Libration points Points Coordinate Determination in Planar N-Body Problem

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A system of N points, each having mass m, forming a planar regular polygon (N-gon), and a central mass M, are considered. Such system, forming a planar central configuration, called a relative equilibrium system [1]. 3N stationary points (libration points) appear in a system. Earliest attempt to calculate libration points coordinates in relative equilibrium system is Ollongren paper [2]. But there is considered very special case 5, 7 and 9 bodies and large central body. It is interesting to solve a problem in more general form. To hold stability (prevent fall into a center), relative equilibrium system must rotate with common angular velocity. Regular n-gon configuration begets a periodic solution in which the bodies rotate uniformly about the central mass with rotation speed [3]:

$$\Omega = \sqrt{\frac{GM}{R^3}} \sqrt{1 + \frac{1}{4} m \left(\sum_{j=1}^{N} \frac{1}{\sin(\frac{1}{2}\alpha_j)} \right)} \tag{1}$$

This equation may be considered as a generalized Kepler Law. Here G is a gravity constant, R is the n-gon radius, and α_j is the angle between the particles of the central configuration. Denote x a distance between the test particle and the central configuration. The main equations to determine stationary points was derived in our previous papers [4] with the helps of computer algebra. The equations for the collinear libration points are:

$$-(R+x)\,\Omega^2 = \pm\frac{GM}{x^2} - \frac{GM}{R^2} - \sum_{j=1}^{N-1} \frac{Gm_j \left(2R\sin\left(\frac{1}{2}\alpha_j\right)^2 + x \right)}{\left(\left(x^2 + 4R^2\sin\left(\frac{1}{2}\alpha_j\right)^2 \right) \left(1+\frac{x}{R}\right) \right)^{3/2}} \tag{2}$$

$$\alpha_j = 2\,\frac{\pi\, j}{N} \tag{3}$$

The sign + is for libration point inside central configuration, and - for outside ones. They can be reduced to a fifth degree polynomial. In dimensionless form:

$$(1+A)x^5 + (3R+2AR+B)x^4 + (3R^2+AR^2+2BR)x^3 + (BR^2 \pm C)x^2 \pm CR^2 = 0 \tag{4}$$

where:

$$A = -\frac{1}{8}\frac{Gm}{R^3}\sum_{j=1}^{N-1}\left(\frac{1}{\sin^3(\frac{\pi j}{N})} - \frac{3}{\sin(\frac{\pi j}{N})}\right) \tag{5}$$

and:

$$B = -\frac{Gm}{R^2}\sum_{j=1}^{N-1}\frac{1}{\sin(\frac{\pi j}{N})} \tag{6}$$

Coefficient A determines area, where potential allows a linearization in small vicinity of N-gon, coefficient B determines rotation of N-gon in respect expression (1). at large N true:

$$Gm/R^3\sum_{j}^{N} 1/(2(\sin(\pi\cdot i/N))^3) \to -1.6541137(N^3/(2\pi))^3 Gm/R^3 \tag{7}$$

When A=0 and B=0 we have well known case 3-body problem [5]). The considered equation for determination libration points coordinates always have only one real root. In case small m / M coordinates of libration points may be calculated as a generalization of classical gravitation 3-body problem:

$$x_{L2} = (\frac{m}{3M+A+2B})^{(1/3)} + \frac{1}{3}(\frac{m}{3M+A+2B})^{(2/3)} - \frac{1}{9}\frac{m}{(3M+A+2B)} \tag{8}$$

$$x_{L1} = (\frac{m}{3M+A+2B})^{(1/3)} - \frac{1}{3}(\frac{m}{3M+A+2B})^{(2/3)} - \frac{1}{9}\frac{m}{(3M+A+2B)} \tag{9}$$

for inner point L_1 and for outer point L_2 respectively. These equations are valid for small m/M ratio and when N is not large than few hundreds. The fifth degree equation is valid at conditions $Nm << M$ and $x << 0.25R$. Then, the dependence of libration points coordinates on mass and number of particles is studied. Coefficients A and B have a limit at large N, depends on N and m/M ratio. Accordingly, libration point coordinates have a maximal value, which respect to a case infinitesimal central mass. In all cases, solutions of fifth degree equations above can be obtained numerically, or, for example, with Maple. Fifth degree equation for non-collinear libration points:

$$(1+A)x^5 + (3R+2AR+B)x^4 + (3R^2+AR^2+2BR)x^3 + BR^2x^2 = 0 \tag{10}$$

has trivial solution x=0 and non-trivial solution of cubic equation:

$$(1+A)x^3 + (3R+2AR+B)x^2 + (3R^2+AR^2+2BR)x + BR^2 = 0 \tag{11}$$

which have at least one real root. For case B=0 we have three real roots if $A < -3/4$. There are addition libration points appear in considered system when masses in a vertex of N-gon are sufficiently large.

References

[1] A. Wintner, *The Analytical Foundation of Celestial Mechanics*, **,** Princeton University Press, Princeton New Jersey (1941).

[2] A. Ollongren, *On a restricted (2n+3) body problem*, Celestial Mechanics, **6**, 45, pp.163-168 (1981).

[3] G. Roberts, *Linear Stability of the 1+n-gon*,World Scientific Monograph Series in Mathematics, **6**, 6, pp.303-330 (2000).

[4] A. Rosaev, *The investigation of stationary points in central configuration dynamics,*, in *Proceedings of Libration Point Orbits and Applications Conference*, New Jersey - London - Singapore - Hon Kong, eds. by G. Gomez, M. W. Lo and .J.J. Masdemont, pp. 623-633 (2003)

[5] V.Szebehely,*Theory of Orbits. The Restricted Problem of Three Bodies*, **,** Acad.press New York and London.(1967).