_0) > 0$, or $\widehat{G}_3'(u_0) = \widehat{G}_3''(u_0) = 0$ and $\widehat{P}_3(u_0)\widehat{G}_3'''(u_0) > 0$.

These show that except the invariant line $y = u_0 x$ system $\mathcal{H}$ has either no orbits or infinitely many orbits connecting with the origin along the characteristic directions $\theta = \arctan(u_0)$.

If $\widehat{G}_3'(u_0) = 0$ and $\widehat{G}_3''(u_0) \neq 0$, the singularity $E_0 = (0, u_0)$ is a saddle–node. More precisely, there exist infinitely many orbits of system $\mathcal{H}$ connecting the origin along the direction of the invariant line $y = u_0 x$ if u_0 is a zero of multiplicity 2 of $\widehat{G}_3(u)$.

Step 3. Generalized normal sectors along the direction $\theta = \frac{\pi}{2}$

We should consider the properties of $\mathcal{H}$ at the origin along the characteristic direction $\theta = \frac{\pi}{2}$ separately.

Assume that $\theta = \frac{\pi}{2}$ is a zero of multiplicity m of $\widetilde{G}(\theta) := xQ_3(\cos\theta, \sin\theta) - yP_3(\cos\theta, \sin\theta)$. The following statements hold.

- If $m > 0$ is even, there exist infinitely many orbits connecting the origin of $\mathcal{H}$ and being tangent to the y–axis at the origin.
- If m is odd, there exist either infinitely many orbits if $\widetilde{G}^{(m)}(\frac{\pi}{2})\widetilde{H}(\frac{\pi}{2}) > 0$, or exactly one orbit if $\widetilde{G}^{(m)}(\frac{\pi}{2})\widetilde{H}(\frac{\pi}{2}) < 0$, connecting the origin of $\mathcal{H}$ and being tangent to the y–axis at the origin.

Step 4. Poincaré compactification

Taking respectively the Poincaré transformations $x = 1/z$, $y = u/z$ and $x = v/z$, $y = 1/z$ system $\mathcal{H}$ around the equator of the Poincaré sphere can be written respectively in

$$\dot{u} = G_3(1, u), \quad \dot{z} = -zP_3(1, u),$$

and

$$\dot{v} = -G_3(v, 1), \quad \dot{z} = -zQ_3(v, 1).$$

A singularity I_{u_0} of system $\mathcal{H}$ located at the infinity of the line $y = xu_0$ is
– a saddle if $\widehat{P}_3(u_0)\widehat{G}'_3(u_0) > 0$, or $\widehat{G}'_3(u_0) = \widehat{G}''_3(u_0) = 0$ and $\widehat{P}_3(u_0)\widehat{G}^{(3)}_3(u_0) > 0$
– a node if $\widehat{P}_3(u_0)\widehat{G}'_3(u_0) < 0$, or $\widehat{G}'_3(u_0) = \widehat{G}''_3(u_0) = 0$ and $\widehat{P}_3(u_0)\widehat{G}^{(3)}_3(u_0) < 0$.
–a saddle–node if $\widehat{G}'_3(u_0) = 0$ and $\widehat{G}''_3(u_0) \neq 0$.

A singularity I_y of system $\mathcal{H}$ located at the end of the y–axis is
– a saddle if $c_{12} > 1$, or $c_{12} = 1$, $d_{12} = c_{21}$ and $d_{21} < c_{30}$;
– a stable node if $c_{12} < 1$, or $c_{12} = 1$, $d_{12} = c_{21}$ and $d_{21} > c_{30}$;
– a saddle–node if $c_{12} = 1$ and $d_{12} \neq c_{21}$.

Summarizing the above analysis and going back to the original system X_{111}, the invariant line $y = u_0 x$ of system $\mathcal{H}$ as $u_0 \neq 0$ is an invariant curve of system X_{111}, which is tangent to the y-axis at the origin and connects the origin and the singularity I_{u_0} at infinity. Moreover, the invariant curve is usually a separatrix of hyperbolic sectors, parabolic sectors or elliptic sectors.

The above analysis provide enough preparation for studying global topological phase portraits of the quintic quasi-homogeneous system X_{111}. By the properties of the singularities at infinity, we discuss three cases: $a_{14} > 1$, $a_{14} < 1$ and $a_{14} = 1$, and get 52 global topological phase portraits of quintic quasi–homogeneous system X_{111}.

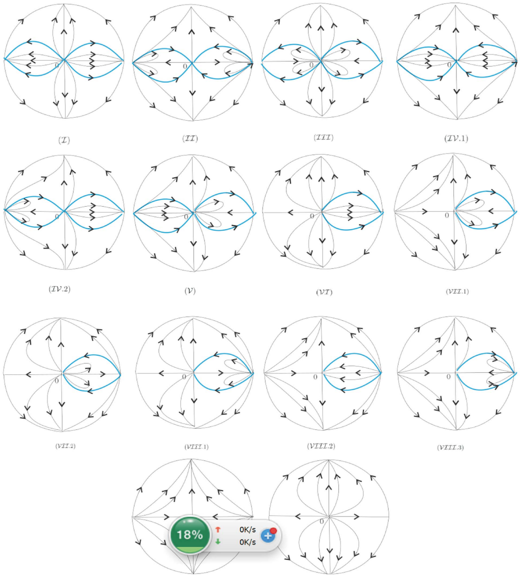

Figure: Global phase portraits of system X_{111} as $a_{14} < 1$.

Remark: Global structures of generic QHS

Theorem

[Tang& Zhang, preprint, 2016] Any quasi–homogeneous but non–homogeneous polynomial differential system (1) *of degree* n *can be transformed into a homogeneous polynomial differential system by an appropriate changes of variables.*

Then, we can investigate global structures of QHS with an arbitrary degree by a similar idea as the study of quintic QHS.

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