Local invariant sets of analytic vector fields

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Introduction

Autonomous differential equations

Consider the autonomous ordinary differential equation

$$\dot{\mathbf{x}} = f(\mathbf{x}), \ \ t \in \mathbb{R},$$

on an open subset $U \subseteq \mathbb{K}^n$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Furthermore let $0 \in U$ be a stationary point of f.

Autonomous differential equations

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Components of vector field $f = (f_1, \dots, f_n)$:

i) $\mathbb{K}[\mathbf{x}]^n$, $\mathbb{K}[\mathbf{x}]$ the polynomial ring over $\mathbb{K}$.

ii) $\mathbb{K}\{\mathbf{x}\}^n$, $\mathbb{K}\{\mathbf{x}\}$ the ring of convergent power series over $\mathbb{K}$.

Later on, we will also need formal power series.

In the following:

$$\mathcal{R} \in \{\mathbb{K}[\mathbf{x}], \mathbb{K}[[\mathbf{x}]], \mathbb{K}\{\mathbf{x}\}\}.$$

Invariant sets

Definition

A subset $V \subseteq U$ is called an invariant set for

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

if for every $x_0 \in V$ the whole trajectory through x_0 is a subset of V.

Invariant sets

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A subset $V \subseteq U$ is called an invariant set for

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

if for every $x_0 \in V$ the whole trajectory through x_0 is a subset of V.

Invariant sets are useful for qualitative analysis, and special solutions of a differential equation.

Invariant sets: Example

Consider the differential equation

$$\begin{aligned}\dot{x} &= -y + x(1 - x^2 - y^2)\\ \dot{y} &= x + y(1 - x^2 - y^2).\end{aligned}$$

The set

$$C := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

is invariant for this equation.

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is invariant for this equation.
Restriction of differential equation to C yields:

$$\dot{x} = -y$$
$$\dot{y} = x$$

Semi-invariants and some of their properties

The Lie derivative

Let $f \in \mathcal{R}^n$. The map:

$$L_f : \mathcal{R} \longrightarrow \mathcal{R},\ \psi \mapsto L_f(\psi) := D(\psi)(\mathbf{x}) \cdot f(\mathbf{x}),$$

is called Lie derivative along f.
Lie derivative plays an important role in study of invariant sets.

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Lie derivative plays an important role in study of invariant sets.
Properties:

i) L_f is linear.

ii) Product rule:

$$L_f(\psi_1\psi_2) = \psi_1 L_f(\psi_2) + \psi_2 L_f(\psi_1).$$

Lie brackets

The $\mathbb{K}$-vector space $\mathcal{R}^n$ becomes a Lie algebra with the following map:

$$[\cdot,\cdot] : \mathcal{R}^n \times \mathcal{R}^n \longrightarrow \mathcal{R}^n,\ (f,g) \mapsto [f,g] := Dg \cdot f - Df \cdot g.$$

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Useful property:
Let $f, g \in \mathcal{R}^n$.
If $\phi \in \mathcal{R}$ one has

$$L_f(L_g(\phi)) - L_g(L_f(\phi)) = L_{[f,g]}(\phi).$$

Semi-invariants and invariant sets

Definition

Let $\phi \in \mathcal{R}$. If there exists $\lambda \in \mathcal{R}$ such that

$$L_f(\phi) = \lambda \cdot \phi$$

holds, then ϕ is called a semi-invariant of f.

Semi-invariants and invariant sets

Definition

Let $\phi \in \mathcal{R}$. If there exists $\lambda \in \mathcal{R}$ such that

$$L_f(\phi) = \lambda \cdot \phi$$

holds, then ϕ is called a semi-invariant of f.

Consequently, ϕ is a semi-invariant iff

$$L_f(\langle \phi \rangle) \subseteq \langle \phi \rangle,$$

for ideal generated by ϕ. Semi-invariants are useful on the study of invariant sets.

Semi-invariants and invariant sets

Lemma

Let $\mathcal{R} = \mathbb{K}[\mathbf{x}]$ or $\mathcal{R} = \mathbb{K}\{\mathbf{x}\}$ and ϕ be a semi-invariant of f. Then, the set

$$\mathcal{V}(\phi) := \{x \in U \mid \phi(x) = 0\}$$

is an invariant set of f.

