1ST JOINT MEETING BRAZIL-SPAIN IN MATHEMATICS

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Revisiting the focus-fold singularity in planar Filippov systems

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Summary

- We consider the family of planar discontinuous piecewise-linear systems with two linearity zones separated by a straight line.
- Three years ago, an example in this family by Huan and Yang was reported to have three nested limit cycles, so breaking a natural conjecture on being two the maximum number of limit cycles.
- We do know that it suffices the presence of one focus in one zone and an invisible tangency in the other to give a general mechanism justifying the existence of three limit cycles.

Summary

- Here, we consider the boundary focus + saddle case. We show how one can get the three limit cycles through simultaneous local and global bifurcations.

- We exploit a reduced Liénard-like canonical form with only five parameters plus two modal ones: the modal parameters define the kind of dynamics in each zone, two more define the divergence and the equilibrium position and the last one characterizes the sliding set.

Planar PWL Filippov Systems
(Utkin 1992, Kuznetsov et al. 2003)

- We consider one discontinuity boundary defined by

$$\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$$

- The boundary induces the partition of the phase plane into

$$S^- = \{(x, y) \in \mathbb{R}^2 : x < 0\},$$

$$S^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}.$$

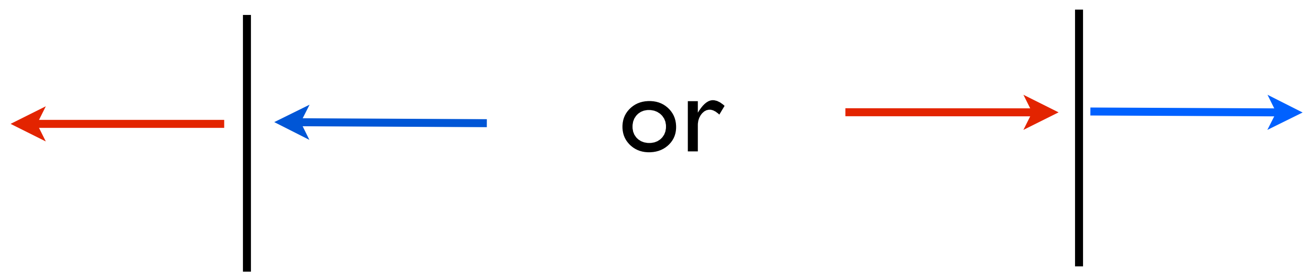
The systems to be studied become

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \begin{cases} \mathbf{F}^+(\mathbf{x}) = \left(F_1^+(\mathbf{x}), F_2^+(\mathbf{x})\right)^T = A^+\mathbf{x} + \mathbf{b}^+, & \text{if } \mathbf{x} \in S^+, \\ \mathbf{F}^-(\mathbf{x}) = \left(F_1^-(\mathbf{x}), F_2^-(\mathbf{x})\right)^T = A^-\mathbf{x} + \mathbf{b}^-, & \text{if } \mathbf{x} \in S^-. \end{cases}$$

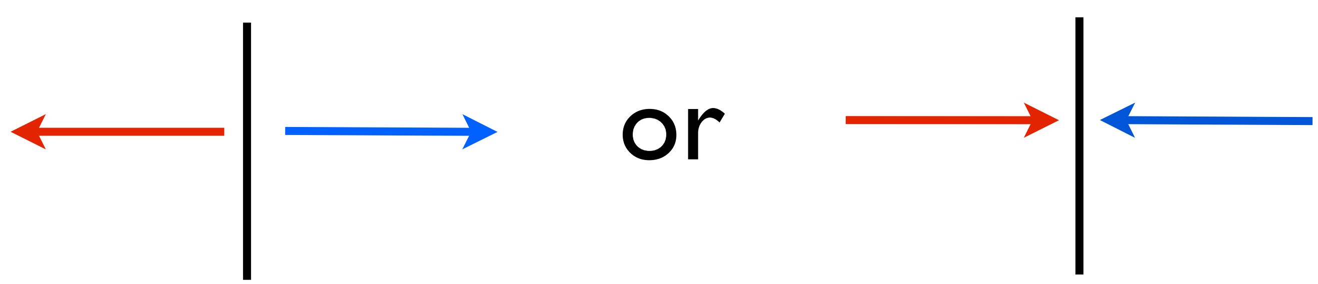
Planar PWL Filippov Systems (cont'd)

- As usual we define the Σ-subsets

$$\Sigma^c = \{(0,y) : F_1^+(0,y)F_1^-(0,y) > 0\} \quad \text{(crossing set)}$$



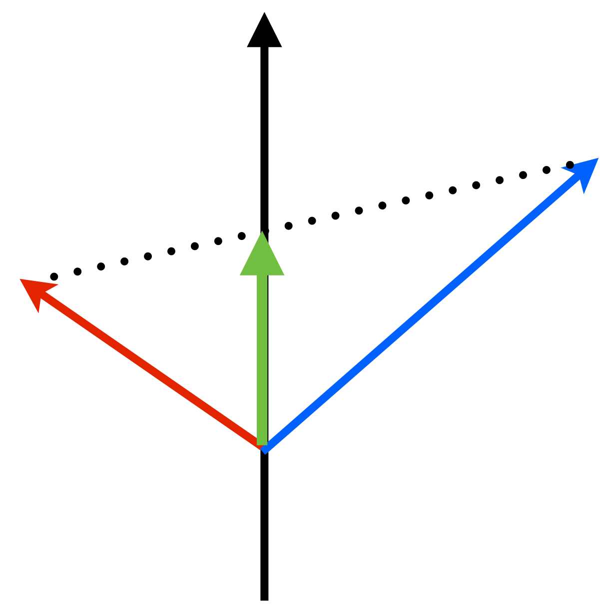
$$\Sigma^s = \{(0,y) : F_1^+(0,y)F_1^-(0,y) \leqslant 0\} \quad \text{(sliding set)}$$



