A Dynamical Systems Approach to Singularities of Ordinary Differential Equations

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Singularities of Differential Equations

Many forms of singular behaviour in the context of differential equations

- (derivatives of) *solutions* become singular $\rightsquigarrow$ “blow-up”, “shock”
- *stationary points* of vector fields
- *bifurcations* in parameter dependent systems
- *singular integrals* (solutions not contained in the “general integral”)
- *multi-valued solutions* (like “breaking waves”)
- ...

here: geometric modelling of differential equations $\rightsquigarrow$ *critical points* of natural projection map $\rightsquigarrow$ **geometric singularities**

- **differential topology**
  - *definition* of singularities (of smooth maps)
  - main emphasis on *classifications* and local *normal forms*
  - hardly any works on (general) *systems*
- **classical analysis**
  - mainly *quasi-linear* systems (including DAEs)
  - rich literature on *scalar* equations
  - *existence*, *uniqueness* and *regularity* of solutions through singularity
  - main techniques: *fixed point theorems*, *sub-* and *supersolutions*
- **differential algebra**
  - main emphasis on *singular integrals*
  - motivating problem for *differential ideal theory*
  - (geometric) singularities *eliminated*
  - useful for *algorithmic* approaches
  - singularities related to *differential Galois theory*

- *fibred manifold:* $\pi : \mathcal{E} \to \mathcal{T}$ with $\dim \mathcal{T} = 1$
    - trivial case: $\mathcal{E} = \mathcal{T} \times \mathcal{U}$, $\pi = \mathrm{pr}_1$
    - adapted local coordinates: $(t, \mathbf{u})$
      (*independent variable* t, *dependent variables* $\mathbf{u}$)
- *section:* smooth map $\sigma : \mathcal{T} \to \mathcal{E}$ with $\pi \circ \sigma = \mathrm{id}$
  (locally: $\sigma(t) = (t, \mathbf{s}(t))$ with function $\mathbf{s} : \mathcal{T} \to \mathcal{U}$)
- *q-jet* $[\sigma]_t^{(q)}$: class of all sections with same Taylor polynomial of degree q around expansion point t
- *jet bundle* $\mathcal{J}_q\pi$: set of all q-jets $[\sigma]_t^{(q)}$
    - local coordinates: $(t, \mathbf{u}^{(q)})$ (derivatives up to order q)
    - natural hierarchy with projections

$$\pi^q_r : \mathcal{J}_q\pi \longrightarrow \mathcal{J}_r\pi \qquad 0 \leq r < q$$

$$\pi^q : \mathcal{J}_q\pi \longrightarrow \mathcal{T}$$

Definition

ordinary differential equation of order q $\rightsquigarrow$
submanifold $\mathcal{R}_q \subseteq \mathcal{J}_q\pi$ such that $\operatorname{im} \pi^q|_{\mathcal{R}_q}$ dense in $\mathcal{T}$

- more general definition than usual in geometric theory
- no conditions on independent variable allowed
- no distinction *scalar equation* or *system*
- basic assumption: equation *formally integrable*
  (no "hidden" integrability conditions)

prolongation of section $\sigma : \mathcal{T} \to \mathcal{E} \quad \rightsquigarrow \quad$ section $j_q\sigma : \mathcal{T} \to \mathcal{J}_q\pi$

$$j_q\sigma(t) = \bigl(t, \mathbf{s}(t), \dot{\mathbf{s}}(t), \dots, \mathbf{s}^{(q)}(t)\bigr)$$

Definition

classical solution $\quad \rightsquigarrow \quad$ section $\sigma : \mathcal{T} \to \mathcal{E}$ such that $\operatorname{im}(j_q\sigma) \subseteq \mathcal{R}_q$

Definition

contact distribution $\mathcal{C}_q \subset T(\mathcal{J}_q\pi)$ generated by vector fields

$$C^{(q)}_{\text{trans}} = \partial_t + \sum_{\alpha=1}^{m}\sum_{j=0}^{q-1} u^\alpha_{j+1}\partial_{u^\alpha_j}$$

$$C^{(q)}_{\alpha} = \partial_{u^\alpha_q} \qquad 1 \leq \alpha \leq m$$

Proposition

section $\gamma : \mathcal{T} \to \mathcal{J}_q\pi$ *of the form* $\gamma = j_q\sigma \iff T\mathrm{im}(\gamma) \subset \mathcal{C}_q$

Proof.

chain rule! □

Vessiot Distribution

Consider prolonged solution $j_q\sigma$ of equation $\mathcal{R}_q \subseteq \mathcal{J}_q\pi$:

- *integral elements* $\rightsquigarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr)$ for $\rho \in \mathrm{im}(j_q\sigma)$
- solution $\Longrightarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr) \subseteq T_\rho\mathcal{R}_q$
- prolonged section $\Longrightarrow$ $T_\rho\bigl(\mathrm{im}(j_q\sigma)\bigr) \subseteq \mathcal{C}_q|_\rho$

