RESONANCE PHENOMENA in
HOMOGENEOUS PIECEWISE-
LINEAR AREA PRESERVING MAPS

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(*) Introduction: Normal Form for Continuous Piecewise Linear Maps.

(*) The Homogeneous Area Preserving Maps

(*) The associate circle map. Rotation number.

(*) The Bifurcation Diagram. Pockets with constant rotation number.
We assume a continuous planar piecewise-linear map and a partition of the phase plane in two regions

\[ x_{n+1} = \begin{cases} 
A^- x_n + B^-, & \text{if } x_n \in \Sigma^-, \\
A^+ x_n + B^+, & \text{if } x_n \in \Sigma^+. 
\end{cases} \]

\( A^{\pm} \) are 2 \( \times \) 2 constant matrices

\( B^{\pm} \) constant vectors in \( \mathbb{R}^2 \)

In principle we have 12 parameters
The continuity implies:

\[
A^- \begin{pmatrix} 0 \\ y \end{pmatrix} + B^- = A^+ \begin{pmatrix} 0 \\ y \end{pmatrix} + B^+
\]

\[
A^- = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix}, \quad A^+ = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix},
\]

\[
B^+ = B^- = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}
\]

We have only 8 parameters
If \( a_{12} \neq 0 \) our map is conjugate to the normal form

\[
x_{n+1} = \begin{cases} 
\left( \begin{array}{cc} T^- & -1 \\ D^- & 0 \end{array} \right)x_n + \left( \begin{array}{c} 0 \\ b \end{array} \right) & \text{if } x_n \in \Sigma^- \\
\left( \begin{array}{cc} T^+ & -1 \\ D^+ & 0 \end{array} \right)x_n + \left( \begin{array}{c} 0 \\ b \end{array} \right) & \text{if } x_n \in \Sigma^+ \cup \Sigma
\end{cases}
\]

where \( T^\pm, D^\pm \) stand for traces and determinant of matrices \( A^\pm \), \( b \in \{0, 1\} \)

This normal form has only 5 parameters
Homogeneous Area Preserving CPWL Maps

Particular case: \[ \begin{cases} D^+ = D^- = 1, \\ b = 0. \end{cases} \]

\[
x_{n+1} = G(x_n) = \begin{cases} 
A(T^-)x_n = \begin{pmatrix} T^- & -1 \\ 1 & 0 \end{pmatrix} x_n, & \text{if } x_n \in \Sigma^- \\
A(T^+)x_n = \begin{pmatrix} T^+ & -1 \\ 1 & 0 \end{pmatrix} x_n, & \text{if } x_n \in \Sigma^+ \cup \Sigma 
\end{cases}
\]
In 1992 Nusse and Yorke gave a piecewise affine approximation with only five parameters for a piecewise map having a border collision bifurcation. We note that the quoted approximation is essentially the canonical form for continuous piecewise linear maps just stated.

In 2005 Lagarias and Rains published a extensive study of this canonical form.

The iterations of a fixed map \( \mathbf{G} \) encodes the solutions of the second-order nonlinear recurrence

\[
x_{n+2} = \frac{T^+ - T^-}{2} |x_{n+1}| + \frac{T^+ + T^-}{2}
\]
Homogeneous Area Preserving CPWL Maps

(a) The map transforms rays into rays because $G(\lambda x) = \lambda G(x)$

(b) The inverse map is $G^{-1}(x_n) = \begin{cases} 
\begin{pmatrix} 0 & 1 \\ -1 & T^- \end{pmatrix} x_n, & \text{if } y_n < 0 \\
\begin{pmatrix} 0 & 1 \\ -1 & T^+ \end{pmatrix} x_n, & \text{if } y_n \geq 0 
\end{cases}$

(c) The map is invariant under the change $(x_n, y_n, T^-, T^+) \rightarrow (-x_n, -y_n, T^+, T^-)$

