P4 and desingularization of vector fields in the plane

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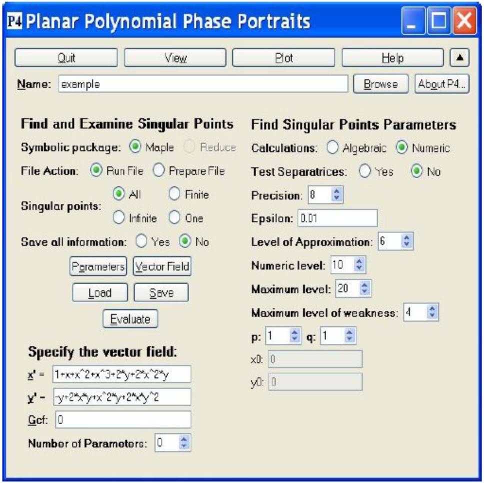
- P4 = Planar Polynomial Phase Portraits
- implemented by C Herssens, J C Artes, J Llibre, F Dumortier
- originally worked for unix with reduce
- Program ported to Qt (windows/unix/mac) with maple by PDM
- P5 = Piecewise P4

Workings of P4 is based on the book Qualitative Theory of Planar Differential Systems by Dumortier, LLibre and Artes.

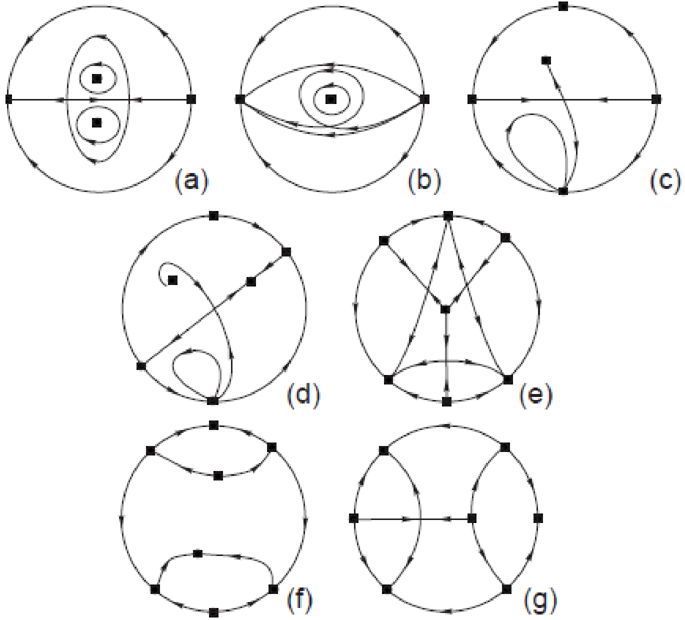

Planar Polynomial Phase Portraits
Quit
View
Plot
Help
Name:
exemple
Browse
About P4...
Find and Examine Singular Points
Symbolic package:
Maple
Reduce
File Action:
Run File
Prepare File
Singular points:
All
Finite
Infinite
One
Save all information:
Yes
No
Parameters
Vector Field
Load
Save
Evaluate
Specify the vector field:
x' =
1+x+x^2+x^3+2*y+2*x^2*y
y' =
-y+2*x*y+x^2*y+2*x*y^2
Gcf:
0
Number of Parameters:
0
Find Singular Points Parameters
Calculations:
Algebraic
Numeric
Test Separatrices:
Yes
No
Precision:
8
Epsilon:
0.01
Level of Approximation:
6
Numeric level:
10
Maximum level:
20
Maximum level of weakness:
4
p:
1
q:
1
x0:
0
y0:
0

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases}$$

Goal: qualitative study of dynamics, disregarding time-related features. This means looking at the phase portrait
Theoretics:

- Poincare-Bendixson, so no chaos
- finite number of singular points when reduced
- study at infinity possible
- singular points have a finite number of sectors (parabolic, hyperbolic, elliptic)
- Separatrix skeleton can be drawn (problem of homoclinic and heteroclinic connections)
- Limit cycles may or may not be present

More than any phase portrait drawing program that one can easily find online!


