Quantitative analysis of competition models

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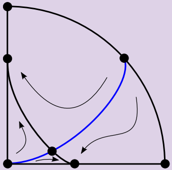
A Lotka-Volterra system

Consider the planar Lotka-Volterra differential system

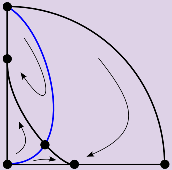
$$\begin{aligned} \dot{x} &= x(\lambda - \alpha_1 x - \alpha_2 y), \\ \dot{y} &= y(\mu - \beta_1 x - \beta_2 y), \end{aligned} \tag{1}$$

where $x, y \geq 0$, $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda, \mu > 0$.

We study the case in which we have a saddle in the open first quadrant.



$\frac{\alpha_2}{\beta_2} > 1$



$\frac{\alpha_2}{\beta_2} \leq 1$

Proposition

System (1) has a saddle in the open first quadrant if and only if

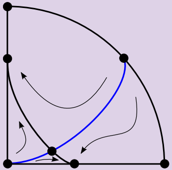
$$\frac{\alpha_1}{\beta_1} < \frac{\lambda}{\mu} < \frac{\alpha_2}{\beta_2}.$$

The rest of the singular points

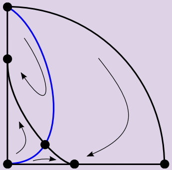
There are three finite nodes at $(0,0)$, $(0,\mu/\beta_2)$, $(\lambda/\alpha_1,0)$.

The characteristic polynomial is

$$xy\big((\alpha_1-\beta_1)x+(\alpha_2-\beta_2)y\big).$$



$\frac{\alpha_2}{\beta_2} > 1$

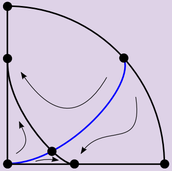


$\frac{\alpha_2}{\beta_2} \leq 1$

Let $\mathcal{S}$ be the *blue* separatrix. We shall provide an index $\kappa = \kappa_{[\mathcal{Y}:\mathcal{X}]}$, the *persistence ratio*, to measure the probability of survival of two species $\mathcal{X}$ and $\mathcal{Y}$:

- in terms of the initial conditions;
- related to area above/below $\mathcal{S}$;
- either in the whole first quadrant or in a finite square;
- depending on the coefficients of the system.

We may need to bound $\mathcal{S}$ by algebraic curves and approximate the areas above and below them.



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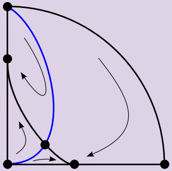


$\frac{\alpha_2}{\beta_2} \leq 1$

Given $R \in (0, \infty]$, consider the square S_R in the first quadrant of sides of length R and a vertex at $(0,0)$. We define

$$A^+(R) = \mu_{\mathcal{L}}\{z_0 \in S_R : \omega(\gamma_{z_0}) = (0, \mu/\beta_2)\},$$
$$A^-(R) = \mu_{\mathcal{L}}\{z_0 \in S_R : \omega(\gamma_{z_0}) = (\lambda/\alpha_1, 0)\}.$$

If $A^+ > A^-$ then the measure of the set of points such that their corresponding orbit has the ω-limit at $(0, \mu/\beta_2)$, that is, that will make $\mathcal{X}$ vanishing, is bigger.

Theorem 1

We have

$$\kappa_{[\mathcal{Y}:\mathcal{X}]} = \begin{cases} 0 & \dfrac{\lambda}{\mu} < \dfrac{\alpha_2}{\beta_2} \leq 1, \\[2ex] \dfrac{\alpha_1(\alpha_2-\beta_2)}{2\beta_2(\beta_1-\alpha_1)-\alpha_1(\alpha_2-\beta_2)} < 1 & 1 < \dfrac{\alpha_2}{\beta_2} < \dfrac{\beta_1}{\alpha_1}, \\[2ex] 1 & 1 < \dfrac{\alpha_2}{\beta_2} = \dfrac{\beta_1}{\alpha_1}, \\[2ex] \dfrac{2\alpha_1(\alpha_2-\beta_2)-\beta_2(\beta_1-\alpha_1)}{\beta_2(\beta_1-\alpha_1)} > 1 & 1 < \dfrac{\beta_1}{\alpha_1} < \dfrac{\alpha_2}{\beta_2}, \\[2ex] \infty & \dfrac{\mu}{\lambda} < \dfrac{\beta_1}{\alpha_1} \leq 1. \end{cases}$$

