Center, weak-focus and ciclicity problems for planar systems with few monomials

Joan Torregrosa

Universitat Autònoma de Barcelona

http://www.gsd.uab.cat

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References

The talk (basically) is based in the next papers:

- A. Gasull, J. Giné & J. Torregrosa. “Center problem for systems with two monomial nonlinearities”. *To appear in Comm. Pure Appl. Math.*.
- A. Gasull, C. Li & J. Torregrosa. “Limit cycles for 3-monomial differential equations”. J. Math. Anal. Appl., 428, 735-749, 2015.
- H. Liang & J. Torregrosa. “Weak-foci of high order and cyclicity”. *Preprint* (2015).
- H. Liang & J. Torregrosa. “Parallelization of the Lyapunov constants and cyclicity for centers of planar polynomial vector fields”. *J. Differential Equations*, 259, 6494–6509 (2015).

Elementary singular points

For differential systems, an elementary singular point is of center-focus type if $\mathrm{tr}\,DX(x_0) = 0$ and $\det DX(x_0) > 0$. Then after a translation and a change of time the system writes as:

$$(x', y') = (-y + P(x, y), x + Q(x, y))$$

and, in complex coordinates ($z = x + iy$),

$$z' = i\,z + \sum_{k+\ell=m} r_{k,\ell}\, z^k \bar{z}^\ell,$$

with $m \geq 2$.

The center-focus problem and related problems

Definition

If $V_{2K+1} \neq 0$ and

$$\Pi(\rho) - \rho = V_{2K+1}\rho^{2K+1} + O(\rho^{2K+2})$$

for $\rho > 0$ close to zero, then V_{2K+1} is called the K-th Lyapunov constant.

- Note that $V_{2K} = 0$.
- V_{2K+1} are polynomials on the coefficients of the system (when the trace vanishes).

Problems

- Characterization of Centers: $\{V_3 = 0, V_5 = 0, \ldots, V_{2K+1} = 0, \ldots\}$.
- Maximum order of a Weak Focus in a concrete family: Highest K?
- Local Cyclicity: Number of limit cycles bifurcating from $\rho = 0$.

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Theorem (GasGinTor2015)

The origin of equation

$$\dot{z} = iz + Az^k\bar{z}^\ell + Bz^m\bar{z}^n$$

is a center when one of the following (nonexclusive) conditions holds:

(a) $k = n = 2$ *and* $\ell = m = 0$ *(quadratic Darboux centers).*

(b) $\ell = n = 0$ *(holomorphic centers).*

(c) $A = -\bar{A}\, e^{i\,\alpha\,\varphi}$ *and* $B = -\bar{B}\, e^{i\,\beta\,\varphi}$ *for some* $\varphi \in \mathbb{R}$ *(reversible centers).*

(d) $k = m$ *and* $(\ell - n)\alpha \neq 0$ *(Hamiltonian or new Darboux centers).*

Here $\alpha = k - \ell - 1$ *and* $\beta = m - n - 1$.

A. Gasull, J. Giné & J. Torregrosa. "Center problem for systems with two monomial nonlinearities". *To appear in Comm. Pure Appl. Math.*.

Centers for systems with few monomials

Theorem (GasGinTor2015)

For equation $\dot z = iz + Az^k\bar z^\ell + Bz^m\bar z^n$, *the list of centers is complete:*

(a) *when* $A\,B = 0$;

(b) *when* $\alpha\,\beta = 0$;

(c) *when* $(\alpha+\beta)(\alpha-\beta) = 0$;

(d) *when* k, ℓ, m *and* n *satisfy* $p\alpha + q\beta = 0$, $(k+\ell-1)Q - (m+n-1)P = 0$, *for some* P, Q, p *and* q, *where* $P \le Q$ *and* $\mathcal{N}(P,Q)$ *are given in the Table and* $(p,q) \in \mathbb{N}\times\mathbb{Z}$ *are such that* $pP + |q|Q \le \mathcal{N}(P,Q)$;

(e) *when the nonlinearities are homogeneous* $(k+\ell = m+n = d)$ *and either* d *is even and* $d \le 34$ *or* d *is odd and* $d \le 57$;

(f) *when* $4 \le k+\ell+m+n \le 36$.

$P\backslash Q$	1	2	3	4	5	6
1	8	10	13	13	15	15
2	-	-	19	-	19	-
3	-	-	-	23	23	-

Values of $\mathcal{N}(P,Q)$ for $P \le Q$ and coprime P and Q

Centers for systems with few monomials

The center-focus problem for equation $\dot{z} = iz + Az^k\bar{z}^\ell + Bz^m\bar{z}^n$ is totally solved when $\alpha\beta = 0$ or $AB = 0$. Consequently, we can reduce our problem to

$$\dot{z} = iz + z^k\bar{z}^\ell + Cz^m\bar{z}^n,$$

with $k + \ell \leq m + n$, $(k,\ell) \neq (m,n)$, $\alpha\beta \neq 0$ and $0 \neq C \in \mathbb{C}$.

