Generic two-parameter families of piecewise smooth systems

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Summary

In this talk we adress some research directions involving qualitative and geometric aspects of non-smooth dynamical systems theory. The purpose is to exhibit the bifurcation diagram of a planar typical cycle.

Joint Paper with: D. Novaes, T. Seara and I. Zeli.

Important Point: In general, one may not reduce dimensions when studying stability problems of NSDS by splitting a non degenerate part. Such approach is very common in the study of singularity of smooth mappings (Thom's splitting lemma) and smooth vector fields (restriction to a center manifold or Lyapunov Schimit reduction).

2- We deal with vector fields Z in R^2 given by

$$\dot{u} = F(u) + sgn\{y\}G(u)$$

where $u = (x, y) \in R^2$, F and G are smooth vector fields on R^2.

$\Sigma = \{y = 0\}$: discontinuity set or switching manifold.

In general one uses the terminology $Z = (X, Y)$ where $X \ and \ Y$ are smooth vector fields on R^2 (derived from $F \ and \ G$.

Some natural questions:

i) Stability and bifurcation problems:

a) Existence of typical limit cycles and separatrices connection.

b) Generic Classification of typical singularities and minimal sets (Local and Global Bifurcations).

ii) *Does it require a new mathematical theory? new tools? adapt known techniques?*

iii) Does the NSDS walk in the vacuum of the smooth theory?

Filippov Systems

$\Omega(n)$: space of all v.f. Z in $\mathbb{R}^2, 0$ s.t

$$Z(x,y) = \begin{cases} X(x,y), & \textit{for} \quad y > 0, \\ Y(x,y), & \textit{for} \quad y < 0. \end{cases} \tag{1}$$

with X, Y being C^∞ vector fields on $\mathbb{R}^2, 0$.

Denote $Z = (X, Y)$, $X = (X^1, X^2)$ and $Y = (Y^1, Y^2)$.

Filippov

Distinguishing open regions on Σ:

1. Σ_1: crossing region *(SWR)*; $X^2.Y^2 > 0$.
2. Σ_2: escaping region *(ESR)*; $X^2 > 0$ and $Y^2 < 0$.
3. Σ_3: sliding region *(SLR)*; $X^2 < 0$ and $Y^2 > 0$.

Filippov

Filippov convention:

In $\Sigma_1 \cup \Sigma_2$ is defined the sliding vector field F associated to Z .

$p \in \Sigma_1 \cup \Sigma_2$ is a Pseudo Equilibrium of Z if $F(p) = 0$.

Cycle at a Fold-Fold singularity

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Let $Z_0 = (X_0, Y_0) \in \Omega$ satisfying:

1- X_0 (respec. Y_0) presents an invisible $fold - singularity$ at p_{X_0} (respec. . p_{Y_0}).

2- there is a separatrix connection, γ_0, (cycle) joining p_{X_0} and p_{Y_0} (see Fig.1).

Denote $\chi = \{C^r - smooth\ vector\ fields\ \ X\ on\ R^2\}$.

Assume that $p_{X_0} = p_{Y_0} = (0, 0)$

Fig. 1

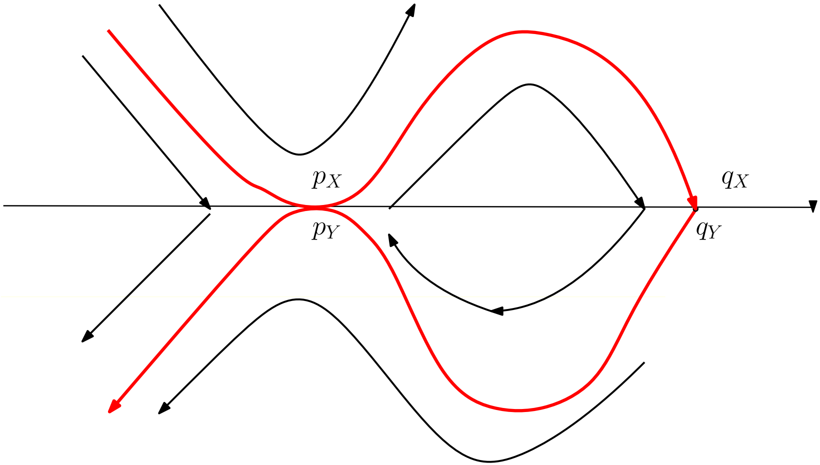


Figure: Degenerate cycle γ_0

Fig. 1

Our goal is to study the behaviour of perturbations of Z_0 in a small annulus around γ_0.

We recall that we also obtained some results involving "non-autonomous" perturbations of the differential system representing Z_0. This issue will not be treated in this talk.