(a)
(b)
(c)
(d)
(e)
(f)
(g)

$$\begin{cases} \dot{x} & = & P(x,y) \\ \dot{y} & = & Q(x,y) \end{cases}$$

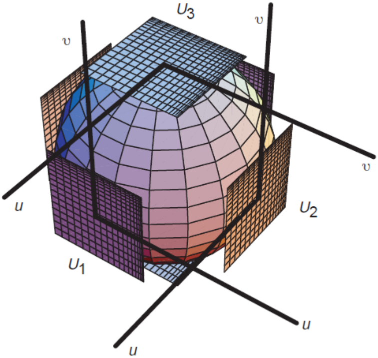
Step 1: Eliminating GCF

This is done using Maple. In the sequel we will assume the GCF has been eliminated.

Step 2: Finding the isolated singular points. Some of them are evaluated algebraically some numerically, but all computations are done with real roots.

Step 3: behaviour at infinity
Consider $S^2 = \{X^2 + Y^2 + Z^2 = 1\}$, and define
$\Delta(x, y) = \sqrt{1 + x^2 + y^2}$,

$$f^{\pm}(x, y) = \pm\left(\frac{x}{\Delta}, \frac{y}{\Delta}, \frac{1}{\Delta}\right) = (X, Y, Z)$$



$\Longrightarrow$ vf is defined on S^2 outside equator

How to extend to the equator? Consider three charts

$$\phi_1(X, Y, Z) = (\frac{Y}{X}, \frac{Z}{X}) = (u, v)$$
$$\phi_2(X, Y, Z) = (\frac{X}{Y}, \frac{Z}{Y})$$
$$\phi_3(X, Y, Z) = (\frac{X}{Z}, \frac{Y}{Z}) = (x, y)$$

Then define the vector field using the relation

$$\begin{aligned}(u, v) &= (\phi_1 \circ \phi_3^{-1})(x, y) \\ &= (y/x, 1/x)\end{aligned}$$

The equator $\{v = 0\}$ corresponds to infinity in the U_3 chart.

Chart U_1:

$$(u,v) = (\phi_1 \circ \phi_3^{-1})(x,y) = (y/x, 1/x) \implies (x,y) = (1/v, u/v)$$

Chart U_2:

$$(u,v) = (\phi_2 \circ \phi_3^{-1})(x,y) = (x/y, 1/y) \implies (x,y) = (u/v, 1/v)$$

They can be joint by 1 formula:

$$(x,y) = \left(\frac{\cos\theta}{v}, \frac{\sin\theta}{v}\right)$$

Chart U_1:

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases}$$

goes to

$$\begin{cases} \dot{u} &= -uP(1/v,u/v)+Q(1/v,u/v) \\ \dot{v} &= -vP(1/v,u/v) \end{cases}$$

and after multiplication to

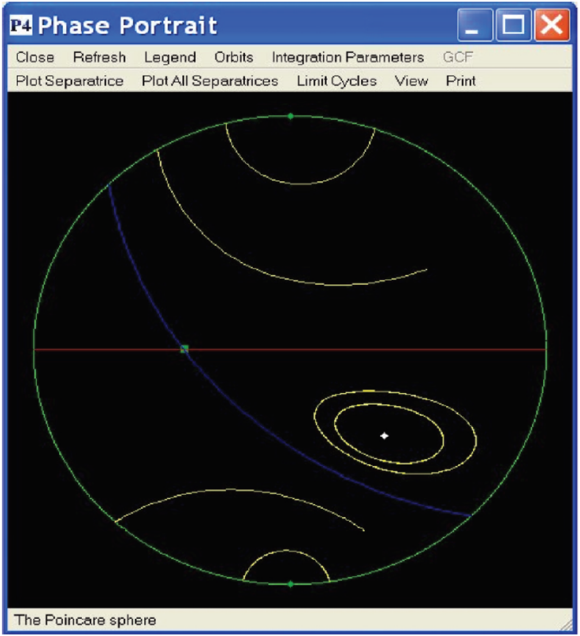
$$\begin{cases} \dot{u} &= v^d\left(-uP(1/v,u/v)+Q(1/v,u/v)\right) \\ \dot{v} &= -v^{d+1}P(1/v,u/v) \end{cases}$$

where d is the degree of the polynomials P, Q. The result is again a polynomial vector field. At $\{v=0\}$:

$$\begin{cases} \dot{u} &= -uP_d(1,u)+Q_d(1,u) \\ \dot{v} &= 0 \end{cases}$$

Equator is invariant with a well-defined dynamics on it!

