Computation of doubly asymptotic solutions

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Abstract

We present a numerical method for the computation and continuation of families of homoclinic and heteroclinic connections of periodic orbits of a given system Hamiltonian system with n degrees of freedom. As an application, we compute families of homoclinic and heteroclinic connections of families of the Lyapunov periodic orbits associated with the collinear equilibrium points, L_1, L_2 and L_3 of the planar restricted three-body problem (RTBP).

It is well known that invariant manifolds as well as homoclinic and heteroclinic connections of hyperbolic objects play an important role in the study of dynamical systems from a global point of view. Particularly interesting are their application to Celestial Mechanics and Astrodynamics, and more specifically, to the design of libration point missions (see for example, [6, 13], or the extended work presented in the series of books [8, 9, 10, 11]), to the transport in Solar System (see [1, 7]) and to galactic dynamics (see [15, 16]).

Taking advantage of the natural dynamics of the problems at hand, it is possible to design new ”low-energy” orbits and station keeping strategies, fitting the required goals, but with lower energy demanding for reaching and following the nominal trajectory. The use of homoclinic and heteroclinic phenomena allows to envisage more complex missions, like low–energy transfers to the Moon [14] and the *Petit Grand Tour* to the moons of Jupiter [12]. Having a better understanding of the underlying homoclinic/heteroclinic structures should allow us to construct spacecraft trajectories with desired characteristics. In this way, it is desirable to construct maps of homoclinic and heteroclinic connections and the methodology described in this talk goes towards this direction.

We consider the simplest model to start with which is the Circular Restricted Three-Body Problem (CRTBP). This problem describes the motion of a massless particle under the gravitational influence of two point masses called primaries, in circular motion around their common center of mass. In a synodical reference system, there exist five equilibrium (or libration) points. Three of them, the collinear ones, are in the line joining the primaries and are usually denoted by L_1, L_2 and L_3, where L_1 is between the two primaries, L_2 is at the left-hand side of the small one, and L_3 is at the right-hand side of the big one. The last two equilibrium points, L_4 and L_5 (called triangular points), form equilateral triangles with the primaries. It is also well known that in position-momenta coordinates, this problem may be described as a Hamiltonian system with 2 degrees of freedom.

Due to the stability character of the collinear equilibrium points –they are center∗saddle points– and applying the Lyapunov theorem, the existence of a Lyapunov family of hyperbolic orbits around each collinear point L_i, $i = 1, 2, 3$, is guaranteed. We can take the value of the Hamiltonian as a parameter of the family. So this is our scenario to apply our numerical method for the computation of: (i) homoclinic orbits to the Lyapunov orbits around L_i, $i = 1, 2, 3$, (ii) heteroclinic orbits to different Lyapunov periodic orbits around different L_i, $i = 1, 2, 3$.

So, first of all we describe the numerical method for these computations. It consists mainly in the continuation of the solution of a system of (nonlinear) equations that has as unknowns the initial conditions of the periodic orbits, the linear approximation of the associated stable and unstable manifolds and a point in a given Poincaré section in which the unstable and stable manifolds match. The resolution of this system involves the computation of the invariant manifolds of the periodic orbits, the integration of the RTBP together with its first and second variational equations (using a Runge-Kutta-Fehlberg method of orders 7 and 8 and comptutations in double precision artimetic) as well as the implementation of multiple shooting strategy to cope with the hyperbolic character of the orbits considered. Moreover, an over-determined system is obtained, and we have used the minimum-norm least-squares (LS) solution for the linear system that gives the Newton correction.

We remark that other authors have computed families of homoclinic/heteroclinic connections either solving systems in terms of boundary-value problems (see [5]), or using semi-analytical techniques (asymtotic expansions with coefficients computed in finite-precision aritmetic, see [4]). In these references, individual connections are found by matching the corresponding manifolds on a surface of section. And families are found by manually repeating this matching process for several values of a parameter (typically the energy).

The advantages of our approach is three-fold: (i) the automation of the process, (ii) to overcome the (effective) convergence restrictions of semi-analytical procedures and (iii) the exploration of the neighborhood of L_3 for which semi-analytical procedures do not give useful approximations. Although L_3 has not been considered for astronomical applications, horseshoe-type motion has drawn some attention, since it is performed by co-orbital satellites of Saturn as Janus and Epimetheus or by near Earth asteroids (see [2, 3]).

Finally we describe some results for different interesting problems, including the Earth–Moon and Sun-Jupiter cases, giving explicit ranges of values of the energy where these homoclinic/heteroclinic connections exist.

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