Previous example

Check invariance for $C := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Example

$$\dot{x} = -y + x(1 - x^2 - y^2)$$
$$\dot{y} = x + y(1 - x^2 - y^2),$$

Let $\phi := x^2 + y^2 - 1$. Then

$$L_f(\phi) = -(2x^2 + 2y^2)\phi.$$

Poincaré-Dulac Normal Forms

Normal Forms

Consider the Taylor expansion of an analytic vector field f, $f(0) = 0$;

$$f(\mathbf{x}) = B\mathbf{x} + \sum_{j=2}^{\infty} f^{(j)}(\mathbf{x}),$$

where $B := D_f(0)$ and $f^{(j)}$ is a homogeneous vector field of degree j.

Normal Forms

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where $B := D_f(0)$ and $f^{(j)}$ is a homogeneous vector field of degree j.

Moreover, decompose $B = \underbrace{B_s}_{\text{semi-simple}} + \underbrace{B_n}_{\text{nilpotent}}$.

Example: For Jordan canonical basis

$\longrightarrow$ B_s diagonal, B_n strict upper triangular matrix.

Normal Forms

Definition

f is in Poincaré-Dulac Normal Form (PDNF) if
$[B_s, f] = Df(\mathbf{x}) \cdot B_s\mathbf{x} - B_s f(\mathbf{x}) = 0$ holds.

Normal Forms

Definition

f is in Poincaré-Dulac Normal Form (PDNF) if $[B_s, f] = Df(\mathbf{x}) \cdot B_s\mathbf{x} - B_s f(\mathbf{x}) = 0$ holds.

The following decomposition will be used later:

$$f(\mathbf{x}) = B_s\mathbf{x} + \underbrace{B_n\mathbf{x} + \sum_{j=2}^{\infty} f^{(j)}(\mathbf{x})}_{=:g}$$

Transformations into Normal Forms

Theorem[H. Poincaré and H. Dulac]

There always exists an invertible formal power series h, which is solution preserving from $\dot{\mathbf{x}} = f(\mathbf{x})$ to $\dot{\mathbf{x}} = \widetilde{f}(\mathbf{x})$, where $\widetilde{f}$ is in PDNF.

Structure of Normal Form depends on the eigenvalues of B.

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Example[Dimension $n = 2$]

Let $B_s = diag(\lambda_1, \lambda_2)$ and assume that λ_1, λ_2 are linearly independent over $\mathbb{Q}$.

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Example[Dimension $n = 2$]

Let $B_s = diag(\lambda_1, \lambda_2)$ and assume that λ_1, λ_2 are linearly independent over $\mathbb{Q}$.
Then
$\widetilde{f} = B_s\mathbf{x}$, i.e. Normal Form is very simple.

Transformations into Normal Forms

Example[Dimension $n = 2$]

Let $B_s = diag(\lambda_1, \lambda_2)$ and $\lambda_1 = -\lambda_2 = 1$. Then

$$\widetilde{f} = B_s\mathbf{x} + \sum_{j \geq 1} \gamma^j(\sigma_j\mathbf{x} + \tau_j B_s\mathbf{x}),$$

where $\gamma := x_1 x_2$ and $\sigma_j, \tau_j \in \mathbb{K}$.

Finding semi-invariants

Theorem[S. Walcher, 2002]

Let f be in PDNF and $\phi \in \mathbb{C}[[\mathbf{x}]]$ be L_f-invariant. Then, there exists an invertible formal power series β such that $\beta\phi$ is L_{B_s}-invariant.

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Example[Dimension $n = 2$]

Let $B_s = diag(\lambda_1, \lambda_2)$.

i) If λ_1, λ_2 are linearly independent over $\mathbb{Q}$ then the only irreducible semi-invariants (up to multiplication with invertible power series) are x_1, x_2.

ii) If $\lambda_1 = -\lambda_2 = 1$ then the only irreducible semi-invariants (up to multiplication with invertible power series) are x_1, x_2.

Operation of B_s on monomials

Let $B_s = diag(\lambda_1, \dots, \lambda_n)$ and $m := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ be a monomial.

Operation of B_s on monomials

Let $B_s = diag(\lambda_1, \ldots, \lambda_n)$ and $m := x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ be a monomial.

$$L_{B_s}(m) = \left(\sum_{j=1}^{n} \alpha_j \lambda_j \right) \cdot m := w(m) \cdot m.$$

Consequently, each monomial lies in the eigenspace of L_{B_s}.

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Consequently, each monomial lies in the eigenspace of L_{B_s}.
For $\phi \in \mathcal{R}$ define $W(\phi) \subseteq \mathbb{C}$ to be the set of all weights which occur in the monomial representation of ϕ.