Planar PWL Filippov Systems (cont'd)

- We can also define the Filippov vector field

$$\dot{x} = 0, \quad \dot{y} = g(y) = \frac{F_1^+(\mathbf{x})F_2^-(\mathbf{x}) - F_1^-(\mathbf{x})F_2^+(\mathbf{x})}{F_1^+(\mathbf{x}) - F_1^-(\mathbf{x})}, \quad \mathbf{x} \in \Sigma^s.$$



- Some other standard definitions follow:

Points $(0, \bar{y}) \in \Sigma^s$ with $g(\bar{y}) = 0$ act in some sense as equilibria of our system and they are called *pseudo-equilibria*.

A double invisible tangency point with close orbits spiraling around it, is called a *pseudo-focus* or *fused focus*.

A pseudo-equilibrium in the attractive part of the sliding set with $g'(y) < 0$ is a stable *pseudo-node*, being a *pseudo-saddle* if $g'(y) > 0$.

Similarly, a pseudo-equilibrium in the repulsive part with $g'(y) > 0)$ is an unstable *pseudo-node*, being again a *pseudo-saddle* if $g'(y) < 0$.

Note that at pseudo-equilibria $(0, \bar{y})$ which are neither boundary equilibrium nor tangency points we have

$$\frac{F_2^-(0, \bar{y})}{F_1^-(0, \bar{y})} = \frac{F_2^+(0, \bar{y})}{F_1^+(0, \bar{y})},$$

and so the two vector fields $\mathbf{F}^+$ and $\mathbf{F}^-$ are anticollinear.

The Huan-Yang example

The planar non-smooth piecewise linear differential system with two zones separated by a straight line corresponding to Example 5.1 of Huan and Yang is

$$\dot{\mathbf{x}} = \begin{cases} A^-\mathbf{x} & \text{if } x < 1, \\ A^+\mathbf{x} & \text{if } x \geq 1, \end{cases}$$

where $\mathbf{x} = (x, y)^T$ with

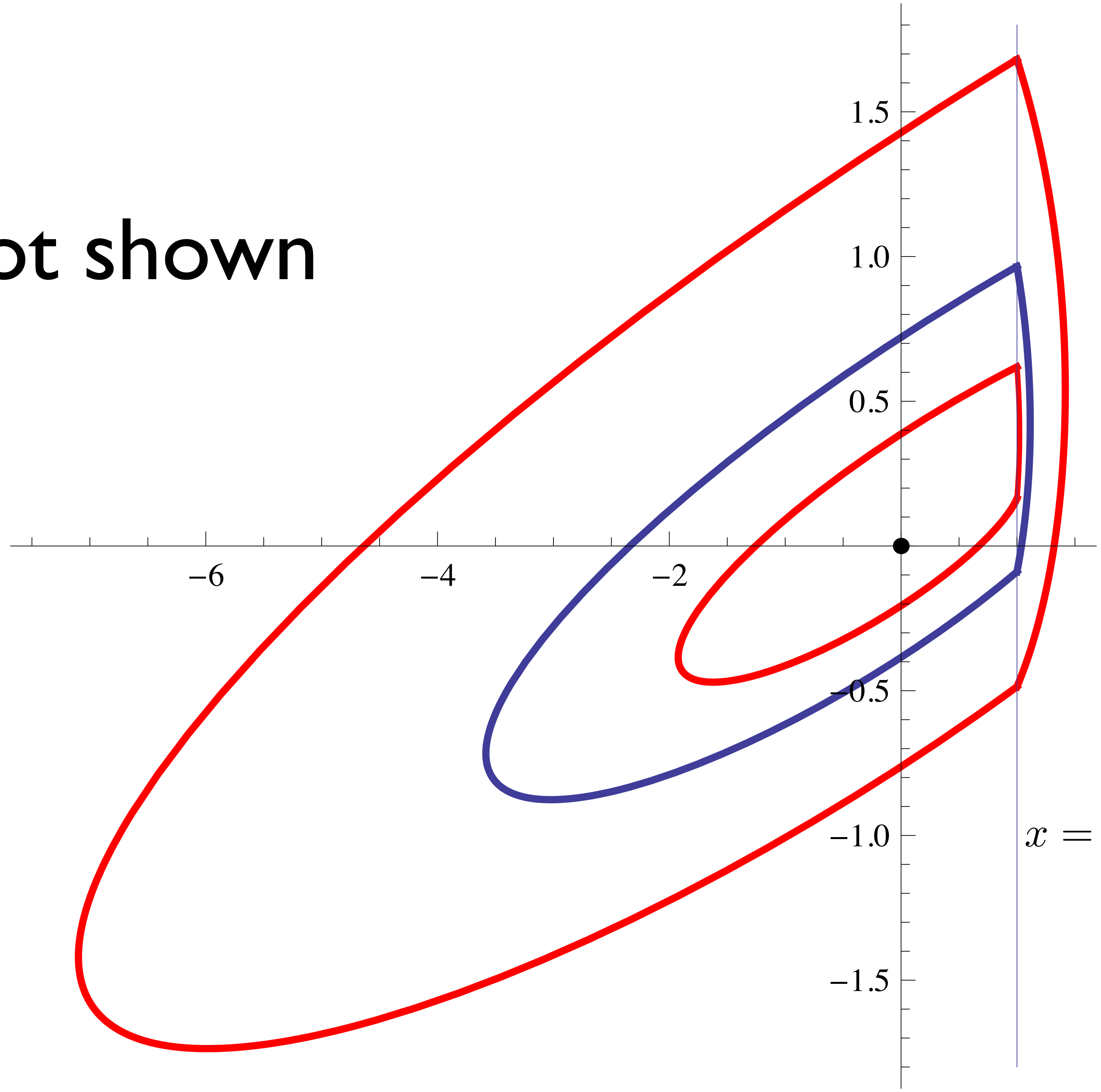
$$A^- = \begin{pmatrix} 1 & -5 \\ \dfrac{377}{1000} & -\dfrac{13}{10} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \dfrac{19}{500} & -\dfrac{1}{10} \\ \dfrac{1}{10} & \dfrac{19}{500} \end{pmatrix}.$$