(d) The map is reversible w.r.t. the involution $x \rightarrow Rx = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$
We denote:

The unit circle as $S^1$

Ray: $\Pi_\theta = \{(x, y) : x = r \sin(2\pi \theta), y = -r \cos(2\pi \theta), 0 \leq \theta < 1, r > 0\}$

Sector: $\Pi(\alpha, \beta) = \{(x, y) : x = r \sin(2\pi \theta), y = -r \cos(2\pi \theta), \alpha < \theta < \beta, r > 0\}$

If we denote $\Pi_{\theta_1} = G(\Pi_{\theta_0})$, we define

$S : S^1 \rightarrow S^1$ such that $\theta_1 = S(\theta_0)$
Rays and Map on the Unit Circle

For $x_0$ belonging to $\Pi_{\theta_0}$

$$x_0 = \begin{pmatrix} r \sin(2\pi \theta_0) \\ -r \cos(2\pi \theta_0) \end{pmatrix} = \begin{pmatrix} 1 \\ -\cot(2\pi \theta_0) \end{pmatrix} x_0 = \begin{pmatrix} 1 \\ \nu_0 \end{pmatrix} x_0,$$

where $x_0 = r \sin(2\pi \theta_0)$, $\nu_0 = -\cot(2\pi \theta_0)$
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where $x_0 = r \sin(2\pi \theta_0), \quad \nu_0 = -\cot(2\pi \theta_0)$.

$$G(x_0) = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix} x_0 = \begin{pmatrix} T - \nu_0 \\ 1 \end{pmatrix} x_0 = \begin{pmatrix} 1 \\ \nu_1 \end{pmatrix} x_1,$$

where $x_1 = (T - \nu_0)x_0, \quad \nu_1 = (T - \nu_0)^{-1} = -\cot(2\pi \theta_1)$.

We define the slope transition map: $h(\nu) = \frac{1}{T - \nu}$. 
The Map on the Unit Circle

\[ S(\theta) = \begin{cases} 
\frac{1}{4}, & \text{if } \theta = 0, \\
\frac{1}{2} - \frac{1}{2\pi} \tan^{-1}(T^+ + \cot(2\pi\theta)), & \text{if } 0 < \theta < 1/2, \\
\frac{3}{4}, & \text{if } \theta = 1/2, \\
1 - \frac{1}{2\pi} \tan^{-1}(T^- + \cot(2\pi\theta)), & \text{if } 1/2 < \theta < \theta_{T-}, \\
0, & \text{if } \theta = \theta_{T-}, \\
-\frac{1}{2\pi} \tan^{-1}(T^- + \cot(2\pi\theta)), & \text{if } \theta_{T-} < \theta < 1,
\end{cases} \]

where \( \theta_{T-} = \frac{1}{2} + \frac{1}{2\pi} \cot^{-1}(-T^-) \)
The Lift of the Map on the Unit Circle

\[
\mathcal{L}_S(\theta) = \begin{cases} 
1/4, & \text{if } \theta = 0, \\
\frac{1}{2} - \frac{1}{2\pi} \tan^{-1}(T^+ + \cot(2\pi \theta)), & \text{if } 0 < \theta < 1/2, \\
3/4, & \text{if } \theta = 1/2, \\
1 - \frac{1}{2\pi} \tan^{-1}(T^+ + \cot(2\pi \theta)), & \text{if } 1/2 < \theta < 1,
\end{cases}
\]

with the natural extension \( \mathcal{L}_S(\theta + 1) = 1 + \mathcal{L}_S(\theta) \)
Invariant Rays

Invariant rays are given by \( S(\theta) = \theta \) or \( h(\nu) = \frac{1}{T - \nu} = \nu \)
or equivalently by \( \nu^2 - T\nu + 1 = 0 \).

