RESONANCE PHENOMENA in HOMOGENEOUS PIECEWISE-LINEAR AREA PRESERVING MAPS

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OUTLINE

(*) Introduction: Normal Form for Continuous Piecewise Linear Maps.

(*) The Homogeneous Area Preserving Maps

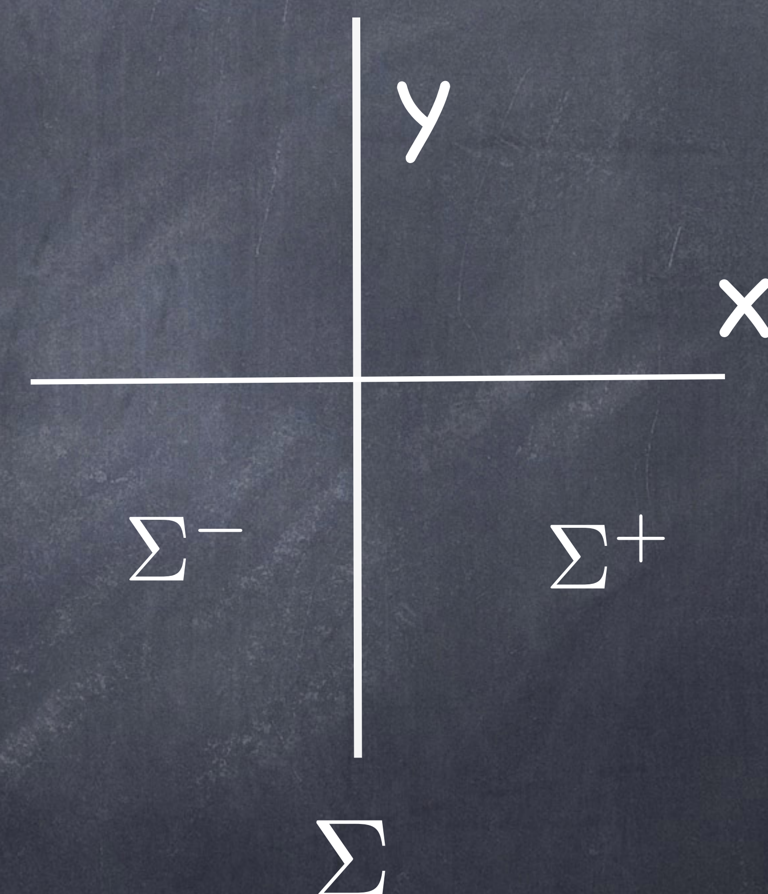
(*) The associate circle map. Rotation number.

(*) The Bifurcation Diagram. Pockets with constant rotation number.

Introduction

We assume a continuous planar piecewise-linear map and a partition of the phase plane in two regions

$$\mathbf{x}_{n+1} = \begin{cases} A^{-}\mathbf{x}_n + B^{-}, \text{ if } \mathbf{x}_n \in \Sigma^{-}, \\ A^{+}\mathbf{x}_n + B^{+}, \text{ if } \mathbf{x}_n \in \Sigma^{+}. \end{cases}$$



$A^{\pm}$ are 2×2 constant matrices

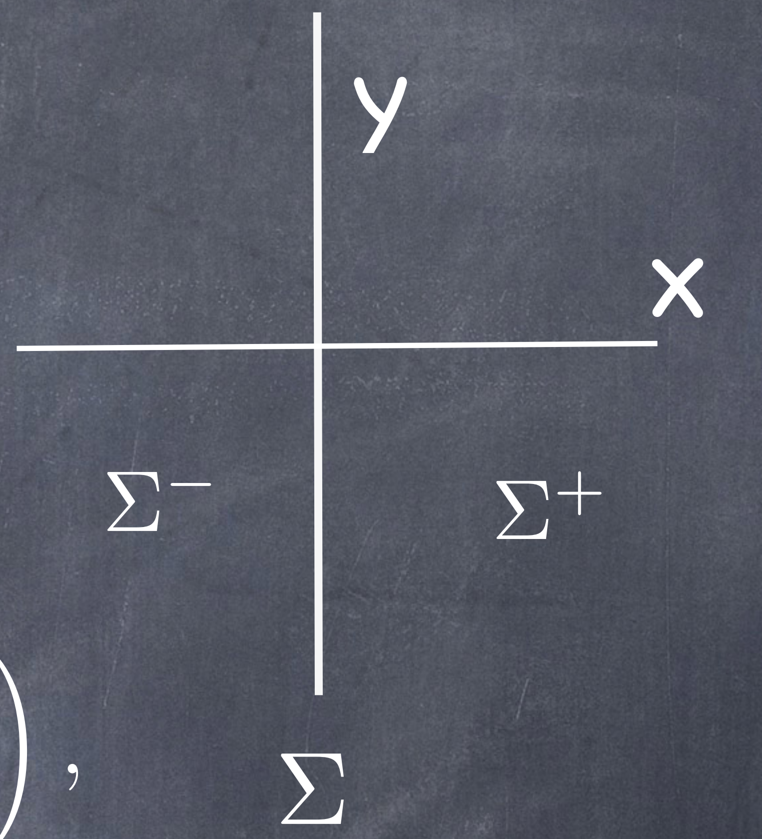
$B^{\pm}$ constant vectors in $\mathcal{R}^2$