The characterization of the reversible centers given in the above result reduces to

$$C^{|q|} + (-1)^{p+|q|+1}\bar{C}^{|q|} = 0,$$

where $(p,q) \in \mathbb{N} \times \mathbb{Z}$ are the coprime values and $p\alpha + q\beta = 0$.

Problems (GasGinTor2015)

- *Is the list of centers of equation with two monomials exhaustive?*
- *In the particular case of homogeneous nonlinearities, is it true that when $k + \ell = m + n \geq 3$ all the centers are reversible?*

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Maximum Cyclicity of a singular point?

For a given family of polynomial vector fields, which is the maximum number of limit cycles that bifurcate from an elementary weak focus or an elementary center?

Theorem

For an analytic general system, the number of limit cycles that bifurcate from a weak focus of order K $(V_{2K+1} \neq 0)$ *is* K.

Problem

The above result could be not true when the family is fixed. For example inside polynomial vector fields of fixed degree.

$M(n) \leq H(n)$

Definition

- $M(n)$ is the number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus in the class of polynomial vector fields of degree n.
- The Hilbert number $H(n)$ is the maximal number of (all) limit cycles in the class of polynomial vector fields of degree n.

Best weak-focus order (Summary)

n	order	References	
$2, 3, 4$	$3, 11, 21$ $(n^2 + 3n - 7)$	[Bau1954,Zol1995,Chr2006, BouSad2008, Gin2012]	P
$5, 6, \ldots, 13$	$n^2 + n - 2$	[LiaTor2015]	P
even	$n^2 - 1$	[QuiYan2009, LliRab2012]	P
odd	$(n^2 - 1)/2$	[QuiYan2009, LliRab2012]	P
even ≤ 34	$n^2 + n - 2$	[QiuYan2009, LiaTor2015]	E
odd ≤ 89	$(n+2)(n-1)/2$	[GasGinTor2015]	E
≤ 77	$(n-1)^2$	[LiaTor2015]	E

$M(n) \geq ?$

Best lower bounds for $M(n)$

The number of small amplitude limit cycles bifurcating from an elementary center or an elementary focus in the class of polynomial vector fields of degree n is

- $M(n) \geq n^2 + 3n - 7$ for $n = 2, 3, 4.$
  [Bau1954,Zol1995,Chr2006,BouSad2008,Gin2012]
- $M(n) \geq n^2 + n - 2$ for $n = 5, 6, \ldots, 13.$
  [LiaTor2015]

$M(n) \geq n^2 + n - 2$ for $4 \leq n \leq 13$

Theorem (LiaTor2015)

For $4 \leq n \leq 13$, equation

$$\dot{z} = iz + z^2 + z^3 + \cdots + z^n + \lambda_1 z + \sum_{k+\ell=2}^{n} \lambda_{k,\ell} z^k \bar{z}^\ell,$$

where $\lambda_1 \in \mathbb{R}$, $\lambda_{k,\ell} \in \mathbb{C}$ are perturbing parameters, has at least $n^2 + n - 2$ small limit cycles bifurcating from the origin.

H. Liang & J. Torregrosa. “Parallelization of the Lyapunov constants and cyclicity for centers of planar polynomial vector fields”. *J. Differential Equations*, 259, 6494–6509 (2015).

Highest order weak foci (all degree)

Theorem (QiuYan2009, LliRab2012)

For every integer $n \geq 3$, there exists a polynomial differential system of degree n having a weak focus of order $n^2 - 1$, when n is even, or $(n^2 - 1)/2$, when n is odd.

J. Llibre & R. Rabanal, "Planar real polynomial differential systems of degree $n > 3$ having a weak focus of high order", *Rocky Mountain J. Math.* 42 (2012), 657–693.

Y. Qiu & J Yang, "On the focus order of planar polynomial differential equations", *J. Differential Equations*, 246 (2009), 3361-3379.