Four possible invariant rays

\[
\nu_{1,2}^+ = \frac{\nu^+ \mp \sqrt{(\nu^+)^2 - 4}}{2}, \quad \nu_{1,2}^- = \frac{\nu^- \mp \sqrt{(\nu^-)^2 - 4}}{2}
\]

which corresponds to

\[
\theta_{1,2}^+ = \frac{1}{2\pi} \cot^{-1}(-\nu_{1,2}^+), \quad \theta_{1,2}^- = \frac{1}{2} + \frac{1}{2\pi} \cot^{-1}(-\nu_{1,2}^-)
\]
The following statements hold for map $G$

(a) Apart from the origin, the only fixed points are the points of the ray $\Pi_{3/8}$ when $T^+ = 2$ and those of the ray $\Pi_{7/8}$ when $T^- = 2$. 

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(b) If $T^+ \geq 2, T^- < 2$, then the two rays $\Pi_{\theta_{1,2}^+}$ and the sector $\Pi(0, \theta_2^+)$ are invariant sets; orbits starting at the sector $\Pi(0, \theta_2^+)$ are unbounded and approaching the ray $\Pi_{\theta_1^+}$. 
Proposition

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(b) If $T^+ \geq 2, T^- < 2$, then the two rays $\Pi_{\theta_{1,2}^+}$ and the sector $\Pi(0, \theta_{2}^+)$ are invariant sets; orbits starting at the sector $\Pi(0, \theta_{2}^+)$ are unbounded and approaching the ray $\Pi_{\theta_{1}^+}$.

(c) If $T^+ < 2, T^- \geq 2$, then the two rays $\Pi_{\theta_{1,2}^-}$ and the sector $\Pi(1/2, \theta_{2}^-)$ are invariant sets; orbits starting at the sector $\Pi(1/2, \theta_{2}^-)$ are unbounded and approaching the ray $\Pi(0, \theta_{1}^-)$.
Proposition

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(a) Apart from the origin, the only fixed points are the points of the ray $\Pi_{3/8}$ when $T^+ = 2$ and those of the ray $\Pi_{7/8}$ when $T^- = 2$.

(b) If $T^+ \geq 2, T^- < 2$, then the two rays $\Pi_{\theta_1^+}$ and the sector $\Pi(0, \theta_2^+)$ are invariant sets; orbits starting at the sector $\Pi(0, \theta_2^+)$ are unbounded and approaching the ray $\Pi_{\theta_1^+}$.

(c) If $T^+ < 2, T^- \geq 2$, then the two rays $\Pi_{\theta_1^-}$ and the sector $\Pi(1/2, \theta_2^-)$ are invariant sets; orbits starting at the sector $\Pi(1/2, \theta_2^-)$ are unbounded and approaching the ray $\Pi(0, \theta_1^-)$.

(d) If $T^+ \geq 2, T^- \geq 2$, then the four rays $\Pi_{\theta_i^\pm}, i = 1, 2$ and the two sectors $\Pi(\theta_2^-, 1) \cup \Pi_0 \cup \Pi(0, \theta_2^+)$, and $\Pi(\theta_2^+, \theta_2^-)$ are invariant sets. Orbits starting at these sectors are unbounded.
Assume that $T^- < 2, T^+ < 2$, then the following statements hold for map $S$

(a) If $T^+ T^- < 4$, then the map $S$ has no 2-periodic orbits.

(b) If $T^+ T^- = 4$, then $T^+ < 0, T^- < 0$ and the map $S$ has only one 2-periodic orbit which is non-hyperbolic.

(c) If $T^+ T^- > 4$, then $T^+ < 0, T^- < 0$ and the map $S$ has two 2-periodic orbits which have opposite stabilities.
The rotation number neither depends on the lift nor the initial point.

If the rotation number is irrational then there are no periodic orbits.
If the rotation number $\rho$ is rational, then the map $S$ has a periodic orbit and one of the following three possibilities occurs.

(i) The map $S$ has exactly one periodic orbit. Then $G$ has exactly one periodic orbit (up to scaling) and the other orbits diverge in modulus to $\infty$ as $n \to \pm\infty$.