In principle we have 12 parameters

Introduction

The continuity implies:

$$A^{-}\begin{pmatrix} 0 \\ y \end{pmatrix} + B^{-} = A^{+}\begin{pmatrix} 0 \\ y \end{pmatrix} + B^{+}$$



$$A^{-} = \begin{pmatrix} a_{11}^{-} & a_{12} \\ a_{21}^{-} & a_{22} \end{pmatrix}, \quad A^{+} = \begin{pmatrix} a_{11}^{+} & a_{12} \\ a_{21}^{+} & a_{22} \end{pmatrix},$$

$$B^{+} = B^{-} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

We have only 8 parameters

A Normal Form for CPWL Map

If $a_{12} \neq 0$ our map is conjugate to the normal form

$$\mathbf{x}_{n+1} = \begin{cases} \begin{pmatrix} T^- & -1 \\ D^- & 0 \end{pmatrix} \mathbf{x}_n + \begin{pmatrix} 0 \\ b \end{pmatrix} \text{if } \mathbf{x}_n \in \Sigma^- \\ \\ \begin{pmatrix} T^+ & -1 \\ D^+ & 0 \end{pmatrix} \mathbf{x}_n + \begin{pmatrix} 0 \\ b \end{pmatrix} \text{if } \mathbf{x}_n \in \Sigma^+ \cup \Sigma \end{cases}$$

where $T^{\pm}, D^{\pm}$ stand for traces and determinant of matrices $A^{\pm}$, $\quad b \in \{0, 1\}$

This normal form has only 5 parameters

Homogeneous Area Preserving CPWL Maps

$$\text{Particular case: } \begin{cases} D^+ = D^- = 1, \\ b = 0. \end{cases}$$

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n) = \begin{cases} A(T^-)\mathbf{x}_n = \begin{pmatrix} T^- & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}_n, \text{ if } \mathbf{x}_n \in \Sigma^- \\ \\ A(T^+)\mathbf{x}_n = \begin{pmatrix} T^+ & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}_n, \text{ if } \mathbf{x}_n \in \Sigma^+ \cup \Sigma \end{cases}$$

Homogeneous Area Preserving CPWL Maps

In 1992 Nusse and Yorke gave a piecewise affine approximation with only five parameters for a piecewise map having a border collision bifurcation. We note that the quoted approximation is essentially the canonical form for continuous piecewise linear maps just stated.

In 2005 Lagarias and Rains published a extensive study of this canonical form.

The iterations of a fixed map map **G** encodes the solutions of the second-order nonlinear recurrence

$$x_{n+2} = \frac{T^+ - T^-}{2}|x_{n+1}| + \frac{T^+ + T^-}{2}$$

Homogeneous Area Preserving CPWL Maps

(a) The map transforms transforms rays into rays because $\mathbf{G}(\lambda\mathbf{x}) = \lambda\mathbf{G}(\mathbf{x})$

(b) The inverse map is $$\mathbf{G}^{-1}(\mathbf{x}_n) = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & T^- \end{pmatrix} \mathbf{x}_n, & \text{if } y_n < 0 \\ \\ \begin{pmatrix} 0 & 1 \\ -1 & T^+ \end{pmatrix} \mathbf{x}_n, & \text{if } y_n \geqslant 0 \end{cases}$$

(c) The map is invariant under the change $(x_n, y_n, T^-, T^+) \rightarrow (-x_n, -y_n, T^+, T^-)$

(d) The map is reversible w.r.t. the involution $\mathbf{x} \longrightarrow R\mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$

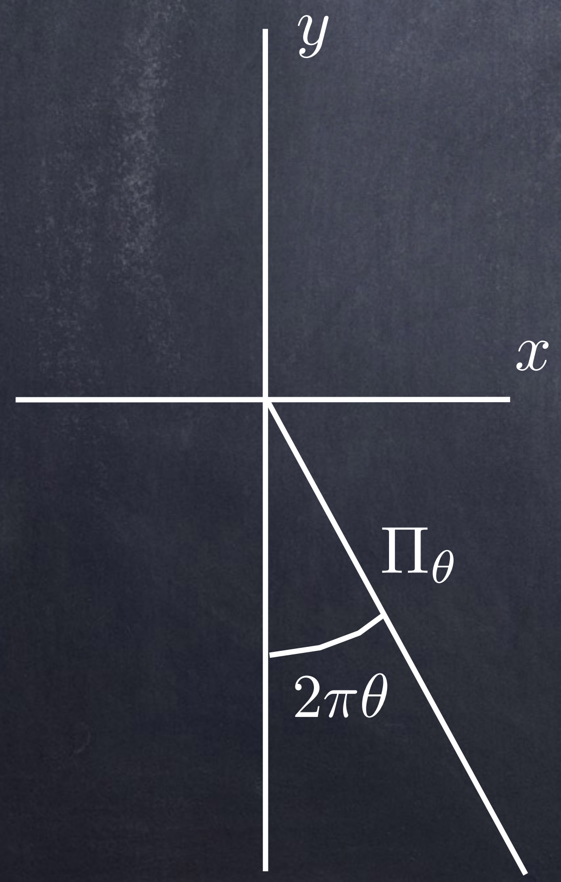
Rays and Map on the Unit Circle

We denote:

The unit circle as S^1

Ray: $\Pi_\theta = \{(x, y) : x = r\sin(2\pi\theta), y = -r\cos(2\pi\theta), 0 \leq \theta < 1, r > 0\}$

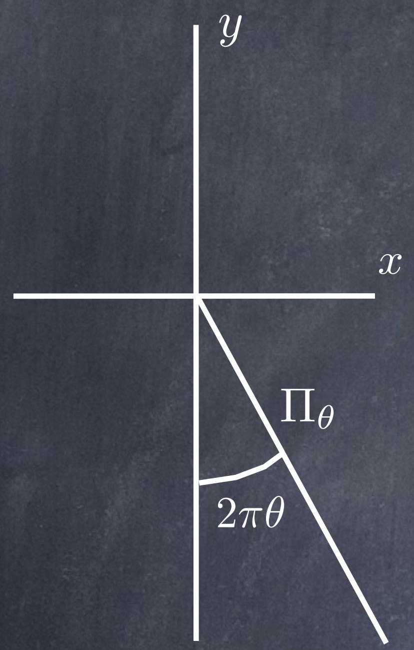
Sector: $\Pi(\alpha, \beta) = \{(x, y) : x = r\sin(2\pi\theta), y = -r\cos(2\pi\theta), \alpha < \theta < \beta, r > 0\}$