Nonexplicit or perturbed weak foci for even degree

Theorem (LliRab2012)

For every $n = 2m$ *there exist* $n+1$ *functions* $(\varepsilon_0(\alpha), \ldots, \varepsilon_n(\alpha))$ *such that the system*

$$\dot{x} = -y(1 - x^{n-1} - \alpha y^{n-1}) + \sum_{j=0}^{m} \varepsilon_{2j}(\alpha) x^{2j} y^{n-2j}$$

$$\dot{y} = x(1 - x^{n-1} - \alpha y^{n-1}) + \sum_{j=0}^{m-1} \varepsilon_{2j+1}(\alpha) x^{2j} y^{n-2j}$$

has a weak focus of order $n^2 - 1$ *at the singular point located at the origin.*

Highest order weak foci (n even) (homogeneous)

Proposition (QiuYan2009 (2..18), LiaTor2015 (20..34))

For every even $n \leq 34$, there exists a real constant C such that equation

$$z' = iz - \frac{n}{n-2} z^n + z\bar{z}^{n-1} + Ci\bar{z}^n$$

has a weak focus at the origin of order $n^2 + n - 2$.

Y. Qiu & J Yang, "On the focus order of planar polynomial differential equations", *J. Differential Equations*, 246 (2009), 3361-3379.

Weak-focus of a system of degree 4

Proposition (HuaWanWanYan2008)

The next system of degree 4 has a weak-focus of order 18 at the origin.

$$z' = iz + 2iz^3 + iz\bar{z}^3 + \sqrt{\frac{52278}{20723}}\bar{z}^4$$

J. Huang, F. Wang, L. Wang & J. Yang. "A quartic system and a quintic system with fine focus of order 18". *Bull. Sci. Math. 132* (2008) 205–217.

Local cyclicity of a system of degree 4

Proposition (HaiTor2015)

The next system of degree 4 has 18 limit cycles bifurcating from the origin.

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Weak foci when n is odd (homogeneous)

Proposition (GasGinTor2015)

For every odd integer $n \leq 89$, there exist c such that the origin of equation

$$z' = i\,z + z^n + c\,z^{n-1}\bar{z}$$

is a weak focus of order $(n+2)(n-1)/2$.

A. Gasull, J. Giné & J. Torregrosa. "Center problem for systems with two monomial nonlinearities". *To appear in Comm. Pure Appl. Math.*.

For example for $n = 89$ we have that $V_3 = V_5 = \ldots = V_{7831} = 0$, $V_{7831} = D_1(E_1 c\bar{c} - E_2)(c^{44} - \bar{c}^{44})$, $V_{7835} = \ldots = V_{8007} = 0$, $V_{8009} = -D_2(c^{44} + \bar{c}^{44})$. Where $D_1 = N_{1225}/N_{220}$, $E_1 = N_{157}$ $E_2 = N_{155}$ $E_1 = N_{2089}/N_{903}$.

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Highest (odd n) order weak foci (nonhomogeneous)

Proposition (LiaTor2015)

For every integer $3 \leq n \leq 77$, the origin of equation

$$z' = i\,z + \bar{z}^{n-1} + z^n$$

is a weak focus of order $(n-1)^2$.

For $n = 77$ the first nonvanishing Lyapunov quantity is $V_{11553} = \frac{N_{3639}}{N_{3551}}$. The computation time is less than 3 hours, in PARI in a Xeon computer (CPU E5-450, 3.0 GHz, RAM 384 Gb) with GNU Linux. But the maximum allocated memory is 153Gb. Higher values for n can not be done.

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The algorithm in PARI

```
wf(N)=
{
  local(t,last,i,j,H,L);
  time=gettime();
  last=2*(N-1)^2+2;
  H=matrix(last+1,last+1);
  L=vector(last+1);
  H[N+1,1]=1;
  H[1,N+1]=1;
  H[N+1,2]=1;
  H[2,N+1]=1;
  for(i=3,last,
    for(j=0,floor((i+1)/2),
      if (j-N+1>=0,
         H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j+1+1,j-N+1+1]*(i-j+1)/(i-2*j+N)/I+H[i-j+1,j+1-N+1]*(j+1-N)/(i-2*j+N-1)/I;
      );
      if(i-j-N+1>=0,
        if(i-2*j-N !=0,
          H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j-N+1+1,j+1+1]*(j+1)/(i-2*j-N)/I;
         );
      if(i-2*j-N+1 !=0,
          H[i-j+1,j+1]=H[i-j+1,j+1]+H[i-j-N+1+1,j+1]*(i-j-N+1)/(i-2*j-N+1)/I;
        );
      );
      if(i-2*j==0,
        L[j+1]=H[i-j+1,j+1];
        if(L[j+1]!=0,
          print("N=",N,", j=",j,", time=",(gettime()-time)/1000.0);
          print(L[j+1]);
         );
      );
    );
   for(j=floor((i+1)/2)+1,i,H[i-j+1,j+1]=conj(H[j+1,i-j+1]););
  );
}
```

Local cyclicity for last example of weak focus

Proposition (HaiTor2015)

Under general polynomial perturbations of degree n, we have that the cyclicity of the origin of system

$$z' = i\,z + \bar{z}^{n-1} + z^n$$

is

(a) $(n-1)^2$ *for* $n = 3, 4, 5$, *and*

(b) *at least* $n^2 - 3n + 6$ *for* $n = 6, 7, 8$.