Auxiliary mappings

Let the following C^r-correspondences:

- $f_1 : \chi, X_0 \to R, 0$: for each X corresponds a fold $f_1(X) = p_X \in \Sigma)$, close to 0.
- $f_2 : \chi, Y_0 \to R, 0$: for each Y corresponds a fold $g_1(Y) = p_Y \in \Sigma)$, close to 0.
- Call $f = f_2 - f_1$. Or equivalently the parameter

  $$\alpha(X, Y) = p_Y - p_X$$

  .

Let q_X (resp. q_Y the point where p_X (resp. p_y) meets Σ for forward (resp. backwards) time. Consider the parameter

$$\beta(X, Y) = q_X - q_Y.$$

Fig. 2

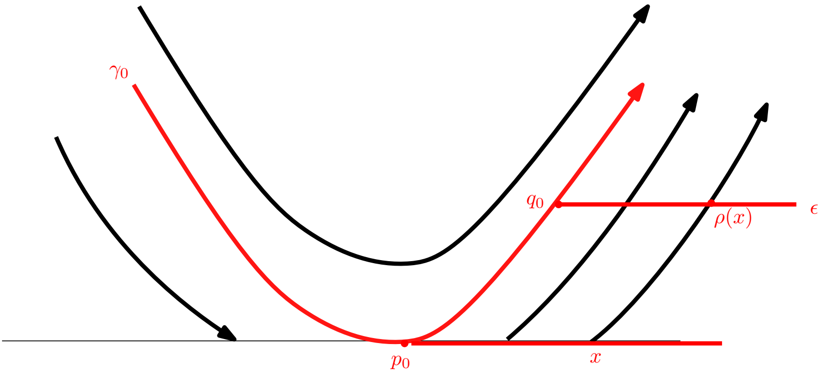


Figure: Fold-mapping

Fold Mapping

Let X_0 be a smooth planar vector field and $p_0 = (0,0) \in \delta$ be a visible fold singularity γ_0 of X_0.

$\delta_+ = [0, \mu)$
τ_0 - transversal section of X_0 at a point $q_0 \in \gamma_0$.

Define $\rho_0 : \delta_+ \to \tau_0$, where $\rho_0(x)$ is the point in τ_0 where the trajectory of X_0 passing through x reaches τ_0.

We may choose coordinates: $X_0(x, y) = (1, x)$ and $\tau_0 = \{y = \epsilon\}$.
$\rho_0(x) = \sqrt{x^2 - 2\epsilon}$ with $q_0 = (\sqrt{2\epsilon}, \epsilon)$.

It can be $C^r - \mathit{extended}$ to $\tilde{\rho}_0(x)$ in a neighbourhood of p_0 in δ.

So $\tilde{\rho}_0(x) = k_1(\epsilon) + k_2(\epsilon)x^2 + h.o.t$ where $\tau_0 = \{y = \epsilon\}$, and each k_i is smooth.

Fig. 3

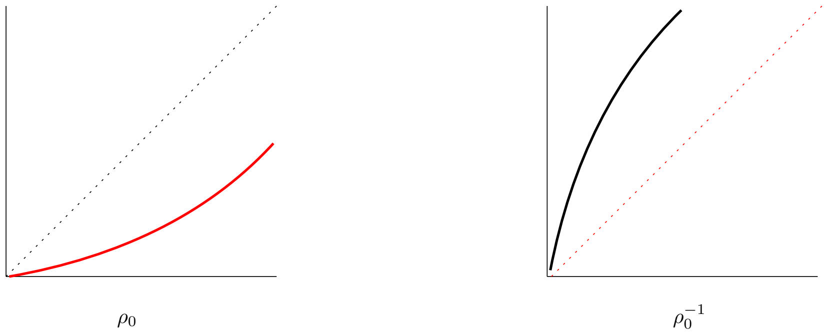
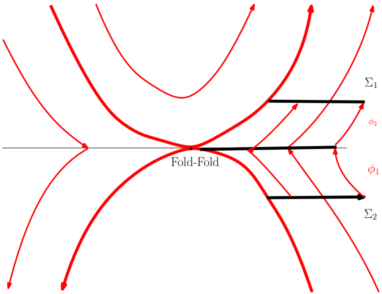


Figure: Fold-mapping

Fold-Fold Cycle

1th Return Mapping



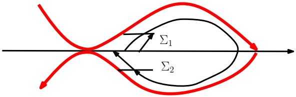


Figure: 1th return mapping

Fig. 4i

Roughly speaking, the orbits of X and Y through 0 (and near 0) can be represented by $y = a_0x^2.....$ and $y = b_0x^2.....$ respectively. The diffeomorphism $h : \Sigma_1 \to \Sigma_2$ is $h(x) = k_0x + ...$ with $x \geq 0$.