P4 shows a view of the sphere from the top:



Poincaré compactification:

$$(x, y) = \left(\frac{\cos\theta}{v}, \frac{\sin\theta}{v} \right)$$

Poincaré-Lyapunov compactification:

$$(x, y) = \left(\frac{\cos\theta}{v^\alpha}, \frac{\sin\theta}{v^\beta} \right)$$

Same idea buth with weights (α, β) (and with a bit more complicated inverted formula)

Step 4: Local study of singular points

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases}$$

Suppose

$$P(x_0,y_0) = Q(x_0,y_0) = 0.$$

Define the jacobian

$$M = \begin{pmatrix} \frac{\partial P}{\partial x}(x_0,y_0) & \frac{\partial P}{\partial y}(x_0,y_0) \\ \frac{\partial Q}{\partial x}(x_0,y_0) & \frac{\partial Q}{\partial y}(x_0,y_0) \end{pmatrix}$$

and consider the linearized equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = M \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

Several cases:

1. Saddle (eigenvalues λ, μ opposite sign)
2. Node (eigenvalues λ, μ same sign and nonzero)
3. Focus (eigenvalues $\alpha \pm i\beta$, $\alpha \neq 0$, $\beta \neq 0$)
4. Center (eigenvalues $\pm i\beta$, $\beta \neq 0$)
5. Semi-elementary (eigenvalues $\lambda, 0$ with $\lambda \neq 0$)
6. nilpotent or degenerate (eigenvalues $0, 0$)

For case 1: we compute invariant manifolds tangent to eigenspace of λ resp. μ.
For cases 4,5,6 we need information from the nonlinear part to determine the type further
Case 4: Lyapunov constants (see talk of Joan Torregrosa). P4 uses a method of Gasull & Torregrosa
Case 5: there exists a smooth 1-dim center manifold which is a graph $y = h(x)$ or $x = k(y)$. Reduction of the dynamics to the center manifold leads to determination of type.
Case 6: desingularization

Consider a singular point at the origin $(0,0)$. We use

$$(x,y) = (r\cos\theta, r\sin\theta) = (r\overline{x}, r\overline{y}).$$

and use (r,θ) as new coordinates. Near $\theta = 0$ we use $\sin\theta \approx \theta$ and $\cos\theta \approx 0$, so

$$(x,y) = (r, r\theta)$$

Better:

$$(x,y) = (r, r\overline{y}) \qquad \text{“chart } \overline{x} = 1\text{”}$$

Near $\theta = \pi/2$ we have $\sin\theta \approx 1$ and $\cos\theta \approx \theta - \pi/2$, so

$$(x,y) = (r(\theta - \pi/2), r)$$

Better:

$$(x,y) = (r\overline{x}, r) \qquad \text{“chart } \overline{y} = 1\text{”}$$

Instead of using (r,θ) we use the charts.

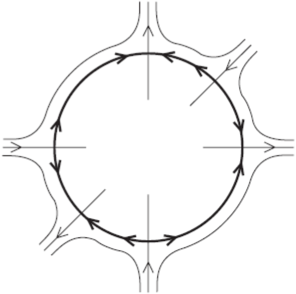
Example:

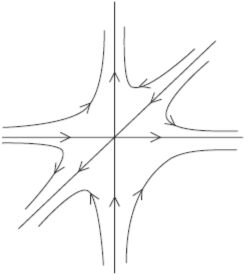
$$\begin{cases} \dot{x} & = & x^2 - 2xy \\ \dot{y} & = & y^2 - xy \end{cases}$$

Leads to

$$\begin{cases} \dot{r} & = & r(\cos^3\theta - 2\cos^2\theta\sin\theta + \dots) + O(r^2) \\ \dot{\theta} & = & \cos\theta\sin\theta(3\sin\theta - 2\cos\theta) + O(r) \end{cases}$$