(ii) The map $S$ has exactly two periodic orbits. Then $G$ has no periodic orbits. All orbits of $G$ diverge in modulus to $\infty$ as $n \to \pm\infty$, with the exception of orbits lying over the two periodic orbits of $S$. These exceptional orbits diverge in modulus to $\infty$ in one direction, and converge to 0 in the other direction.

(iii) The map $S$ has at least three periodic orbits. Then $G$ is of finite order, that is $G^k = I$ for some $k > 1$, and every orbit of $G$ is periodic.
The following statements hold for map $S$

(a) If either $T^+ \geq 2$ or $T^- \geq 2$ then $\rho = 0$. 
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(b) If $T^+ < 0$, $T^- < 0$, and $T^+T^- \geq 4$ then $\rho = 1/2$. 
The following statements hold for map $S$

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(b) If $T^+ < 0$, $T^- < 0$, and $T^+T^- \geq 4$ then $\rho = 1/2$.

(c) If $T^- = 2 \cos(\pi/q)$, where $q \in \mathbb{N}$, $q \geq 2$, and $-2 < T^+ < 2$ so $T^+ = 2 \cos(2\pi\alpha)$ with $0 < \alpha < 1/2$, then $\rho = \frac{2\alpha}{1 + 2\alpha q}$.

In particular when $2\alpha = 1/p$, then $T^+ = 2 \cos(\pi/p)$ and $\rho = \frac{1}{p + q}$. 
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(d) If $T^+ < 0$, $T^- < 0$, and $T^+T^- = 4 \cos^2(\frac{1}{2n})$ then $\rho = \frac{2n - 1}{4n}$.
Some lines with known rotation number

\[ T^+ T^- = 4 \rightarrow \rho = 1/2 \]

\[ (2 \cos (\pi/p), 2 \cos (\pi/q)) \rightarrow \rho = \frac{1}{p + q} \]

\[ (2 \cos (\pi/p), 2 \cos (2\pi\alpha)) \rightarrow \rho = \frac{2\alpha}{1 + 2\alpha p} \]

\[ T^+ T^- = 4 \cos^2 \left( \frac{\pi}{2n} \right) \rightarrow \rho = \frac{2n - 1}{4n} \]
The generalized Fibonacci polynomials are recursively defined as

\[ u_n(x, y) = xu_{n-1}(x, y) + yu_{n-2}(x, y), \quad u_0(x, y) = 0, \quad u_1(x, y) = 1. \]
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By using induction

\[ u_n(x, y) = \frac{\sigma^n - (-y)^n \sigma^{-n}}{\sigma + y\sigma^{-1}}, \quad \sigma(x, y) = \frac{x - \sqrt{x^2 + 4y}}{2}. \]
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By using induction
\[ u_n(x, y) = \frac{\sigma^n - (-y)^n\sigma^{-n}}{\sigma + y\sigma^{-1}}, \quad \text{where} \quad \sigma(x, y) = \frac{x - \sqrt{x^2 + 4y}}{2}. \]

Let us define \( \Psi_n(T) = u_n(T, -1) \), then
\[ \Psi_n(T) = T\Psi_{n-1}(x, y) - \Psi_{n-2}(T), \quad \Psi_0(T) = 0, \quad \Psi_1(x, y) = 1. \]
The power of the matrix $A$

If $A = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix}$ then: $A^n = \begin{pmatrix} \Psi_{n+1}(T) & -\Psi_n(T) \\ \Psi_n(T) & -\Psi_{n-1}(T) \end{pmatrix}$

If $T = 2 \cos \beta$, $0 < \beta < \pi$, then: $A^n = \frac{1}{\sin \beta} \begin{pmatrix} \sin(n+1)\beta & -\sin(n\beta) \\ \sin(n\beta) & -\sin(n-1)\beta \end{pmatrix}$
The power of the matrix $A$