If we denote $\Pi_{\theta_1} = \mathbf{G}(\Pi_{\theta_0})$, we define

$$\mathcal{S} : S^1 \longrightarrow S^1 \text{ such that } \theta_1 = \mathcal{S}(\theta_0)$$

Rays and Map on the Unit Circle



For $\mathbf{x}_0$ belonging to Π_{θ_0}

$$\mathbf{x}_0 = \begin{pmatrix} r\sin(2\pi\theta_0) \\ -r\cos(2\pi\theta_0) \end{pmatrix} = \begin{pmatrix} 1 \\ -\cot(2\pi\theta_0) \end{pmatrix} x_0 = \begin{pmatrix} 1 \\ \nu_0 \end{pmatrix} x_0,$$

where $\quad x_0 = r\sin(2\pi\theta_0), \quad \nu_0 = -\cot(2\pi\theta_0)$

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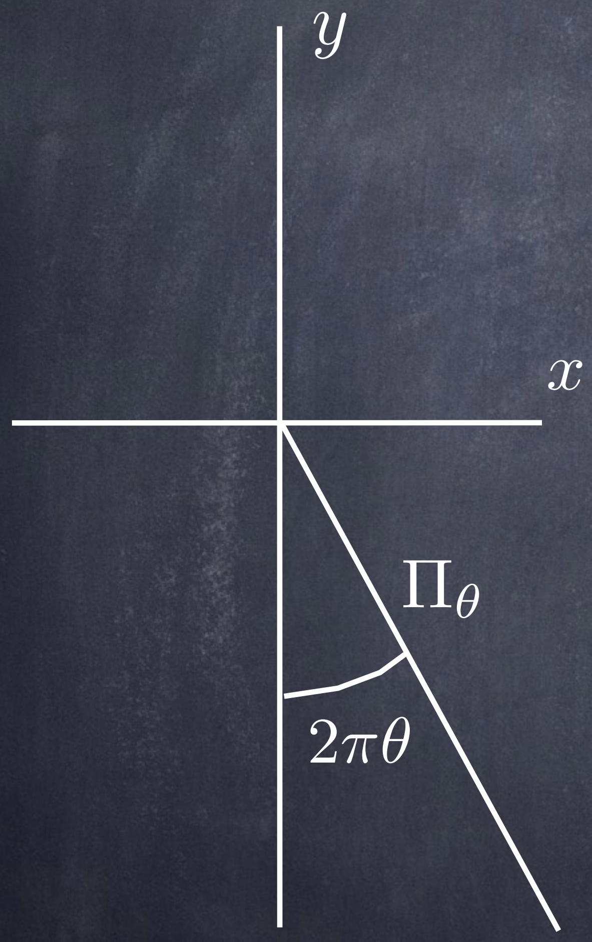
where $\quad x_0 = r\sin(2\pi\theta_0), \quad \nu_0 = -\cot(2\pi\theta_0).$

$$\mathbf{G}(\mathbf{x}_0) = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}_0 = \begin{pmatrix} T-\nu_0 \\ 1 \end{pmatrix} x_0 = \begin{pmatrix} 1 \\ \nu_1 \end{pmatrix} x_1,$$

where $x_1 = (T-\nu_0)x_0, \quad \nu_1 = (T-\nu_0)^{-1} = -\cot(2\pi\theta_1).$

We define the slope transition map: $\qquad h(\nu) = \dfrac{1}{T-\nu}$

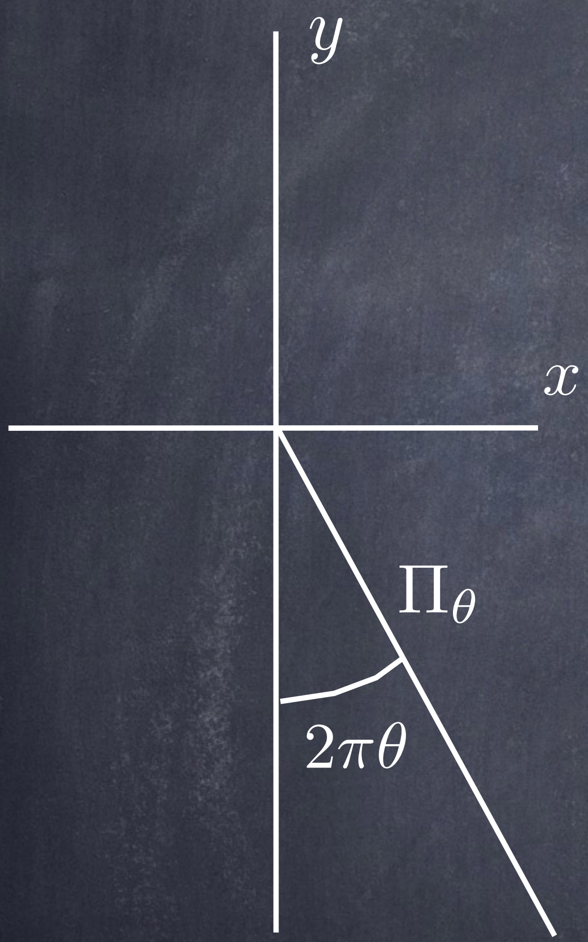
The Map on the Unit Circle



$$\mathcal{S}(\theta) = \begin{cases} \frac{1}{4}, \ \text{if } \theta = 0, \\ \frac{1}{2} - \frac{1}{2\pi}\tan^{-1}(T^{+} + \cot(2\pi\theta)), \ \text{if } 0 < \theta < 1/2, \\ \frac{3}{4}, \ \text{if } \theta = 1/2, \\ 1 - \frac{1}{2\pi}\tan^{-1}(T^{-} + \cot(2\pi\theta)), \ \text{if } 1/2 < \theta < \theta_{T^-}, \\ 0, \ \text{if } \theta = \theta_{T^-}, \\ -\frac{1}{2\pi}\tan^{-1}(T^{-} + \cot(2\pi\theta)), \ \text{if } \theta_{T^-} < \theta < 1, \end{cases}$$