Remarks

- The relation between the ratio of the interspecific and intraspecific competition taxes of each species, that is β_1/α_1 for $\mathcal{X}$ and α_2/β_2 for $\mathcal{Y}$, determines which one has more chances of surviving.
- When the ratio of $\mathcal{Y}$ is bigger than the ratio of $\mathcal{X}$, then $\mathcal{Y}$ has more chances than $\mathcal{X}$ of surviving.
- This statement is coherent with the biological interpretation of the taxes α_j and β_j, $j = 1, 2$.

Lemma

Consider algebraic upper and lower bounds of $\mathcal{S}$ and let A_U^+, A_U^-, A_L^+ and A_L^- be the areas above/below these upper/lower bounds. Then:

$$\frac{A_U^+(R)}{A_U^-(R)} < \kappa(R) = \frac{A^+(R)}{A^-(R)} < \frac{A_L^+(R)}{A_L^-(R)}.$$

Consequently, given algebraic approximations of $\mathcal{S}$ we can estimate the ratio $\kappa(R)$ and hence the probability of survival of the species.

Definition

Consider $f \in \mathbb{C}[x,y]$. $f = 0$ is invariant by a differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf,$$

where $k \in \mathbb{C}[x,y]$ is the cofactor of $f = 0$.

In the case of system (1) we have

$$x(\lambda - \alpha_1 x - \alpha_2 y)\frac{\partial f}{\partial x} + y(\mu - \beta_1 x - \beta_2 y)\frac{\partial f}{\partial y} = (k_0 + k_1 x + k_2 y)f,$$

where $k(x,y) = k_0 + k_1 x + k_2 y$ is the cofactor of $f = 0$.

Theorem 2 (based on Mou2001, see also CaiGiaLli2003)

The families of systems (1) with $\mu \geq \lambda$ having a saddle in the first quadrant whose stable manifold $\mathcal{S}$ is contained into an algebraic curve of degree N are the ones satisfying:

(i) $\mu = \lambda$, $\alpha_1 - \beta_1 < 0$, $\alpha_2 - \beta_2 > 0$, $N = 1$.

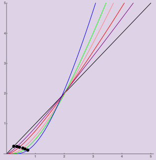
(ii) $\mu = 2\lambda$, $\beta_1 = (2\alpha_1\alpha_2 - 3\alpha_1\beta_2)/(\alpha_2 - 2\beta_2)$, $\alpha_2 > 2\beta_2$, $N = 2$.

(iii) $\mu = 3\lambda$, $\alpha_2 = 7\beta_2/3$, $\beta_1 = 5\alpha_1$, $N = 3$.

(iv) $\mu = 4\lambda$, $\alpha_2 = 9\beta_2/4$, $\beta_1 = 6\alpha_1$, $N = 4$.

(v) $\mu = 3\lambda/2$, $\alpha_2 = 8\beta_2/3$, $\beta_1 = 7\alpha_1/2$, $N = 4$.

(vi) $\mu = 6\lambda$, $\alpha_2 = 13\beta_2/6$, $\beta_1 = 8\alpha_1$, $N = 6$.



Theorem

Moreover:

1. The families (i) and (ii) are Liouville integrable.
2. The families (iii) to (vi) are rationally integrable.

After a change of variables and time, we have

$$\begin{aligned} \frac{dx}{dt} =& \dot{x} = x(1 - x - ay), \\ \frac{dy}{dt} =& \dot{y} = y(s - bx - y), \end{aligned} \tag{2}$$

where $a = \alpha_2/\beta_2 > 0$, $b = \beta_1/\alpha_1 > 0$, $s = \mu/\lambda > 0$.

