Greedy trajectories of Plancherel processes on two dimensional Young and Schur graphs

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Outline

• The asymptotics of maximum dimensions of linear and projective representations of symmetric group;
• Growth and oscillations of normalized dimensions;
• The sequences of greedy branching and their properties;
• 3D case. The analogue of Plancherel process (asymptotical centrality).
The Young diagrams

**Standard Young diagram**
- Size = 17 boxes.
- \(17 = 6 + 5 + 2 + 2 + 1 + 1\) (the heights of columns)

**Irreducible representations**

**Strict Young diagram**
- Size = 13 boxes.
- \(13 = 6 + 4 + 2 + 1\) (the heights of columns)

**Projective representations**

**Skewed Young diagram**
- The columns are shifted

Some columns have the same height

The height of each column is different

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An example of 3D Young diagram
The Young tableaux

A **Young tableau** is a Young diagram with boxes filled by integers. The integers in each row and each column are increasing.
The skewed Young tableaux

The skewed Young tableaux are used for strict diagrams.
Dimension of the diagram

Dimension of the diagram – the number of Young tableaux in the diagram.

Hook formula for dimensions of standard Young diagrams:

$$\dim(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} h(i,j)}$$

Analogous formula for dimensions of strict diagrams:

$$\dim(\lambda) = \prod_{i<j} \frac{l_i - l_j}{l_i + l_j} \cdot \frac{n!}{\prod l_i!}$$

\(n\) – size of the diagram \(\lambda\) (the number of boxes), \(l_i\) is the height of column number \(i\).

In 3D case, the formula for dimension is unknown.
The Young graph
(first 6 levels)
The Young graph
(first 6 levels)

The Schur graph
(contains strict diagrams only)
The Schur graph
(first 6 levels)
The 3D Young graph
(first 2 levels)
The 3D Young graph
(first 3 levels)
The 3D Young graph
(first 4 levels)
The 3D Young graph
(first 5 levels)
The 3D Young graph
(first 10 levels)
The Plancherel process

The probability of the next standard diagram is calculated as

\[
p(\lambda_1 \to \lambda_2) = \prod_{i=0}^{x-1} \frac{h(\lambda_1, i, y)}{h(\lambda_1, i, y) + 1} \prod_{j=0}^{y-1} \frac{h(\lambda_1, x, j)}{h(\lambda_1, x, j) + 1}
\]

, where \( \lambda_1, \lambda_2 \) – Young diagrams, 
\( x, y \) – coordinates of a box to be added, 
\( h \) – hook length formula.

The probability of the next strict diagram is calculated as

\[
p(\lambda_1 \to \lambda_2) = \frac{2^{l(\lambda_1) - l(\lambda_2) + 1}}{y} \prod_{i \neq x} \frac{y - l_i}{y + l_i} \cdot \frac{y + l_i - 1}{y - l_i - 1},
\]

, where \( \lambda_1, \lambda_2 \) – Young diagrams, 
\( x, y \) – coordinates of a box to be added, 
\( l(\lambda) \) – the width of the diagram \( \lambda \), 
\( l_i \) – the height of the column \( i \).
An analogue of Plancherel process in 3D case

- An analogue of formula for transition probability:

\[ f(\lambda, x, y, z) = \prod_{i=0}^{x-1} \frac{h(\lambda, i, y, z)}{h(\lambda, i, y, z) + 1} \prod_{j=0}^{y-1} \frac{h(\lambda, x, j, z)}{h(\lambda, x, j, z) + 1} \prod_{k=0}^{z-1} \frac{h(\lambda, x, y, k)}{h(\lambda, x, y, k) + 1}. \]

- The sum of values \( \neq 1 \)
- It can be used as a weight function.
- This process is not central but it has an asymptotic centrality.
The sequences of greedy branching
A normalized dimension

A normalized dimension for **standard** diagrams:

\[ c(\lambda) = -\frac{1}{\sqrt{n}} \ln \frac{\dim \lambda}{\sqrt{n!}}, \]

A normalized dimension for **strict** diagrams:

\[ c(\lambda) = -\ln \frac{\dim \lambda}{\ln \sqrt{n!}} + \frac{\ln 2}{2} \cdot n \]

where \( n \) – diagram size, \( \dim \lambda \) - exact dimension of the diagram \( \lambda \).

The greater the exact dimension, the smaller the normalized dimension.

The boundaries \([c_0, c_1]\) of the normalized dimension of maximal dimension Young diagrams:

\[ c_0 = \frac{2}{\pi} - \frac{4}{\pi^2} \approx 0.2313, \quad c_1 = \frac{2\pi}{\sqrt{6}} \approx 2.5651 \]

It was introduced in

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Normalized dimensions of the diagrams of Plancherel and greedy sequences
Normalized dimensions of diagrams of greedy sequence (a segment)
Estimation of limit of normalized dimensions of the greedy sequence of standard diagrams

The estimated value: 0.7001
Merging conjecture

• Given two random Young Diagrams.

• We build two greedy sequences of diagrams starting from each of them.

• After some time these sequences will merge into one sequence.
The “Plancherel” diagram

Size: 50000
Corners: 1511
The diagram of a greedy sequence

Size: 50000
Corners: 1854
The operations on Young diagrams

• Set theory operations over Young diagrams
• Adding and removing the boxes
• Calculation of the transition probabilities in various Markov processes
• Calculation of the exact and normalized dimensions
• Calculation of probabilities of paths and diagrams
• Comparison of two diagrams dimensions
Implementation of Young diagrams

Rows: \{4, 3, 2, 2, 1\}
Columns: \{5, 4, 2, 1\}
Corners: \{1, 2, 4, 5\}
Points: \{1, 2, 3, 5, 6\}
The Young tableau corresponds to a certain path in the Young graph. We can store the tableau as an array of numbers of lines where the boxes were added to:

Tableau: \{1, 2, 3, 2, 1, 1\}
Comparison of dimensions

The **following steps** must be done in order to compare the dimensions of two arbitrary diagrams:

1. Obtain the **intersection** of the diagrams
2. Build **two paths** from the intersection to reach both of the diagrams
3. The product of dimension ratios between each pair of adjacent diagrams in these paths is equal to the increase rate of the initial dimension.
4. The diagram with the higher value of product has the **larger dimension**.

\[ R = \frac{\text{dim}(D1)}{\text{dim}(D2)} = \frac{p1 \cdot p2 \cdot p3}{q1 \cdot q2 \cdot q3} \]
Conclusions

• The algorithms implemented in this project can be applied to solve many other problems of asymptotical representation theory and asymptotical combinatorics;
• Future investigations:
  • Analogue of Plancherel process on 3D Schur graph;
  • Limit shape of 3D diagrams;
  • Asymptotically central processes on 3D graphs.
Thank you!