$$\text{where } \theta_{T^-} = \frac{1}{2} + \frac{1}{2\pi}\cot^{-1}(-T^{-})$$

The Lift of the Map on the Unit Circle



$$\mathcal{L}_{\mathcal{S}}(\theta) = \begin{cases} 1/4, \ \text{if } \theta = 0, \\ \frac{1}{2} - \frac{1}{2\pi} \tan^{-1}(T^+ + \cot(2\pi\theta)), \ \text{if } 0 < \theta < 1/2, \\ 3/4, \ \text{if } \theta = 1/2, \\ 1 - \frac{1}{2\pi} \tan^{-1}(T^+ + \cot(2\pi\theta)), \ \text{if } 1/2 < \theta < 1, \end{cases}$$

with the natural extension $\mathcal{L}_{\mathcal{S}}(\theta + 1) = 1 + \mathcal{L}_{\mathcal{S}}(\theta)$

Invariant Rays

Invariant rays are given by $\mathcal{S}(\theta) = \theta$ or $h(\nu) = \dfrac{1}{T - \nu} = \nu$

or equivalently by $\nu^2 - T\nu + 1 = 0.$

Four possible invariant rays

$$\nu^+_{1,2} = \frac{T^+ \mp \sqrt{(T^+)^2 - 4}}{2}, \quad \nu^-_{1,2} = \frac{T^- \mp \sqrt{(T^-)^2 - 4}}{2}$$

which corresponds to

$$\theta^+_{1,2} = \frac{1}{2\pi}\cot^{-1}(-\nu^+_{1,2}), \quad \theta^-_{1,2} = \frac{1}{2} + \frac{1}{2\pi}\cot^{-1}(-\nu^-_{1,2})$$

Proposition

The following statements hold for map $\mathbf{G}$

(a) Apart from the origin, the only fixed points are the points of the ray $\Pi_{3/8}$ when $T^+ = 2$ and those of the ray $\Pi_{7/8}$ when $T^- = 2$.

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(b) If $T^+ \geqslant 2, T^- < 2,$ then the two rays $\Pi_{\theta^+_{1,2}}$ and the sector $\Pi(0, \theta^+_2)$ are invariant sets; orbits starting at the sector $\Pi(0, \theta^+_2)$ are unbounded and approaching the ray $\Pi_{\theta^+_1}$.

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(c) If $T^+ < 2, T^- \geqslant 2$, then the two rays $\Pi_{\theta^-_{1,2}}$ and the sector $\Pi(1/2, \theta^-_2)$ are invariant sets; orbits starting at the sector $\Pi(1/2, \theta^-_2)$ are unbounded and approaching the ray $\Pi(0, \theta^-_1)$.

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(d) If $T^+ \geqslant 2, T^- \geqslant 2$, then the four rays $\Pi_{\theta^\pm_i}, i = 1, 2$ and the two sectors $\Pi(\theta^-_2, 1) \cup \Pi_0 \cup \Pi(0, \theta^+_2),$ and $\Pi(\theta^+_2, \theta^-_2)$ are invariant sets. Orbits starting at these sectors are unbounded.

Proposition

Assume that $T^- < 2, T^+ < 2$, then the following statements hold for map $\mathcal{S}$

(a) If $T^+T^- < 4,$ then the map $\mathcal{S}$ has no 2-periodic orbits.

(b) If $T^+T^- = 4,$ then $T^+ < 0, T^- < 0$ and the map $\mathcal{S}$ has only one 2-periodic orbit which is non-hyperbolic.

(c) If $T^+T^- > 4,$ then $T^+ < 0, T^- < 0$ and the map $\mathcal{S}$ has two 2-periodic orbits which have opposite stabilities.

Rotation Number

$$\rho = \lim_{n\to\infty} \frac{\mathcal{L}_S^n(\theta) - \theta}{n} = \lim_{n\to\infty} \frac{\mathcal{L}_S^n(\theta)}{n}$$

The rotation number neither depends on the the lift nor the initial point.

If the rotation number is irrational then there are no periodic orbits.

Proposition

If the rotation number ρ is rational, then the map $\mathcal{S}$ has a periodic orbit and one of the following three possibilities occurs.

(i) The map $\mathcal{S}$ has exactly one periodic orbit. Then $\mathbf{G}$ has exactly one periodic orbit (up to scaling) and the other orbits diverge in modulus to ∞ as n $\to \pm\infty$.

(ii) The map $\mathcal{S}$ has exactly two periodic orbits. Then $\mathbf{G}$ has no periodic orbits. All orbits of $\mathbf{G}$ diverge in modulus to ∞ as n $\to \pm\infty$, with the exception of orbits lying over the two periodic orbits of $\mathcal{S}$. These exceptional orbits diverge in modulus to ∞ in one direction, and converge to 0 in the other direction.

(iii) The map $\mathcal{S}$ has at least three periodic orbits. Then $\mathbf{G}$ is of finite order, that is $\mathbf{G}^k = I$ for some $k > 1$, and every orbit of $\mathbf{G}$ is periodic.

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(c) If $T^- = 2\cos(\pi/q)$, where $q \in \mathbb{N}, q \geqslant 2$, and $-2 < T^+ < 2$ so $T^+ = 2\cos(2\pi\alpha)$ with $0 < \alpha < 1/2$, then $\rho = \dfrac{2\alpha}{1+2\alpha q}$.

In particular when $2\alpha = 1/p$, then $T^+ = 2\cos(\pi/p)$ and $\rho = \dfrac{1}{p+q}$.