Proposition

System (2) has a saddle in the open first quadrant if and only if

$$\frac{1}{b} < \frac{1}{s} < a.$$

- We fix in system (2) the values $a = b = 3$, $s = \frac{1567}{807} \sim 1.94$.
- We shall construct two rational functions $y = R^n_{1,2}(x)$ of degree $(n, n-1)$, $n > 2$, approximating the Taylor series of $\mathcal{S}$ up to order $2n-3$ that bound $\mathcal{S}$ above and below.
- $R^n_{1,2}(x)$ will be asymptotic to a straight line at infinity.

An algorithm to build $R^n_{1,2}(x)$

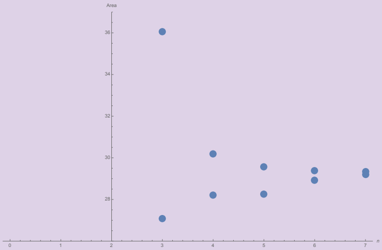
- Take $y = R^n_{1,2}(x) = \sum_{i=0}^{n} a_i x^i / \sum_{i=0}^{n-1} b_i x^i$ and compute its power series expansion at the saddle.
- Compute the power series expansion of $\mathcal{S}$ at the saddle from $P(x, y(x))y'(x) - Q(x, y(x)) = 0$.
- Equaling the coefficients of both power series, we compute all the a_i and also $b_0, \ldots, b_{n-4}$.

- We get b_{n-3} from $a_n/b_{n-1} = \mu$, $\mu \in \{(c+1)/c, c/(c+1)\}$. Thus $\lim_{x\to\infty} \frac{R^n_{1,2}(x)}{x} = \mu$. We recall that there is an infinite singular point in the direction $y/x = 1$.
- We fix b_{n-2}, b_{n-1}, c in such a way that

$$M_{R^n_i(x)} = [(P, Q) \cdot (-(R^n_i)'(x), 1)]_{y=R^n_i(x)}$$

  has constant sign on $x > 0$, $i = 1, 2$. Indeed $M_{R^n_1(x)} > 0$ and $M_{R^n_2(x)} < 0$, on $x > 0$.
- The gradients $(-(R^n_i)'(x), 1)$ point upwards and $\operatorname{sign}(M_{R^n_1}) \cdot \operatorname{sign}(M_{R^n_2}) < 0$.

Areas below the upper and lower bounds of $\mathcal{S}$ for some values of n in a square S_{10}:



n	3	4	5	6	7
A_U^-	36.05	30.18	29.57	29.38	29.33
A_L^-	27.08	28.22	28.25	28.93	29.20

From the relations

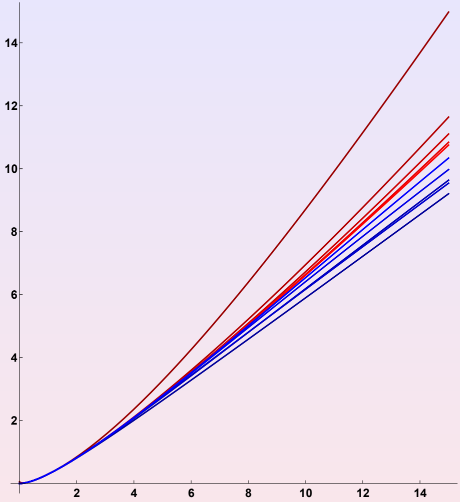
$$\frac{A_U^+(R)}{A_U^-(R)} < \kappa(R) = \frac{A^+(R)}{A^-(R)} < \frac{A_L^+(R)}{A_L^-(R)}.$$

provided before we get, using the bounds R_1^7 and R_2^7,

$$\kappa(10) \in (2.40919, 2.42485).$$

Hence, with bigger probability, the species $\mathcal{X}$ will disappear.

An algorithm to build $R^n_{1,2}(x)$

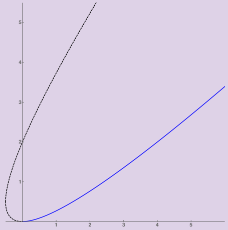


Upper (red) and lower (blue) bounds of the separatrix $\mathcal{S}$ when $a = b = 3$ and $s = \frac{1567}{807}$ for $n = 3, 4, 5, 6, 7$.