Theorem (J. Llibre & E.P.) The above planar non–smooth piecewise linear differential system with two zones has 3 limit cycles surrounding its unique equilibrium point located at the origin.

S.-M. Huan and X.-S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Discrete and Continuous Dynamical Systems-A **32** (2012) pp. 2147–2164.

J. Llibre and E. P., *Three Nested Limit Cycles In Discontinuous Piecewise Linear Differential Systems With Two Zones*, Dynamics of Continuous, Discrete and Impulsive Systems-B **19** (2012) pp. 325–335.

Sliding set not shown



$$\boxed{\lambda = -\frac{1}{5} \pm i}$$

$$\boxed{\lambda = \frac{19}{50} \pm i}$$

$$A^- = \begin{pmatrix} \frac{4}{3} & -\frac{20}{3} \\ \frac{377}{750} & -\frac{26}{15} \end{pmatrix}, \quad \text{and} \quad A^+ = \begin{pmatrix} \frac{19}{50} & -1 \\ 1 & \frac{19}{50} \end{pmatrix}.$$

(after rescaling time, differently in each side)

Tangencies and sliding set

$$\Sigma = \{(x,y) \in \mathbb{R}^2 : x = 0\}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- x + a_{12}^- y + b_1^- \\ a_{21}^- x + a_{22}^- y + b_2^- \end{pmatrix} \quad \Bigg| \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ x + a_{12}^+ y + b_1^+ \\ a_{21}^+ x + a_{22}^+ y + b_2^+ \end{pmatrix}$$

We will assume $a_{12}^-, a_{12}^+ \neq 0$ to avoid 'wall' cases.

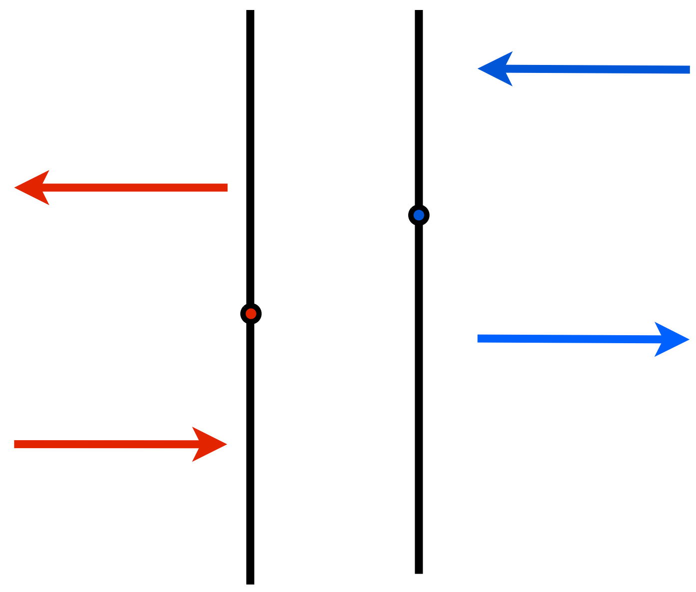
We have a tangency point in Σ when $\dot{x}|_{x=0} = a_{12}y + b_1$ vanishes.

At tangency points, we speak of visible (invisible) tangency depending on the sign of $\ddot{x}$. Since $\ddot{x}|_{x=0} = a_{11}(a_{12}y + b_1) + a_{12}(a_{21}y + b_2)$, we obtain

$$\ddot{x}|_{\dot{x}=0} = a_{12}b_2 - a_{21}b_1$$

Tangencies and sliding set (cont'd)

Assuming $a_{12}^{-} < 0$, there are two possibilities for a_{12}^{+}:



($a_{12}^{+} < 0$: bounded sliding)

Tangencies and sliding set (cont'd)

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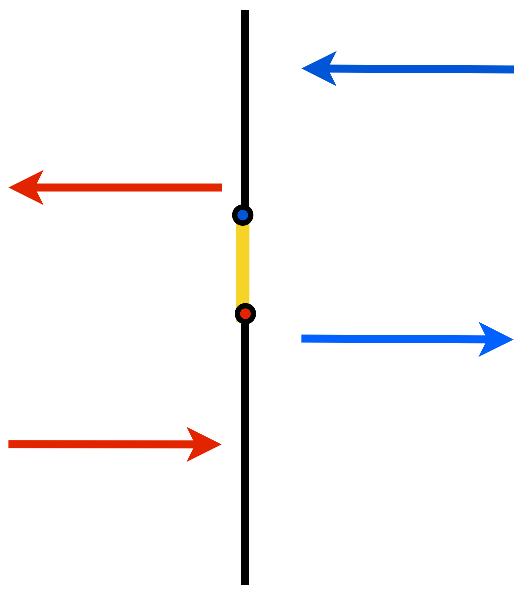