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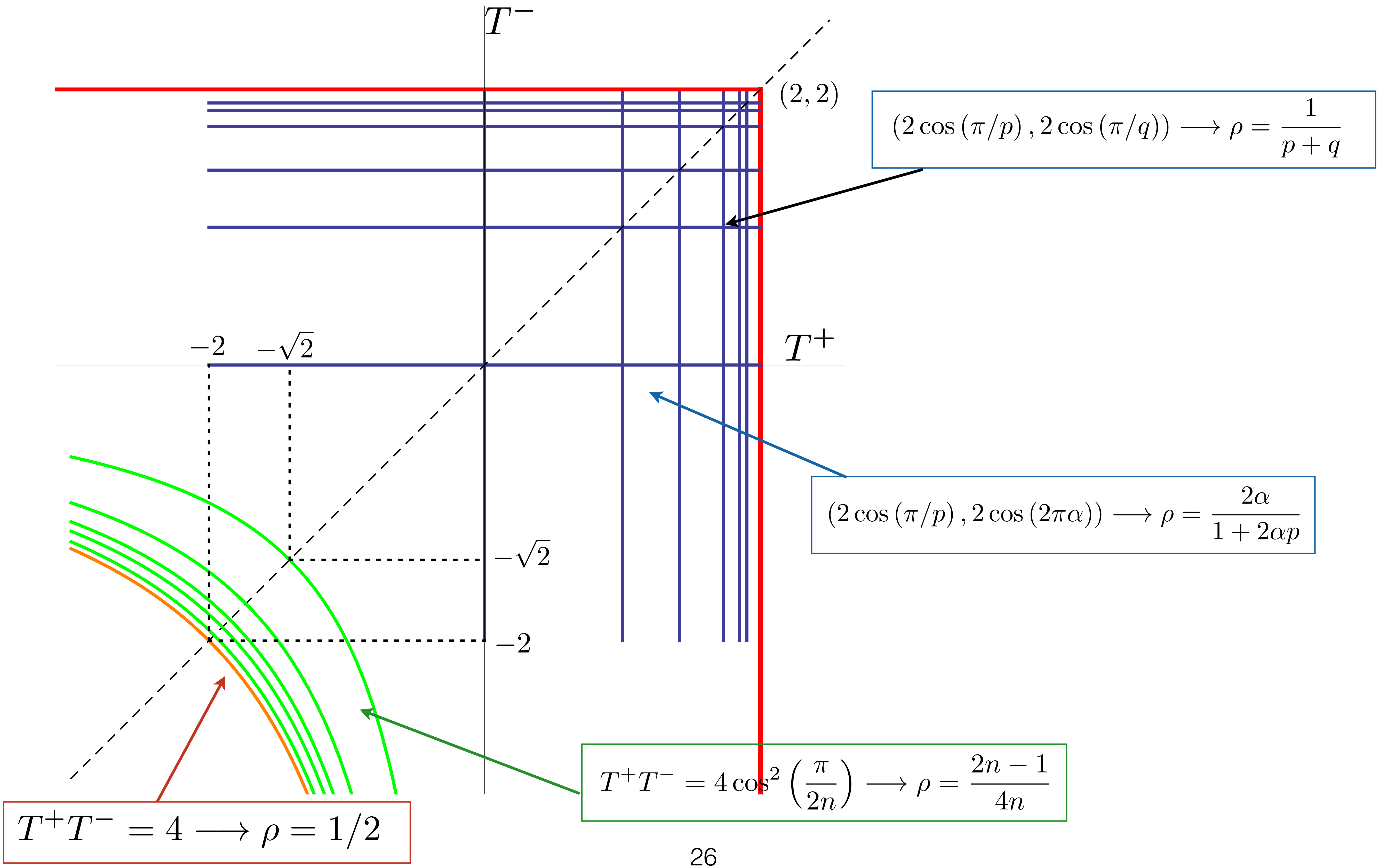
(d) If $T^+ < 0, T^- < 0,$ and $T^+T^- = 4\cos^2(\frac{1}{2n})$ then $\rho = \dfrac{2n-1}{4n}$.

Some lines with known rotation number



Generalized Fibonacci Polynomials

The generalized Fibonacci polynomials are recursively defined as

$$u_n(x,y) = xu_{n-1}(x,y) + yu_{n-2}(x,y), \qquad u_0(x,y) = 0, \quad u_1(x,y) = 1.$$

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By using induction

$$u_n(x,y) = \frac{\sigma^n - (-y)^n \sigma^{-n}}{\sigma + y\sigma^{-1}}, \qquad \sigma(x,y) = \frac{x - \sqrt{x^2+4y}}{2}$$

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$$u_n(x,y) = \frac{\sigma^n - (-y)^n \sigma^{-n}}{\sigma + y\sigma^{-1}}, \qquad \text{where} \quad \sigma(x,y) = \frac{x - \sqrt{x^2+4y}}{2}$$

Let us define $\Psi_n(T) = u_n(T,-1)$, then

$$\Psi_n(T) = T\Psi_{n-1}(x,y) - \Psi_{n-2}(T), \qquad \Psi_0(T) = 0, \quad \Psi_1(x,y) = 1.$$

The power of the matrix A

If $A = \begin{pmatrix} T & -1 \\ 1 & 0 \end{pmatrix}$ then: $A^n = \begin{pmatrix} \Psi_{n+1}(T) & -\Psi_n(T) \\ \Psi_n(T) & -\Psi_{n-1}(T) \end{pmatrix}$

If $T = 2\cos\beta, \quad 0 < \beta < \pi,$ then: $A^n = \frac{1}{\sin\beta} \begin{pmatrix} \sin(n+1)\beta & -\sin(n\beta) \\ \sin(n\beta) & -\sin(n-1)\beta \end{pmatrix}$