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If $T_n = 2 \cos(\pi/n)$, $\hat{T}_n = 2 \cos \left( \frac{2\pi}{2n+1} \right)$

$\Psi_{n-1}(T_n) = 1$, $\Psi_n(T_n) = 0$, $\Psi_{n+1}(T_n) = -1$,

$\Psi_{2n}(\hat{T}_n) = 1$, $\Psi_{2n+1}(\hat{T}_n) = 0$, $\Psi_{2n+2}(\hat{T}_n) = 1$.

and so, $A^n(T_n) = -I$, $A^{2n+1}(\hat{T}_n) = I$. 
The Dynamics near the point \((T_p, T_q)\)

If \(T^+ = 2 \cos(\pi/p)\) and \(T^- = 2 \cos(\pi/q)\), it can be shown that \(G^{p+q} = I\).

In particular for \(x_0 = (0, -1)\) we have

\[ G^p(x_0) = A^p(T_p)x_0 = -x_0, \quad G^q(-x_0) = -A^q(T_q)x_0 = x_0, \]  

so \(G^{p+q}(x_0) = x_0\).
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\]
so \(G^{p+q}(x_0) = x_0\)

Due the continuity

\[
G^{p+q}(x_0) = A^q(T_q)A^p(T_p)x_0 = A^{q-1}(T_q)A^{p+1}(T_p)x_0 = A^{q+1}(T_q)A^{p-1}(T_p)x_0
\]
The Dynamics near the point \((T_p, T_q)\)

If \(T^+ = 2 \cos(\pi/p)\) and \(T^- = 2 \cos(\pi/q)\), it can be shown that \(G^{p+q} = I\).

In particular for \(x_0 = (0, -1)\) we have

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G^p(x_0) = A^p(T_p)x_0 = -x_0, \quad G^q(-x_0) = -A^q(T_q)x_0 = x_0, \quad \text{so} \quad G^{p+q}(x_0) = x_0
\]

Due the continuity

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G^{p+q}(x_0) = A^q(T_q)A^p(T_p)x_0 = A^{q-1}(T_q)A^{p+1}(T_p)x_0 = A^{q+1}(T_q)A^{p-1}(T_p)x_0
\]

Since \(\det(A(T_p)) = \det(A(T_q)) = 1\), the three equations

\[
\text{tr}(A^q(T^-)A^p(T^+)) = \text{tr}(A^{q-1}(T^-)A^{p+1}(T^+)) = \text{tr}(A^{q+1}(T^-)A^{p-1}(T^+)) = 2
\]

define regions with constant rotation number \(\rho = 1/(p + q)\)
Pockets with constant rotation number
Some bifurcation points with $T^+ > T^-$

\[(T_{p-1}, T_{q+1})\]

\[\text{trace}(A^q(T^-)A^p(T^+)) = 2\]

\[\text{trace}(A^{q+1}(T^-)A^{p-1}(T^+)) = 2\]

\[(T_p, T_q)\]

\[\text{trace}(A^{q-1}(T^-)A^{p+1}(T^+)) = 2\]

\[(T_{p+1}, T_{q-1})\]
Diagonal in the parametric plane (1)

\[(T_p, T_{p+1})\]

\[\text{trace}(A^{p+1}(T^-)A^p(T^+)) = 2\]

\[\hat{T}_p = 2\cos\frac{\pi}{p + \frac{1}{2}}\]

\[\text{trace}(A^p(T^-)A^{p+1}(T^+)) = 2\]
Diagonal in the parameter plane (2)

\[(T_{p-1}, T_{p+1})\] \hspace{2cm} \[(T_{p+1}, T_{p+1})\]

\[(T_{p-1}, T_{p-1})\] \hspace{2cm} \[(T_{p+1}, T_{p-1})\]

\[\text{trace}(A^{p+1}(T^-)A^{p-1}(T^+)) = 2\]

\[\text{trace}(A^p(T^-)A^p(T^+)) = 2\]
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