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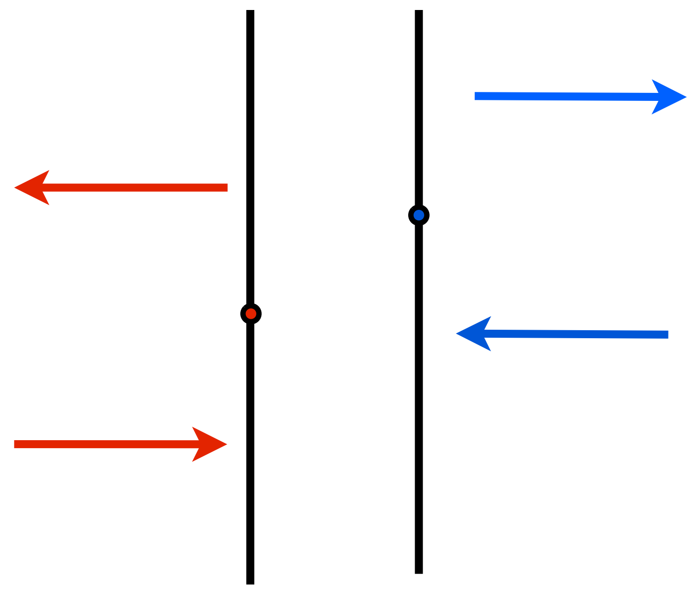

($a_{12}^{+} > 0$: bounded crossing)

Tangencies and sliding set (cont'd)

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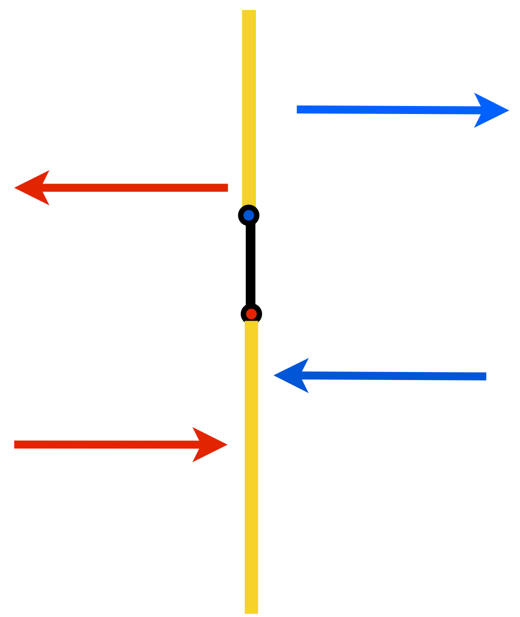


($a_{12}^{+} > 0$: bounded crossing)

For a non-smooth system... a non-smooth change!

We do a continuous piecewise linear change of variables $\mathbf{u} = f(\mathbf{x})$, where

$$\mathbf{u} = -a_{12}^{+} \begin{pmatrix} x \\ a_{22}^{-}x - a_{12}^{-}y \end{pmatrix} + a_{12}^{+} \begin{pmatrix} 0 \\ b_1^{-} \end{pmatrix}, \quad x < 0,$$

and

$$\mathbf{u} = -a_{12}^{-} \begin{pmatrix} x \\ a_{22}^{+}x - a_{12}^{+}y \end{pmatrix} + a_{12}^{+} \begin{pmatrix} 0 \\ b_1^{-} \end{pmatrix}, \quad x > 0,$$

and afterwards rename the variable $\mathbf{u}$ to $\mathbf{x}$.

This change is a global homeomorfism that conjugates the vector field in each halfplane, separately. Such a conjugacy cannot be extended to the sliding vector field (but it works for our purpose)

M. Guardia, T.M. Seara, and M. A. Teixeira, *Generic bifurcations of low codimension of planar Filippov Systems*, Journal of Differential Equations **250** (2011) 1967–2023.

The discontinuous canonical form

Liénard canonical form for DPWL systems. Assume that $a_{12}^{+}a_{12}^{-} > 0$ (bounded sliding set). Then the system can be written in the form,

$$\dot{\mathbf{x}} = \begin{pmatrix} T^{-} & -1 \\ D^{-} & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a^{-} \end{pmatrix} \text{ if } \mathbf{x} \in S^{-},$$

$$\dot{\mathbf{x}} = \begin{pmatrix} T^{+} & -1 \\ D^{+} & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a^{+} \end{pmatrix} \text{ if } \mathbf{x} \in S^{+},$$

where T,D stand for trace and determinant, and

$$a^{-} = a_{12}^{+}(a_{12}^{-}b_{2}^{-} - a_{22}^{-}b_{1}^{-}), \quad a^{+} = a_{12}^{-}(a_{22}^{+}b_{1}^{+} - a_{12}^{+}b_{2}^{+}), \quad b = a_{12}^{+}b_{1}^{-} - a_{12}^{-}b_{1}^{+}.$$