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If $T_n = 2\cos(\pi/n), \quad \hat{T}_n = 2\cos\left(\frac{2\pi}{2n+1}\right)$

$$\Psi_{n-1}(T_n) = 1, \qquad \Psi_n(T_n) = 0, \qquad \Psi_{n+1}(T_n) = -1,$$

$$\Psi_{2n}(\hat{T}_n) = 1, \quad \Psi_{2n+1}(\hat{T}_n) = 0, \quad \Psi_{2n+2}(\hat{T}_n) = 1.$$

and so, $A^n(T_n) = -I, \qquad A^{2n+1}(\hat{T}_n) = I.$

The Dynamics near the point (T_p , T_q)

If $T^+ = 2\cos(\pi/p)$ and $T^- = 2\cos(\pi/q),$ it can be shown that $\mathbf{G}^{p+q} = I.$

In particular for $\mathbf{x}_0 = (0, -1)$ we have

$\mathbf{G}^p(\mathbf{x}_0) = A^p(T_p)\mathbf{x}_0 = -\mathbf{x}_0, \quad \mathbf{G}^q(-\mathbf{x}_0) = -A^q(T_q)\mathbf{x}_0 = \mathbf{x}_0,$ so $\mathbf{G}^{p+q}(\mathbf{x}_0) = \mathbf{x}_0$

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Due the continuity

$$\mathbf{G}^{p+q}(\mathbf{x}_0) = A^q(T_q)A^p(T_p)\mathbf{x}_0 = A^{q-1}(T_q)A^{p+1}(T_p)\mathbf{x}_0 = A^{q+1}(T_q)A^{p-1}(T_p)\mathbf{x}_0$$

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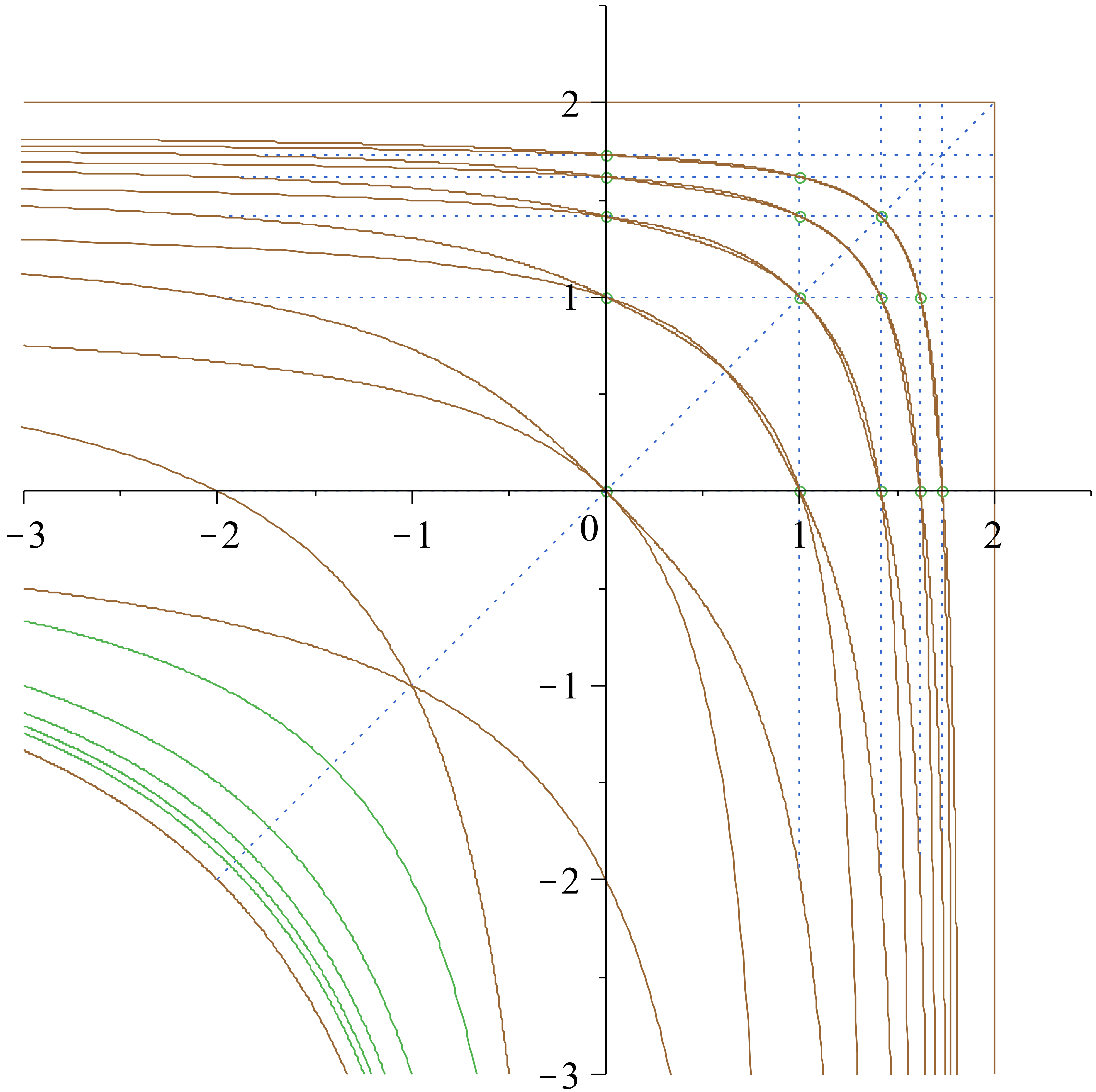
Since $\det(A(T_p)) = \det(A(T_q)) = 1$, the three equations