Definition

Vessiot space at point $\rho \in \mathcal{R}_q$: $\mathcal{V}_\rho[\mathcal{R}_q] = T_\rho\mathcal{R}_q \cap \mathcal{C}_q|_\rho$

- generally: $\dim \mathcal{V}_\rho[\mathcal{R}_q]$ depends on ρ $\rightsquigarrow$ regular distribution only on open subset of $\mathcal{R}_q$
- computing Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ corresponds to "projective" form of prolonging from $\mathcal{R}_q$ to $\mathcal{R}_{q+1}$
- computation requires only linear algebra

Vessiot Distribution

Consider *square* first-order ordinary differential equation $\mathcal{R}_1 \subset \mathcal{J}_1\pi$ with local representation $\mathbf{\Phi}(t, \mathbf{u}, \dot{\mathbf{u}}) = 0$ where $\mathbf{\Phi} : \mathcal{J}_1\pi \to \mathbb{R}^m$

- define $m \times m$ matrix A and m-dimensional vector $\mathbf{d}$

$$A = \mathbf{C}^{(1)}\mathbf{\Phi} = \frac{\partial \mathbf{\Phi}}{\partial \dot{\mathbf{u}}} \qquad \mathbf{d} = C^{(1)}_{\mathrm{trans}}\mathbf{\Phi} = \frac{\partial \mathbf{\Phi}}{\partial t} + \frac{\partial \mathbf{\Phi}}{\partial \mathbf{u}} \cdot \dot{\mathbf{u}}$$

  assume A almost everywhere non-singular
- compute determinant $\delta = \det A$ and adjugate $C = \operatorname{adj} A$
- $\mathcal{V}[\mathcal{R}_1]$ almost everywhere generated by single vector field

$$X = \delta C^{(1)}_{\mathrm{trans}} - (C\mathbf{d})^T \mathbf{C}^{(1)}$$

  (X essentially lift of "evolutionary vector field" associated to given differential equation to $\mathcal{J}_1\pi$)

Definition

ordinary differential equation $\mathcal{R}_q \subseteq J_q\pi$

- *generalised solution* $\rightsquigarrow$ integral curve $\mathcal{N} \subseteq \mathcal{R}_q$ of $\mathcal{V}[\mathcal{R}_q]$
- *geometric solution* $\rightsquigarrow$ projection $\pi^q_0(\mathcal{N})$ of generalised solution $\mathcal{N}$

- geometric solution in general *not* image of a section (thus *no* interpretation as a function!)
- geometric solution $\pi^q_0(\mathcal{N})$ is classical solution $\iff$ $\mathcal{N}$ everywhere transversal to π^q
- geometric solutions allow for modelling of *multi-valued* solutions

Ordinary differential equation $\mathcal{R}_q \subset \mathcal{J}_q\pi$

- local description: $\mathbf{\Phi}(t, \mathbf{u}^{(q)}) = 0$ $(\dim \mathbf{u} = m)$
  (not necessarily square $\rightsquigarrow$ "DAEs" included)
- Assumptions:
  - equation *formally integrable*
  - equation of *finite type* $\rightsquigarrow$ $\partial\mathbf{\Phi}/\partial\mathbf{u}_q$ has almost everywhere rank m
- second assumption $\Longrightarrow$ almost everywhere $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1$

Definition

point $\rho \in \mathcal{R}_q \subset \mathcal{J}_q\pi$ is

- *regular* $\rightsquigarrow$ $\mathcal{V}_\rho[\mathcal{R}_q]$ 1-dimensional and transversal to π^q
- *regular singular* $\rightsquigarrow$ $\mathcal{V}_\rho[\mathcal{R}_q]$ 1-dimensional and *not* transversal to π^q
- *irregular singular* (*s-singular*) $\rightsquigarrow$ $\dim \mathcal{V}_\rho[\mathcal{R}_q] = 1 + s$ with $s > 0$

(singular points $\rightsquigarrow$ critical points of $\pi^q_0|_{\mathcal{R}_q}$)

Proposition

point $\rho \in \mathcal{R}_q \subset \mathcal{J}_q\pi$

- ρ *regular* $\iff$ $\operatorname{rank}\left(\mathbf{C}^{(q)}\mathbf{\Phi}\right)_\rho = m$
- ρ *regular singular* $\iff$ ρ *not regular and*

$$\operatorname{rank}\left(\mathbf{C}^{(q)}\mathbf{\Phi} \mid C^{(q)}_{\text{trans}}\mathbf{\Phi}\right)_\rho = m$$