H. Liang & J. Torregrosa. "Weak-foci of high order and cyclicity". *Preprint* (2015).

Limit cycles for families with few monomials

Clearly, equations with one monomial

$$\dot{z} = Az^u\bar{z}^v$$

have NO limit cycles because they are homogeneous.
We are now studying equations with two monomials,

$$\dot{z} = Az^u\bar{z}^v + Bz^k\bar{z}^l,$$

where $A, B \in \mathbb{C}$ and $u, v, k, l \in \mathbb{N} \cup \{0\}$, trying to give a uniform bound for their number of limit cycles.
For instance, consider

$$\dot{z} = (1+i)\,z - z^2\bar{z}.$$

This equation with two monomials has the circle $|z| = 1$ as limit cycle, because, in polar coordinates, writes as $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$.

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This equation with two monomials has the circle $|z| = 1$ as limit cycle, because, in polar coordinates, writes as $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$.

p limit cycles in three monomial families

Proposition

For $3 \leq p \in \mathbb{N}$, consider the 2-parameter family of systems

$$\dot{z} = (a+i)\,z + (b+i)\,z|z|^{2(p-2)} - \frac{5i}{2}\,\bar{z}^{p-1},$$

with $a, b \in \mathbb{R}$, $3 \leq p \in \mathbb{N}$. Then there exist values for a and b for which the above equation has at least p limit cycles.

The unperturbed system

Equation

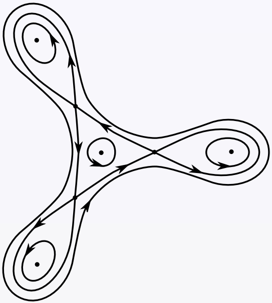
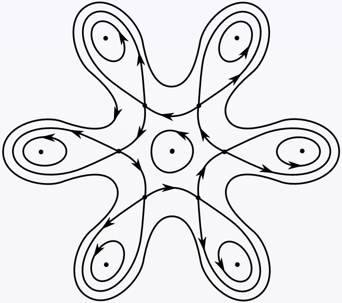
$$\dot{z} = (a+i)\,z + (b+i)\,z^{p-1}\bar{z}^{p-2} - \frac{5i}{2}\,\bar{z}^{p-1},$$

when $a = b = 0$ is Hamiltonian, with Hamiltonian function

$$H(r,\theta) = \frac{r^2}{2} - \frac{5}{2p} r^p \cos(p\,\theta) + \frac{r^{2(p-1)}}{2(p-1)} - \tilde{\rho},$$

where $\tilde{\rho} = \dfrac{(p-2)(p-5)}{2p(p-1)}\, 2^{\frac{2}{p-2}}$.

The phase portraits of the unperturbed system

Centers when $a = b = 0$ for the cases $p = 3$ and $p = 6$.

Idea of the proof I

The differential equation in polar coordinates is

$$dH(r,\theta) - (a\, r^2 + b\, r^{2(p-1)})\, d\theta = 0.$$

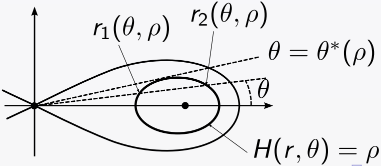
Writing $a = \varepsilon\,\alpha$ and $b = \varepsilon\,\beta$, for $\alpha, \beta \in \mathbb{R}$ and ε small enough, the associated first order Melnikov function is

$$M(\rho) = \alpha\, I_2(\rho) + \beta\, I_{2(p-1)}(\rho),$$

where

$$I_j(\rho) = \int_{H=\rho} r^j\, d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^j(\theta,\rho) - r_1^j(\theta,\rho) \right)\, d\theta,$$

for $j = 2, 2(p-1)$ and $\rho \in (\rho^*, 0)$.



Idea of the proof II

Then, we introduce the auxiliary analytic function

$$J(\rho) = \frac{I_{2(p-1)}(\rho)}{I_2(\rho)}, \quad \rho \in (\rho^*, 0)$$

and we write

$$M(\rho) = I_2(\rho)\big(\alpha + \beta J(\rho)\big).$$

Notice that $I_2(\rho) > 0$ because this function gives the double of the area surrounded by a connected component of the curve $H(r, \theta) = \rho$.

The proof continues showing that $J(\rho)$ is not constant and, in fact, $M(\rho)$ has a simple zero and finishes using the symmetry of the perturbed system.

We remark that there is a limit cycle for each petal, and they are p.