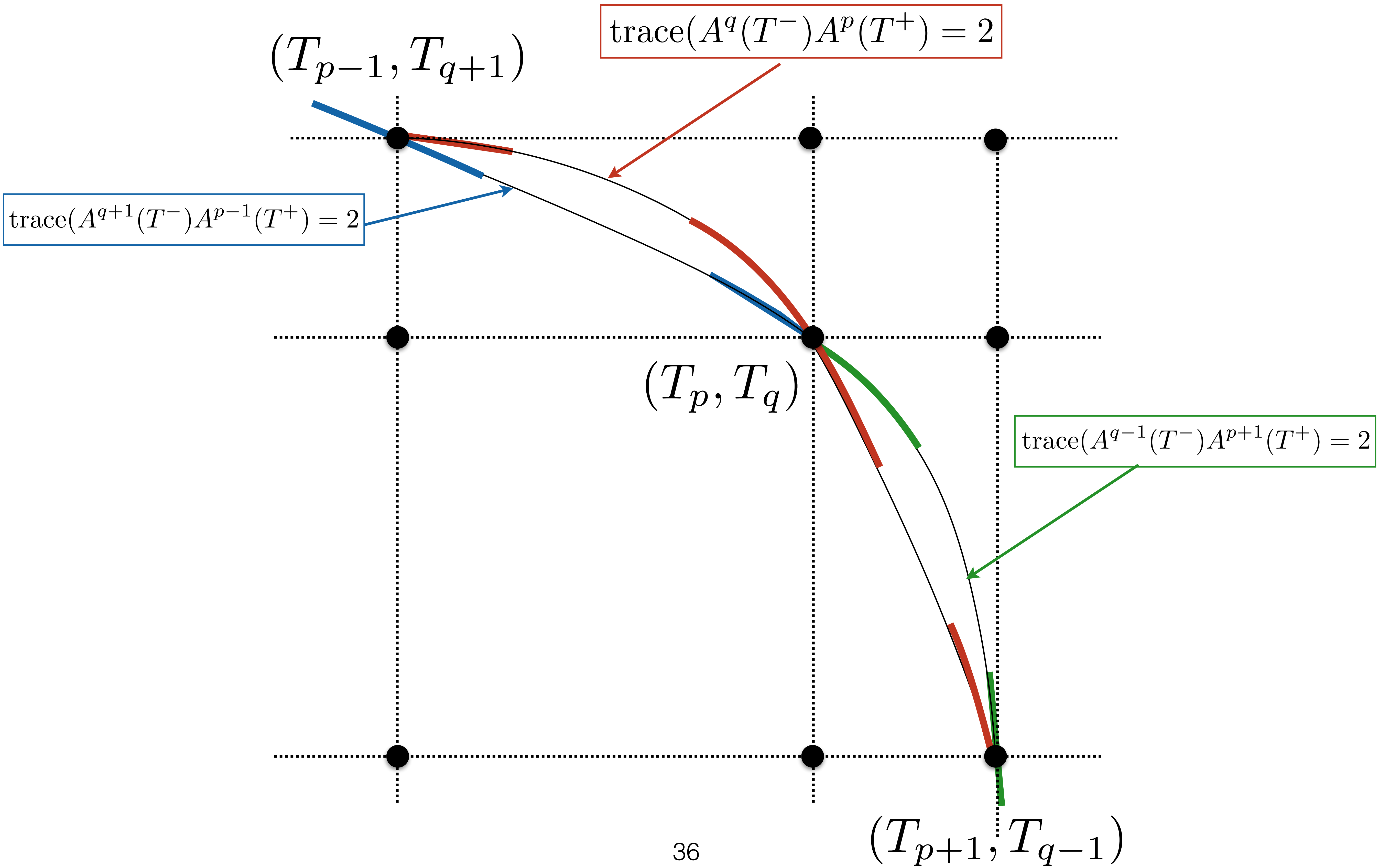
$$\mathrm{tr}(A^q(T^-)A^p(T^+)) = \mathrm{tr}(A^{q-1}(T^-)A^{p+1}(T^+)) = \mathrm{tr}(A^{q+1}(T^-)A^{p-1}(T^+)) = 2$$