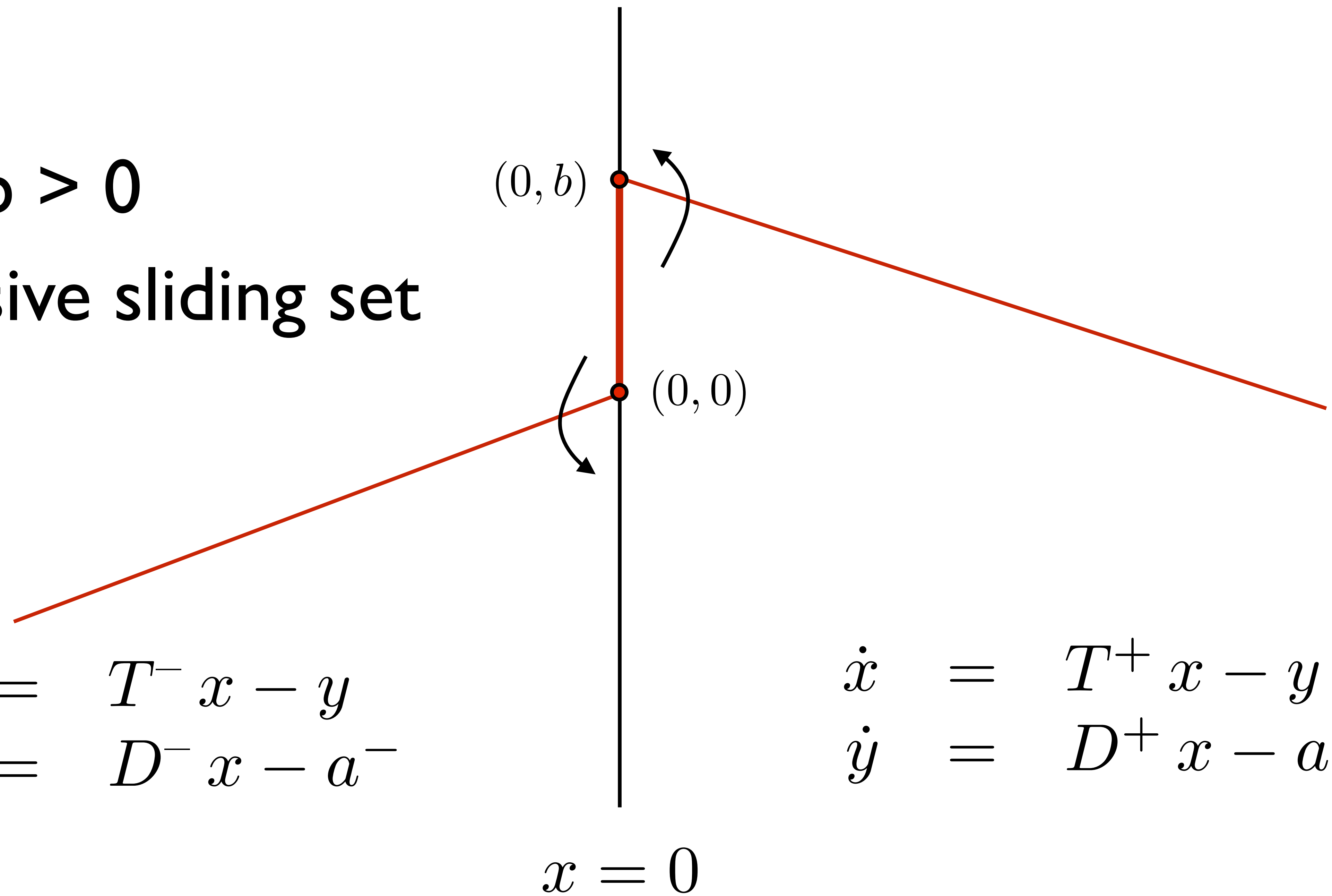
This system has as its tangency points (0,0) and (0,b).

Apart from the linear invariants, the other three parameters are associated to the x-coordinates of the equilibrium points (a^{+} and a^{-}) and the size and stability of the sliding set (b).

The discontinuous canonical form

b > 0

Repulsive sliding set



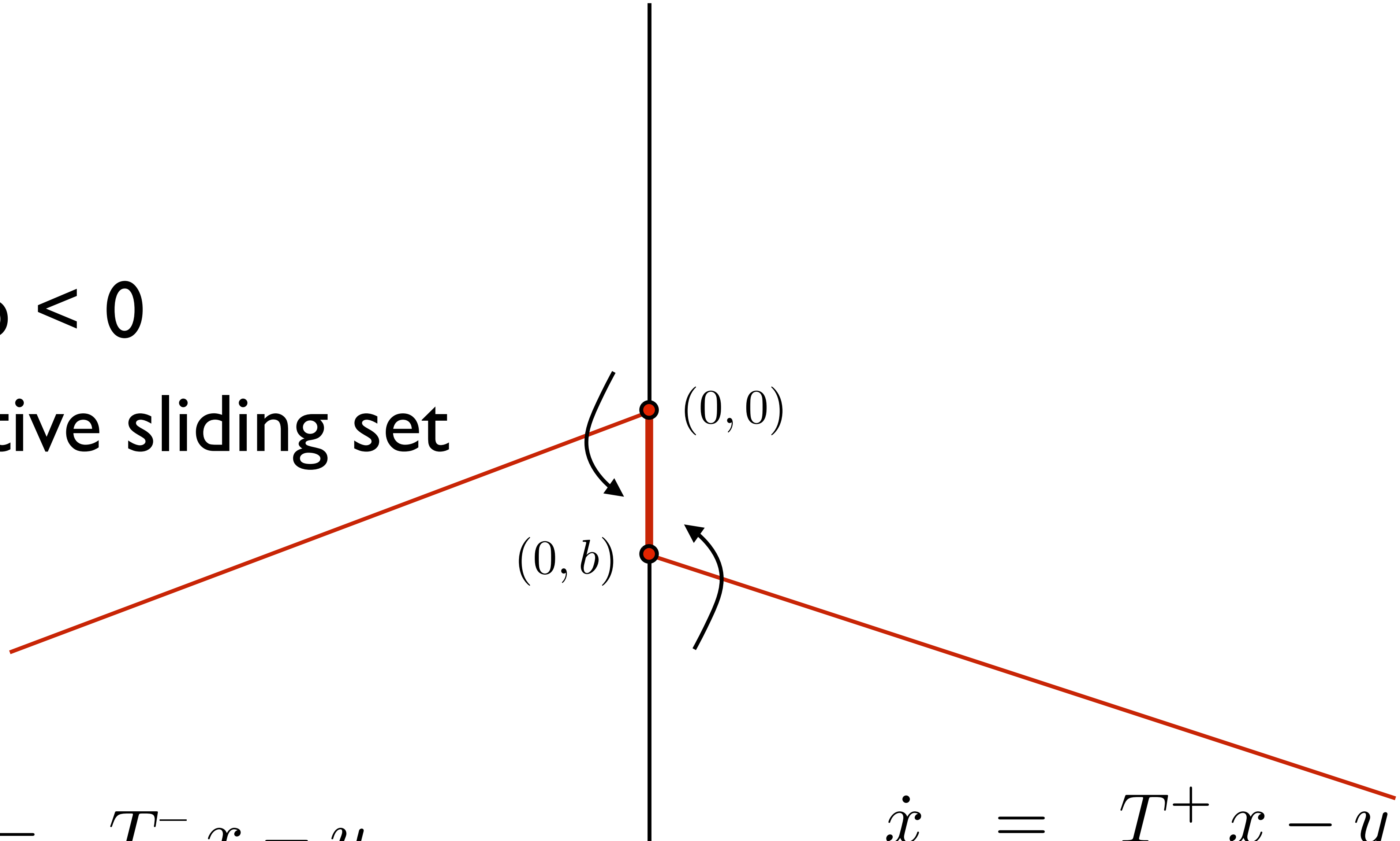
$$\begin{aligned} \dot{x} &= T^{-} x - y \\ \dot{y} &= D^{-} x - a^{-} \end{aligned}$$