Theorem

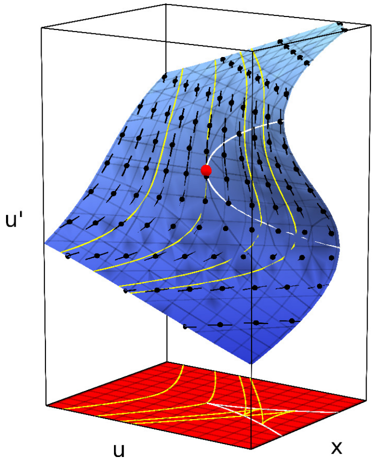
$\mathcal{R}_q \subset \mathcal{J}_q\pi$ *equation* without *irregular singularities*

- $\rho \in \mathcal{R}_q$ *regular point* $\Longrightarrow$
  - **(i)** *unique classical solution* σ *exists with* $\rho \in \operatorname{im} j_q\sigma$
  - **(ii)** *solution* σ *can be extended in any direction until* $j_q\sigma$ *reaches either boundary of* $\mathcal{R}_q$ *or a regular singularity*
- $\rho \in \mathcal{R}_q$ *regular singularity* $\Longrightarrow$ *dichotomy*
  - **(i)** *either two classical solutions* σ_1, σ_2 *exist with* $\rho \in \operatorname{im} j_q\sigma_i$ *(both ending or both starting in* ρ*)*
  - **(ii)** *or one classical solution* σ *exists with* $\rho \in \operatorname{im} j_q\sigma$ *whose derivative of order* $q+1$ *blows up at* $t = \pi^q(\rho)$

Proof.

- $\mathcal{V}[\mathcal{R}_q]$ locally generated by vector field X
- ρ regular singularity $\Longrightarrow$ X vertical wrt π^q
- dichotomy $\rightsquigarrow$ ∂_t-component of vector field X does or does not change sign at ρ □

Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*

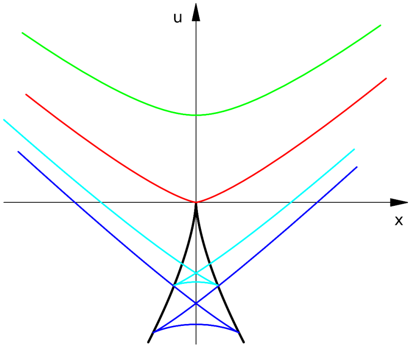


singularity manifold (*criminant*):

$3\dot{u}^2 + u = 0$

(visible part contains only regular singularities)

Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*



second derivative of geometric solution touching "tip" of *discriminant* (projection of criminant) blows up

below intersections of criminant and generalised solutions geometric solutions "change direction" let $\rho \in \mathcal{R}_q$ be an *irregular* singularity

- consider simply connected open set $\mathcal{U} \subset \mathcal{R}_q$ without any irregular singularities such that $\rho \in \overline{\mathcal{U}}$
- in $\mathcal{U}$ Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ generated by single vector field X

Proposition

Generically *any smooth extension of X vanishes at ρ*

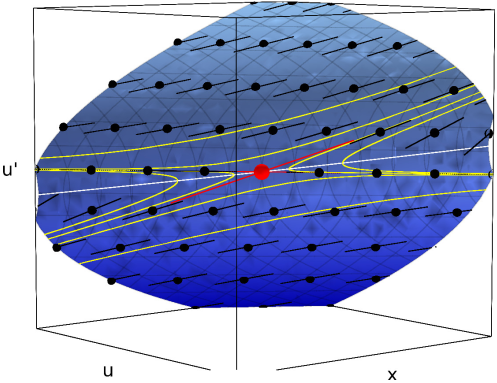
Proof.

- elementary linear algebra of adjugate matrix
- problem: do components of X possess common divisor? □

Conjecture: *not* true, if and only if ρ lies on *singular integral*

Example: $\dot{u}^3 + u\dot{u} - t = 0$ *(hyperbolic gather)*

neighbourhood of an irregular singularity



stable/unstable manifolds define *intersecting* generalised solutions!

Conclusions

Comparison with dynamical systems theory

- use of Vessiot distribution transforms *implicit* differential equation into an *explicit* (and autonomous) one
- one-dimensional distribution defines only *direction*, not an *arrow*
  $\rightsquigarrow$ X and $-42X$ define same distribution!
- *absolute* signs of (real parts of) eigenvalues meaningless; only relative signs matter
- different (smooth) *centre manifolds* yield different generalised solutions
- generalised solutions through irregular singularity are one-dimensional *invariant manifolds* of vector field X with discrete α and ω limit sets
  $\rightsquigarrow$ consist of orbits separated by isolated stationary points

Open problems/questions:

- determine number of generalised solutions through irregular singularity: none, finitely many, infinitely many
- *regularity* theory $\rightsquigarrow$ possible via *prolongations*
- going beyond scalar first-order equations requires local phase portraits in *more than two dimensions*
- (un)stable/centre manifold *higher-dimensional* $\rightsquigarrow$ does it always contain *one-dimensional* invariant manifolds with discrete α and ω limit set $\rightsquigarrow$ requires tangential information
- what about *complex* differential equations?
- do everything *algorithmically* (at least for polynomial differential equations)