Generalization of semi-invariants

Definition

Let $I \subseteq \mathcal{R}$ be an ideal. If $L_f(I) \subseteq I$ holds then I is called L_f-invariant.

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Let $I \subseteq \mathcal{R}$ be an ideal. If $L_f(I) \subseteq I$ holds then I is called L_f-invariant.

Lemma

If I is L_f-invariant the vanishing set $\mathcal{V}(I)$ is an invariant set of f.

Example of invariant ideal

If $f \in \mathbb{K}[\mathbf{x}]^n$ is homogeneous, then

$$I_{2\times 2} := \langle 2 \times 2 \text{ minors of } \begin{bmatrix} f_1 & x_1 \\ f_2 & x_2 \\ \vdots & \vdots \\ f_n & x_n \end{bmatrix} \rangle$$

is an invariant ideal of f.

Lemma

If I is radical, i.e.

$$I = \sqrt{I},$$

and $\mathcal{V}(I)$ is an invariant set of $\dot{\mathbf{x}} = f(\mathbf{x})$, then I is L_f-invariant.

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Example

The ideal

$$I := \langle x_1, x_2, \ldots, x_n \rangle$$

is radical and $\mathcal{V}(I) = \{0\}$ is an invariant set because 0 is a stationary point. Therefore, I is L_f-invariant.

Invariant ideals of vector fields in PDNF

Theorem [K., 2016]

Let f be in PDNF. If $I \subseteq \mathbb{K}[[\mathbf{x}]]$ is L_f-invariant, then I is L_{B_s}-invariant.

Why useful?

Invariant ideals of vector fields in PDNF

Theorem [K., 2016]

Let f be in PDNF. If $I \subseteq \mathbb{K}[[\mathbf{x}]]$ is L_f-invariant, then I is L_{B_s}-invariant.

Why useful?
The L_{B_s}-invariant ideals are easier to compute.

Invariant ideals of vector fields in PDNF

Structure of proof:

Take $\phi \in I$ and make use of the decomposition

$$\phi = \sum_{w \in W(\phi)} \phi_w.$$

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$$\phi = \sum_{w \in W(\phi)} \phi_w.$$

Next, use that f is in PDNF, i.e. $[B_s, f] = 0$. This implies:

$$L_f^m(\phi) := \Big(\underbrace{L_f \circ \cdots \circ L_f}_{\text{m times}}\Big)(\phi) = \sum_{j=0}^{m} \binom{m}{j} L_{B_s}^{(j)}(L_g^{(m-j)}(\phi)),$$

since L_{B_s} and L_g commute.

Invariant ideals of vector fields in PDNF

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since L_{B_s} and L_g commute. Approximate ϕ by its residue class $[\phi]_{\mathbb{K}[[\mathbf{x}]]/\langle \mathbf{x} \rangle^i}$, which can be represented by a polynomial. This leads to a finite dimensional linear algebra problem. Finally, keep in mind that ideals are closed sets under the **x**-adic topology.

L_{B_s}-invariant ideals

Structure of L_{B_s}-invariant ideals?

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Proposition [K., 2016]

All L_{B_s}-invariant ideals can be generated by semi-invariants.

L_{B_s}-invariant ideals

Structure of L_{B_s}-invariant ideals?

Proposition [K., 2016]

All L_{B_s}-invariant ideals can be generated by semi-invariants.

Example[Dimension $n = 2$]

If f is in PDNF, and $B_s = diag(\lambda_1, \lambda_2)$, where λ_1, λ_2 are either linearly independent over $\mathbb{Q}$ or $\lambda_1 = -\lambda_2 = 1$, the only invariant prime ideals are

$$\langle x_1 \rangle, \langle x_2 \rangle, \langle x_1, x_2 \rangle.$$

Application to polynomial vector fields

Computing Poincaré Transforms of a polynomial

Goal: Include behaviour "at infinity".

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For simplicity, consider dimension two:

Let $\phi := \sum_{j=0}^{r} \phi_j \in \mathbb{K}[x, y]$, $\deg(\phi_j) = j$ or $\phi_j = 0$, $\phi_r \neq 0$, and let

$$\phi^{hom} := \sum_{j=0}^{r} \phi_j z^{r-j}$$

be its homogenization with respect to z.

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be its homogenization with respect to z.

Substituting $x = 1$ leads to a Poincaré Transform of ϕ with respect to the vector $e_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

$$\phi^* := \phi^*_{e_1} = \sum_{j=0}^{r} \phi_j(1, y) z^{r-j}.$$

Computing Poincaré Transforms of a vector field

Let $f := \sum_{j=0}^{m} f^{(j)}$, $f^{(j)}$ homogeneous of degree j or zero and $\deg(f) = m$.