Seems somewhat complicated trigonometry but is in fact not so hard





It is better to use the charts instead of (r, θ):

$$(x, y) = (r, r\overline{y}) \qquad \text{“chart } \overline{x} = 1\text{”}$$

$$\begin{cases} \dot{x} &= x^2 - 2xy \\ \dot{y} &= y^2 - xy \end{cases}$$

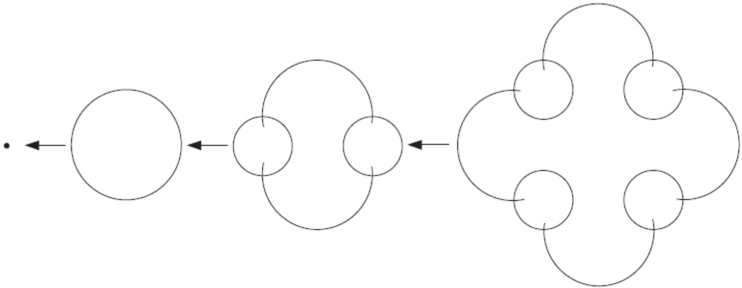
Leads to

$$\begin{cases} \dot{r} &= r(1 - 2\overline{y}) \\ \dot{\overline{y}} &= 3\overline{y}^2 - 2\overline{y} \end{cases}$$

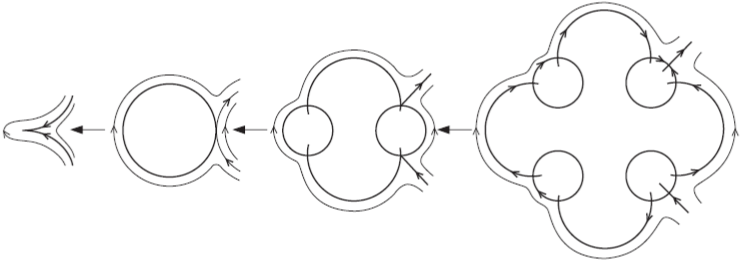
$\Longrightarrow$ polynomial character is retained.

Of course to get information on the full circle we need to complement with additional charts.

Sometimes more than one blow-up is necessary:



Theorem: any singular point of an analytic planar vector field can be blown up after a finite number of blowups so that on the blow-up locus only elementary or semi-elementary singular points are found



For each of these (semi)elementary points one can compute separatrices.

$\Longrightarrow$ for any singular point there is an algorithm to divide the neighbourhood in sectors (hyperbolic, eliptic, parabolic) and to compute the type of the singular point.

P4 actually implements Quasi-homogeneous blow-up

$$(x, y) = (r^\alpha \cos\theta, r^\beta \sin\theta) = (r^\alpha \overline{x}, r^\beta \overline{y}).$$

How to choose the weights (α, β)?
Let

$$\dot{x} = P(x, y) = \sum a_{ij} x^i y^j, \dot{y} = Q(x, y) = \sum b_{ij} x^i y^j$$

$$S = \{(i-1, j) : a_{ij} \neq 0\} \cup \{(i, j-1) : b_{ij} \neq 0\}$$

The newton polygon is the convex hull of the set

$$\mathcal{P} = \cup_{(r,s)\in S}\{(r', s') \colon r' \geq r, s' \geq s\}.$$

One of the borders of the Newton polygon is a straight line with equation

$$r\alpha + s\beta = m$$

then (α, β) is a suitable choice

Lemma: if we proceed this way, then after blowing up, the north and south poles are either nonsingular or (semi)elementary $\implies$ iterated blow-ups are only necessary in the horizontal directions.

This reduces the computational work.

Conclusion: besides determining homoclinic, heteroclinic connections and limit cycles, P4 offers a full global study of planar vector fields.
P5: same thing but with piecewise polynomial systems, defined in regions by algebraic inequalities

Possible extensions to P4/P5:

- computing saddle quantities
- alternative algorithms for numerical integration
- beter sewing in P5
- period computation, computing abelian integrals, Melnikov integrals, ...
- report in Latex/pdf
- alternative symbolic math programs
- ...