The first return mapping has the shape $\mu(x) = m_0x \ + \ hot$ with m_0 depending on a_0, b_0 *and* k_0 with $x \geq 0$. It is assumed that $m_0 \neq 1$. The perturbations of such mappings must be carefully analysed.

We now proceed with some codimension-one bifurcations close to γ_0.

Closed orbit and codimension-One loop

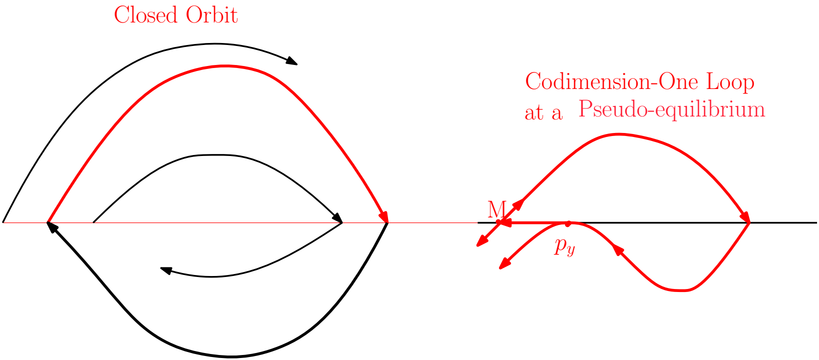


Figure: Loop at a Pseudo-equilibrium

Codimension-One loop

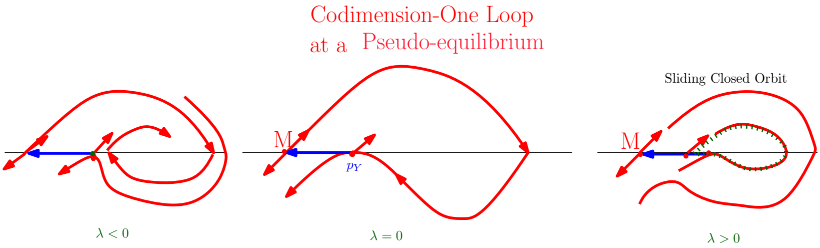


Figure: Loop at a Pseudo-equilibrium

Fig. 5

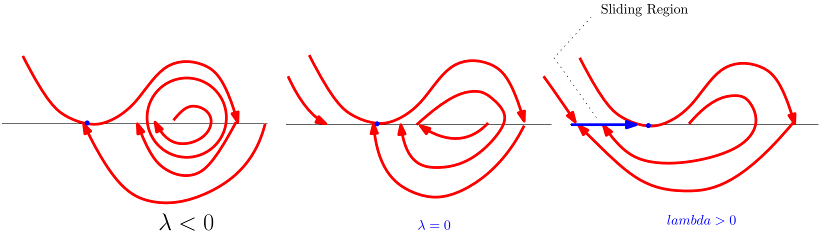


Figure: Codimension one

Fig. 6

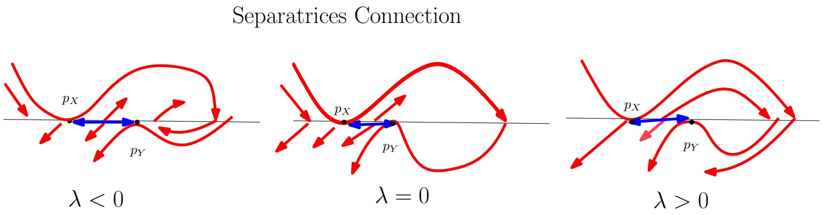


Figure: Codimension one

Fig. 7a

The complete bifurcation diagram is now exhibited.

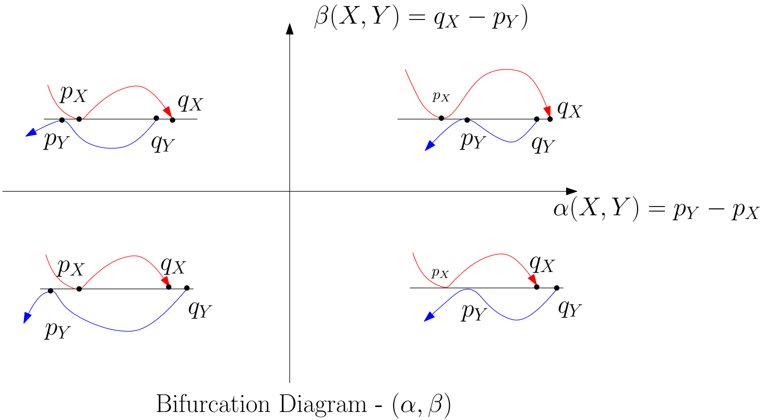


Figure: Bifurcation Diagram of Z_0

Fig. 7b