There is a machinery to compute a Poincaré Transform of f with respect to e_1 which uses homogenization and projection.

Computing Poincaré Transforms of a vector field

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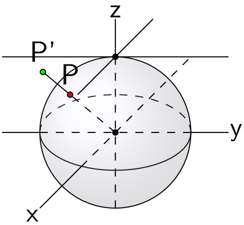
There is a machinery to compute a Poincaré Transform of f with respect to e_1 which uses homogenization and projection.
This leads to a vector field $f^* := f^*_{e_1} \in \mathbb{K}[y, z]^2$.

Poincaré sphere $\longrightarrow$ projective plane:

Geometric motivation/interpretation.

Poincaré sphere $\longrightarrow$ projective plane:



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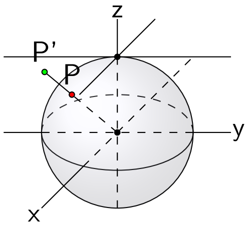

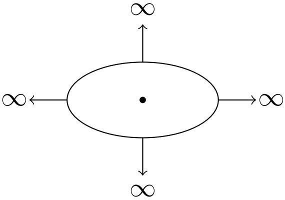

Some properties

Polynomials:

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Vector Fields:

i) One has $f^*_{e_1}(0) = 0$ iff $f^{(m)}_{e_1}(v) \in \mathbb{C}v$.

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Vector Fields:

i) One has $f_{e_1}^*(0) = 0$ iff $f_{e_1}^{(m)}(v) \in \mathbb{C}v$.

ii) In case that ϕ is L_f-invariant, one gets L_{f^*}-invariance of ϕ^*.

Bounding total degrees

Definition

If $v \in \mathbb{C}^2 \setminus \{0\}$ fulfills $f^{(m)}(v) \in \mathbb{C}v$ one calls v a stationary point at infinity.

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For dimension $n = 2$, a stationary point at infinity v is called nondegenerate, if not both eigenvalues of $Df_v^*(0)$ are equal to zero.

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For dimension $n = 2$, a stationary point at infinity v is called nondegenerate, if not both eigenvalues of $Df_v^*(0)$ are equal to zero.

Theorem[S. Walcher, 2000]

Assume that all stationary points at infinity of $\dot{x} = f(x)$ are nondegenerate, and none of them is a rational node (i.e. $\frac{\lambda_2}{\lambda_1} \notin \mathbb{Q}_{>0}$). In case that ϕ is a irreducible semi-invariant of f, its total degree is at most $m + 1$.

Bounding total degrees

A stationary point at infinity v is called generic if the eigenvalues of $Df_v^*(0)$ are linearly independent over $\mathbb{Q}$. Generalization of Theorem:

Theorem[K., 2016]

Assume that all stationary points at infinity of $\dot{x} = f(x)$ are generic. Assume further, that $\phi_1, \ldots, \phi_{n-1}$ are different irreducible semi-invariants of f, where all terms of highest degree are relatively prime. Then, the product of their total degrees is at most $\frac{m^n-1}{m-1}$.

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Compare to dimension $n = 2$

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Compare to dimension $n = 2 \longrightarrow \deg(\phi) \leq m + 1 = \frac{m^2-1}{m-1}$.

A small example

Special Lotka-Volterra-system (J. Chavarriga, H. Giacomini, M. Grau, 2005))

$$\dot{x} = x(ax + by + 1)$$
$$\dot{y} = y(x + y),$$

where $0 < a < 1$ and $b > 1$.
Compute all stationary points at infinity:

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Compute all stationary points at infinity:
One has $m = 2$ and $f^{(m)} = \begin{pmatrix} x(ax + by) \\ y(x + y) \end{pmatrix}$.
Computing $\det(f^{(m)}, \mathbf{x})$ yields 3 stationary points at infinity:

$$v_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; v_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; v_3 := \begin{bmatrix} 1 - b \\ a - 1 \end{bmatrix}.$$

A small Example

Special Lotka-Volterra-system

All stationary points at infinity are nondegenerate in case that

$$\frac{a-b}{(a-1)\cdot(b-1)}$$

is irrational. Applying our previous results yields

$$\deg(\phi) \leq m+1 = 3$$

if ϕ is a possible irreducible semi-invariant.

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if ϕ is a possible irreducible semi-invariant.
Solving the corresponding linear system of equations gives the only irreducible semi-invariants

$$x, y.$$

The end

Thank you for your attention