Assume that $a > 2$ and consider

$$\mathcal{F}(x,y) = y - \frac{1}{2}\left(\frac{x}{a-2} - y\right)^2 = 0.$$

We note that $\mathcal{F} = 0$ is invariant for (2) if $s = 2$, $b = (2a-3)/(a-2)$, $a > 2$.



Lemma

If $s \neq 2$ and $b = (2a-3)/(a-2)$ then the vector field crosses the right branch of $f = 0$ always in the same direction.

We fix $a = b = 3$, and therefore we have $1 < s < 3$.

We use the relation

$$s = \frac{t^2 - 24t + 236}{t^2 + 92}.$$

Then $1 < s < 2$ is equivalent to $2 < t < 6$ and $2 < s < 3$ is equivalent to $-2 < t < 2$.

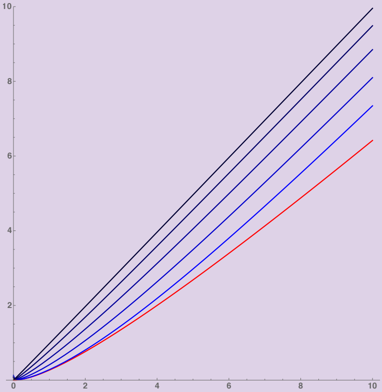
Proposition

If $a = b = 3$ and $1 < s < 2$, then $\mathcal{S}$ is bounded below by $\mathcal{F} = 0$ and bounded above by

$$R_1(x, y) = y - \frac{r_0(t) - x + r_2(t)x^2 + r_3(t)x^3}{1 + r_1(t)x + r_3(t)x^2} = 0,$$

$r_i \in \mathbb{Q}(t)$.

Results



Graph of the right branch of $\mathcal{F} = 0$ (red) and $R_1 = 0$ (blues), for $x \in (0, 10)$ and some values of $t \in (2, 6)$.

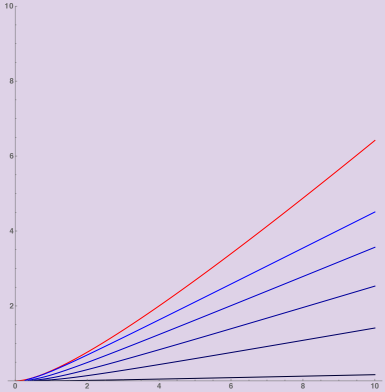
Proposition

If $a = b = 3$ and $2 < s < s^* \sim 2.9999$, then $\mathcal{S}$ is bounded above by $\mathcal{F} = 0$ and bounded below by

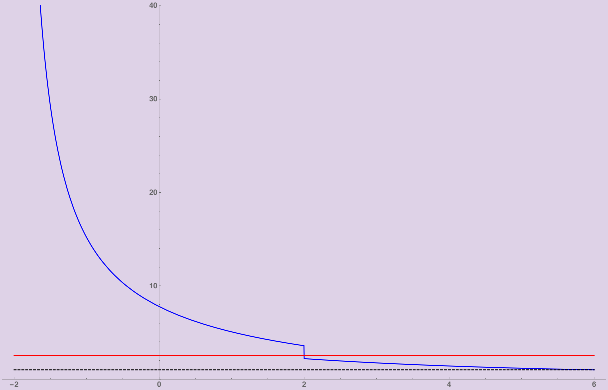
$$R_2(x, y) = y - \frac{r_0(t) + r_1(t)x + r_2(t)x^2 + r_3(t)x^3 + r_4(t)x^4}{r_4(t)(3x^3 + 10^6x^2 + 1)} = 0,$$

$r_i \in \mathbb{Q}[t]$.

Results



Graph of the right branch of $\mathcal{F} = 0$ (red) and $R_2 = 0$ (blues), for $x \in (0, 10)$ and some values of $t \in (-2, 2)$.



Graph of the range of $\kappa(10)$ in terms of $t \in \{t^*, 6\}$. The black dashed straight line is the value 1. The value $\kappa(10)$ lays between the red and the blue curves. Thus $\kappa > 1$.

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