$$\begin{aligned} \dot{x} &= T^{+} x - y + b \\ \dot{y} &= D^{+} x - a^{+} \end{aligned}$$

The discontinuous canonical form

b < 0

Attractive sliding set



$$\begin{aligned}\dot{x} &= T^{-}x - y\\ \dot{y} &= D^{-}x - a^{-}\end{aligned}$$

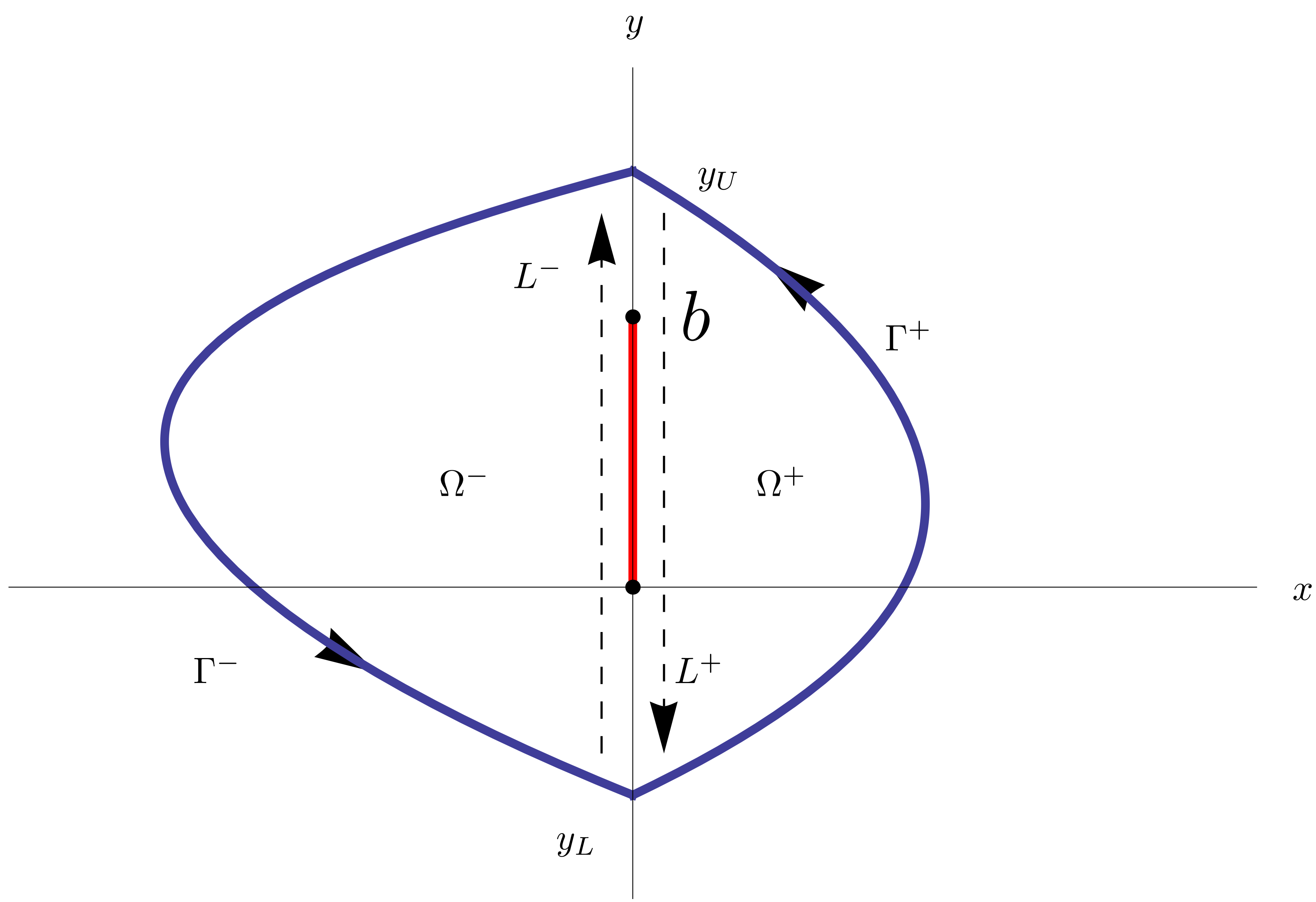
$$x = 0$$

$$\begin{aligned}\dot{x} &= T^{+}x - y + b\\ \dot{y} &= D^{+}x - a^{+}\end{aligned}$$

