Invariant varieties for rational control systems

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- Controlled invariant varieties

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Motivation: Generalize concept of "controlled and conditioned invariant subspaces for linear control systems"

Notations

- $k, m, n, p \in \mathbb{N}$
- $K \in \{\mathbb{R}, \mathbb{C}\}$
- $R = K[x_1, \ldots, x_n]$ polynomial ring
- $Q = \{\frac{p}{q} \mid p, q \in R, q \neq 0\}$ quotient field
- $\mathcal{I} = \langle p_1, \ldots, p_k \rangle$ ideal of R
- $V = \mathcal{V}(\mathcal{I}) = \{x \in K^n \mid p_i(x) = 0 \text{ for } i = 1, \ldots, k\} \subseteq K^n$ variety
- $\mathcal{J}(V) = \{p \in R \mid p(x) = 0 \text{ for all } x \in V\}$ vanishing ideal

<u>Assumption:</u> $\quad \mathcal{J}(\mathcal{V}(\mathcal{I})) = \mathcal{I}$

For $h \in R^p$:

- $K[\underline{h}] := K[h_1, \ldots, h_p] \subseteq R$ subalgebra
- $K(\underline{h}) := K(h_1, \ldots, h_p) \subseteq Q$ subfield

Invariant varieties for autonomous systems

Let $U \subseteq K^n$ be open, $x_0 \in U$ and $F \in \mathcal{C}^1(U, K^m)$. Consider

$$\dot{x}(t) = F(x(t)), \quad x(0) = x_0. \tag{1}$$

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Theorem (Global existence and uniqueness)

For all $x_0 \in U$ there is a unique solution

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In this case: F **vector field on** V.

Module of polynomial vector fields on $\mathcal{V}(\mathcal{I})$

Given: $\mathcal{I} = \langle p_1, \ldots, p_k \rangle$ an ideal of R ideal satisfying

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The following statements are equivalent:

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is an R-module (computable with Gröbner bases) ✓

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$$\sum_{i=1}^{n} \partial_i p_j \cdot F_i \in \mathcal{J}(V \setminus \mathcal{V}(d)) \quad \textit{for all} \quad j = 1, \ldots, k.$$

Insertion: Algebraic geometry

Definition

For ideals $\mathcal{I}, \mathcal{K} \subseteq R$ define the **ideal quotient** of $\mathcal{I}$ by $\mathcal{K}$:

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Consider a rational control system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{2}$$

$x(t)$ state at time t	$f \in Q^n$ autonomous part
$g \in Q^{n\times m}$ control matrix	u input function
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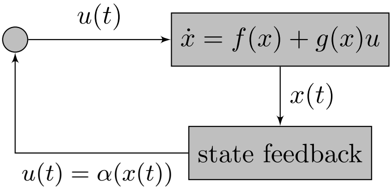
Use **state feedback** to make V invariant for (2)



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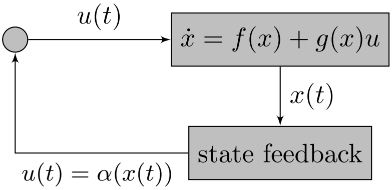

Definition

We call a variety V **controlled invariant for (2)** if there is a state feedback $u(t) = \alpha(x(t))$ such that the closed loop $F := f + g\alpha$ is a vector field on V.

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Rational control system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad \text{where } f \in Q^n, \quad g \in Q^{n\times m}. \quad (2)$$

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Thus, for rational state feedbacks $u(t) = \alpha(x(t))$, where $\alpha \in Q^n$, we may assume w.l.o.g. that (2) takes the form

$$\dot{x}(t) = (\frac{1}{e} \cdot f)(x(t)) + g(x(t))u(t),$$

where $f \in R^n, \ e \in R \setminus \{0\}, \ g \in R^{n\times m}$.

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Rational state feedback making V invariant:

$$\alpha = \frac{z}{d} \in Q^2, \text{ where } z = \begin{pmatrix} 2-w \\ 2w \end{pmatrix}.$$

Controlled and conditioned invariant varieties

Rational control system with polynomial output:

$$\dot{x}(t) = (\frac{1}{e} \cdot f)(x(t)) + g(x(t))u(t), \quad y(t) = h(x(t)) \qquad (4)$$

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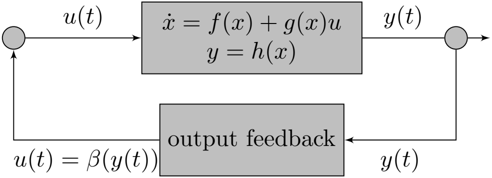
Use **output feedback** to make V invariant for (4)



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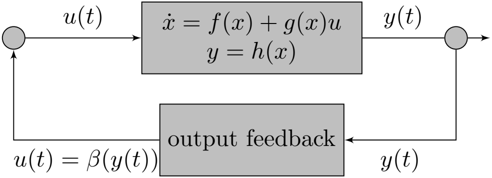

Definition

We call a variety V **controlled and conditioned invariant for (4)** if there is an output feedback $u(t) = \beta(y(t))$ such that the closed loop system $f + g \cdot \beta(h)$ is a vector field on V.

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Given: $f \in R^n, \quad g \in R^{n\times m}, \quad e \in R \setminus \{0\}, \quad h \in R^p, \quad \mathcal{I} \subseteq R$ ideal

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- $\mathcal{M} = \{F \in R^n \mid F \text{ vector field on } V\}$
- $\mathcal{F} = \{(d, z) \in R^{1+m} \mid fd + egz \in \mathcal{M}\}$

1. V controlled invariant $\Leftrightarrow$ There is $(d, z) \in \mathcal{F}$ with $ed \notin \mathcal{I}$
2. V controlled and conditioned invariant
   $\Leftrightarrow$ There is $(d, z) \in \mathcal{F} \cap K[\underline{h}]^{1+m}$ with $ed \notin \mathcal{I}$

Summary

Given: $f \in R^n, \quad g \in R^{n\times m}, \quad e \in R \setminus \{0\}, \quad h \in R^p, \quad \mathcal{I} \subseteq R$ ideal

Assumption: $\mathcal{J}(\mathcal{V}(\mathcal{I})) = \mathcal{I}$ is prime

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Thank you!