define regions with constant rotation number $\rho = 1/(p+q)$

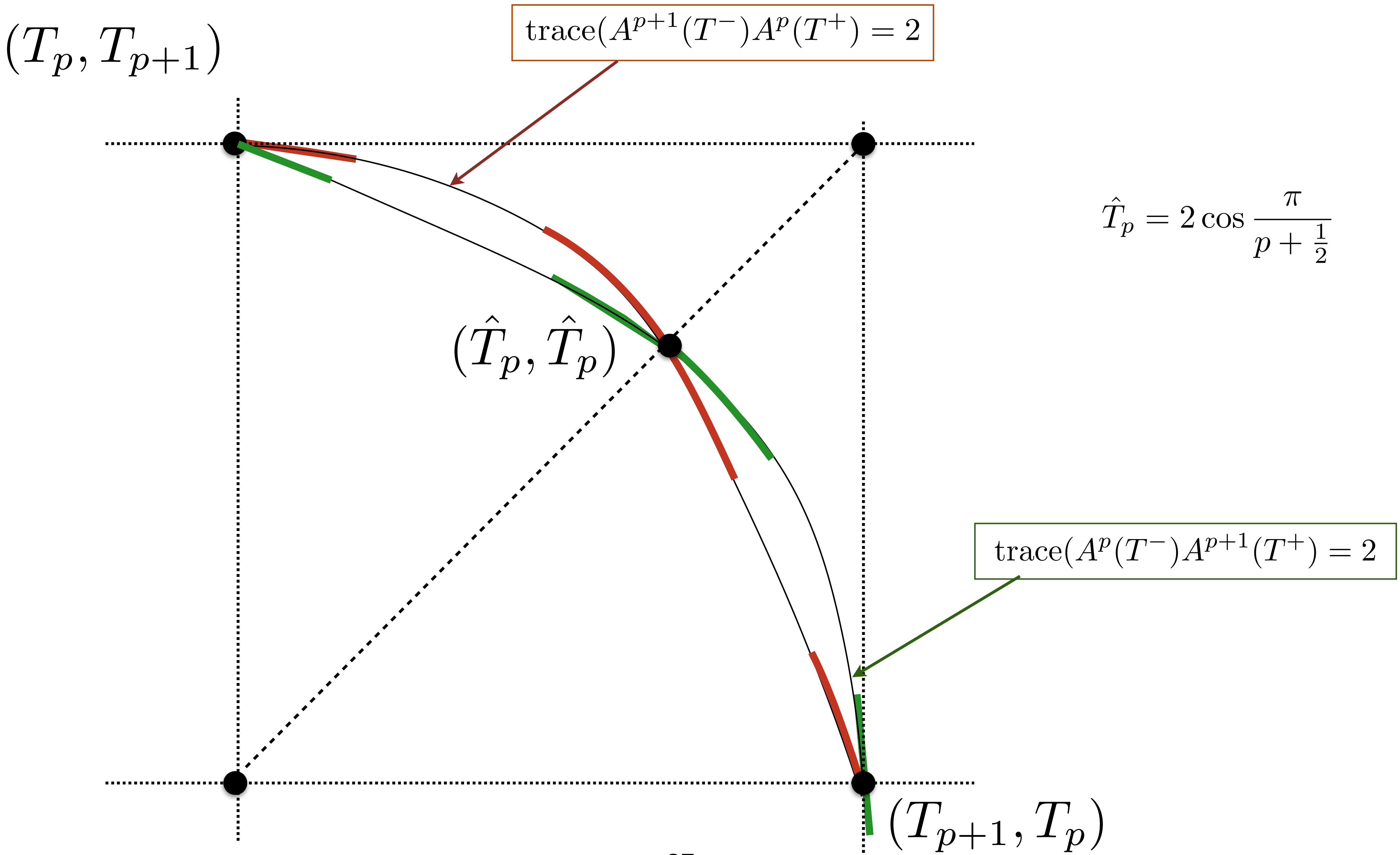
Pockets with constant rotation number



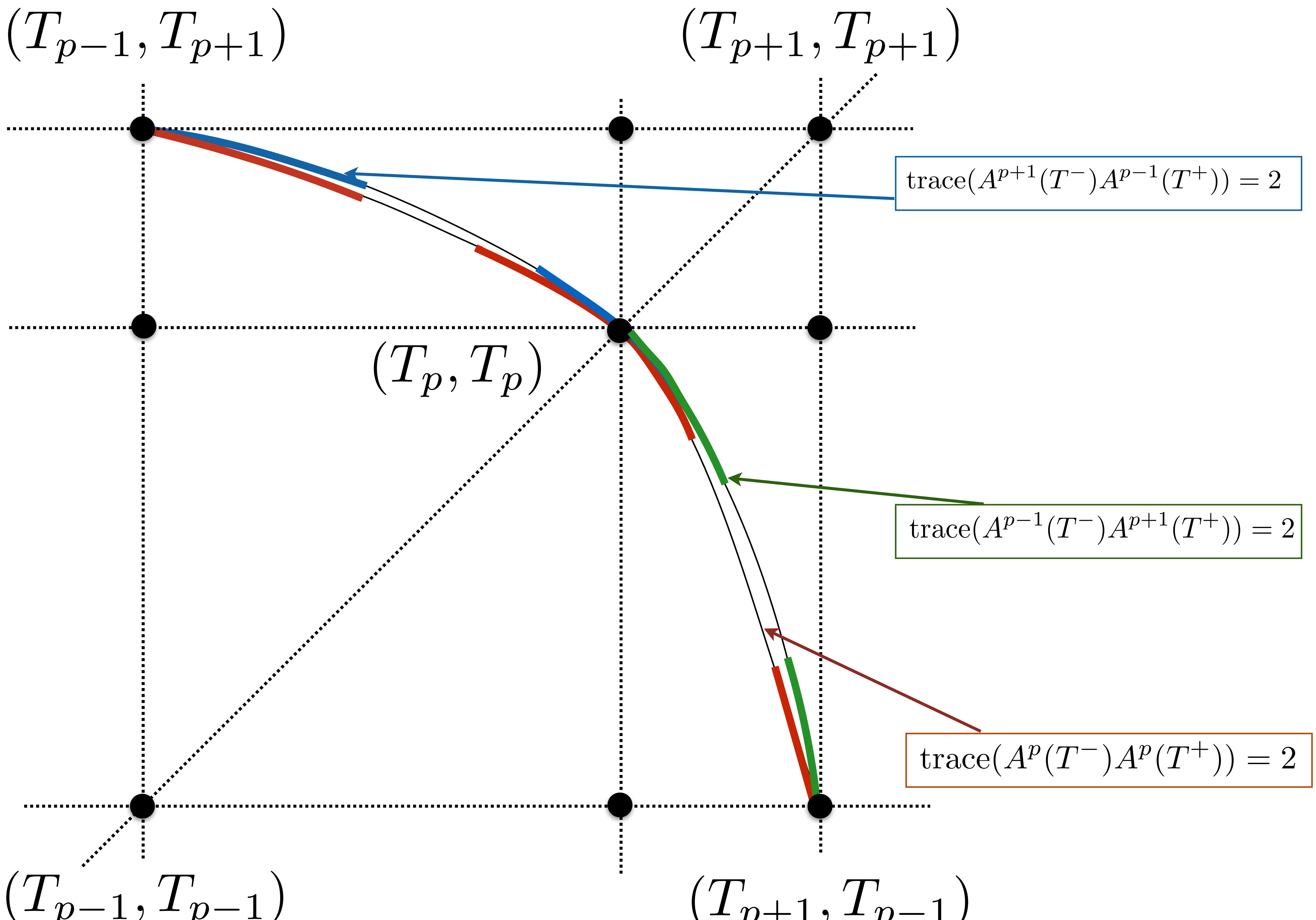
Some bifurcation points with $T^+>T^-$



Diagonal in the parametric plane (1)



Diagonal in the parameter plane (2)



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