A necessary condition for crossing periodic orbits

Proposition Defining the values $\sigma^- = \text{area } (\Omega^-)$, $\sigma^+ = \text{area } (\Omega^+)$ and $h = y_U - y_L$, then we have

$$T^- \sigma^- + T^+ \sigma^+ + bh = 0.$$



The universal canonical form with modal parameters

Proposition. Our canonical form can be rewritten as

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 - m_L^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} \text{ if } \mathbf{x} \in S^-,$$

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - m_R^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix} \text{ if } \mathbf{x} \in S^+,$$

where the modal parameters $m_{\{R,L\}} \in \{i, 0, 1\}$ are defined for each zone by

$$m = \begin{cases} i & \text{if} \quad T^2 - 4D < 0 \ (\textbf{focus}), \\ 0 & \text{if} \quad T^2 - 4D = 0 \ (\textbf{improper node}), \\ 1 & \text{if} \quad T^2 - 4D > 0 \ (\textbf{node/saddle}), \end{cases}$$

the symbol i standing for the imaginary unit of the complex plane ($i^2 = -1$).

Accordingly, the new constant terms $a_{\{R,L\}}$ and normalized semi-traces $\gamma_{\{R,L\}}$ are

$$a_{\{R,L\}} = \frac{a^\pm}{\omega_{\{R,L\}}}, \quad \gamma_{\{R,L\}} = \frac{T^\pm}{2\omega_{\{R,L\}}}, \quad \text{where } \omega_{\{R,L\}} = \begin{cases} \sqrt{\left| \dfrac{(T^\pm)^2}{4} - D^\pm \right|} & \text{if} \quad m_{\{R,L\}} \neq 0, \\ 1 & \text{if} \quad m_{\{R,L\}} = 0. \end{cases}$$

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$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 - m_L^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix} \text{ if } \mathbf{x} \in S^-,$$

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - m_R^2 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix} \text{ if } \mathbf{x} \in S^+,$$

where the modal parameters $m_{\{R,L\}} \in \{i, 0, 1\}$ are defined for each zone by

$$(\lambda = \gamma \pm m) \qquad m = \begin{cases} i & \text{if} \quad T^2 - 4D < 0 \ (\textbf{focus}), \\ 0 & \text{if} \quad T^2 - 4D = 0 \ (\textbf{improper node}), \\ 1 & \text{if} \quad T^2 - 4D > 0 \ (\textbf{node/saddle}), \end{cases}$$

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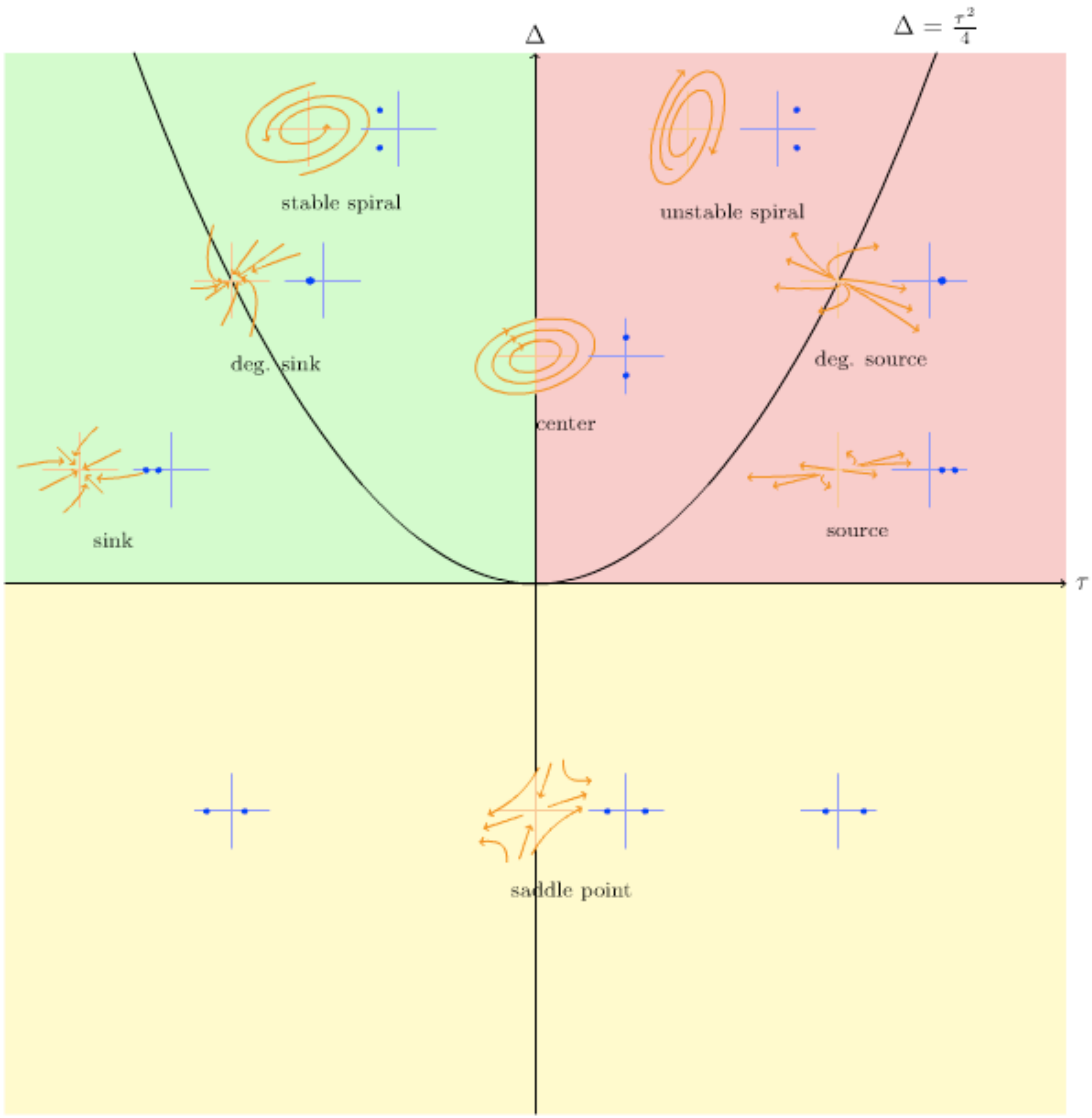
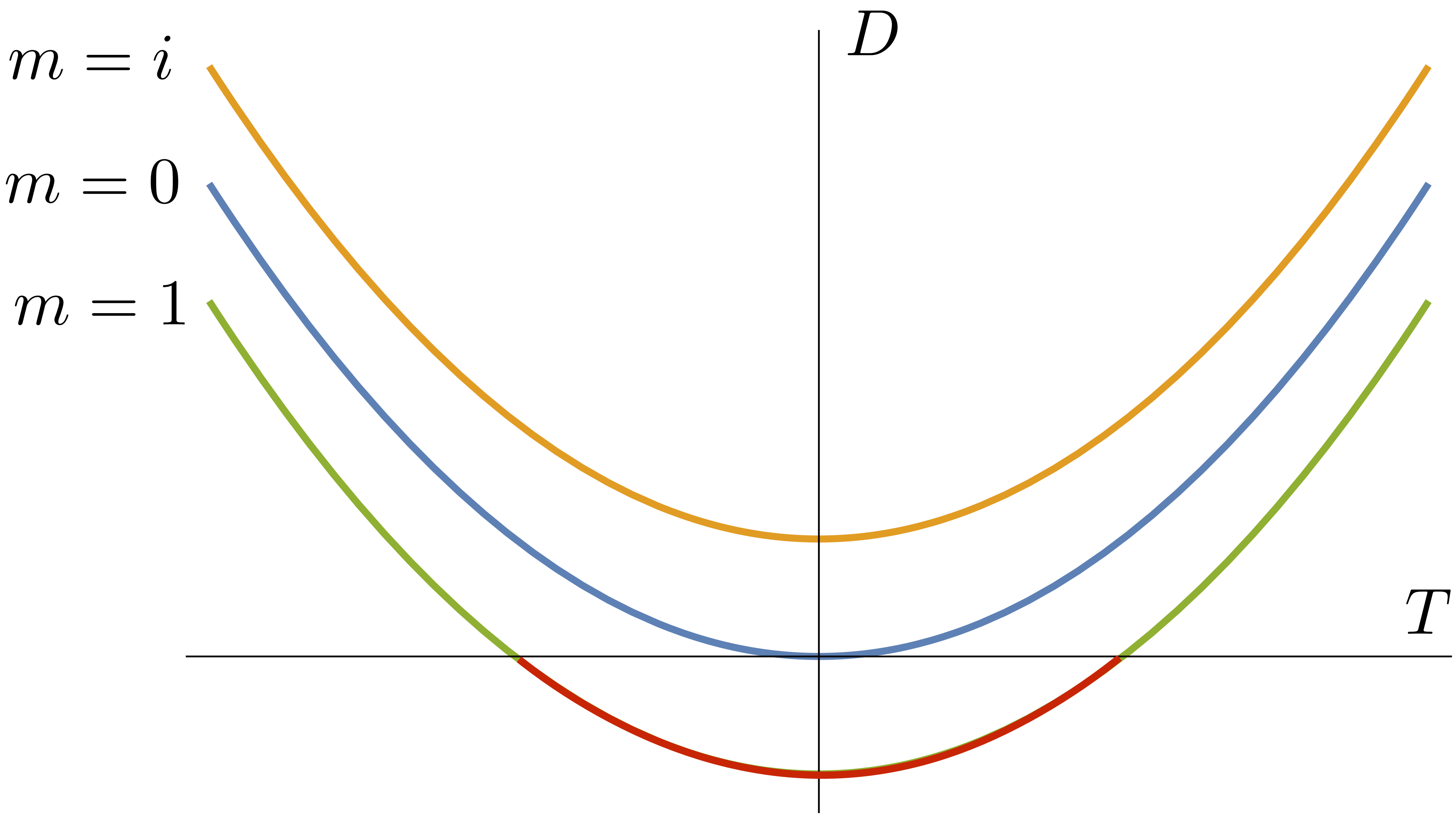
Accordingly, the new constant terms $a_{\{R,L\}}$ and normalized semi-traces $\gamma_{\{R,L\}}$ are

$$a_{\{R,L\}} = \frac{a^\pm}{\omega_{\{R,L\}}}, \quad \gamma_{\{R,L\}} = \frac{T^\pm}{2\omega_{\{R,L\}}}, \quad \text{where } \omega_{\{R,L\}} = \begin{cases} \sqrt{\left| \dfrac{(T^\pm)^2}{4} - D^\pm \right|} & \text{if} \quad m_{\{R,L\}} \neq 0, \\ 1 & \text{if} \quad m_{\{R,L\}} = 0. \end{cases}$$

The trace-determinant plane



The universal canonical form in the trace-determinant plane



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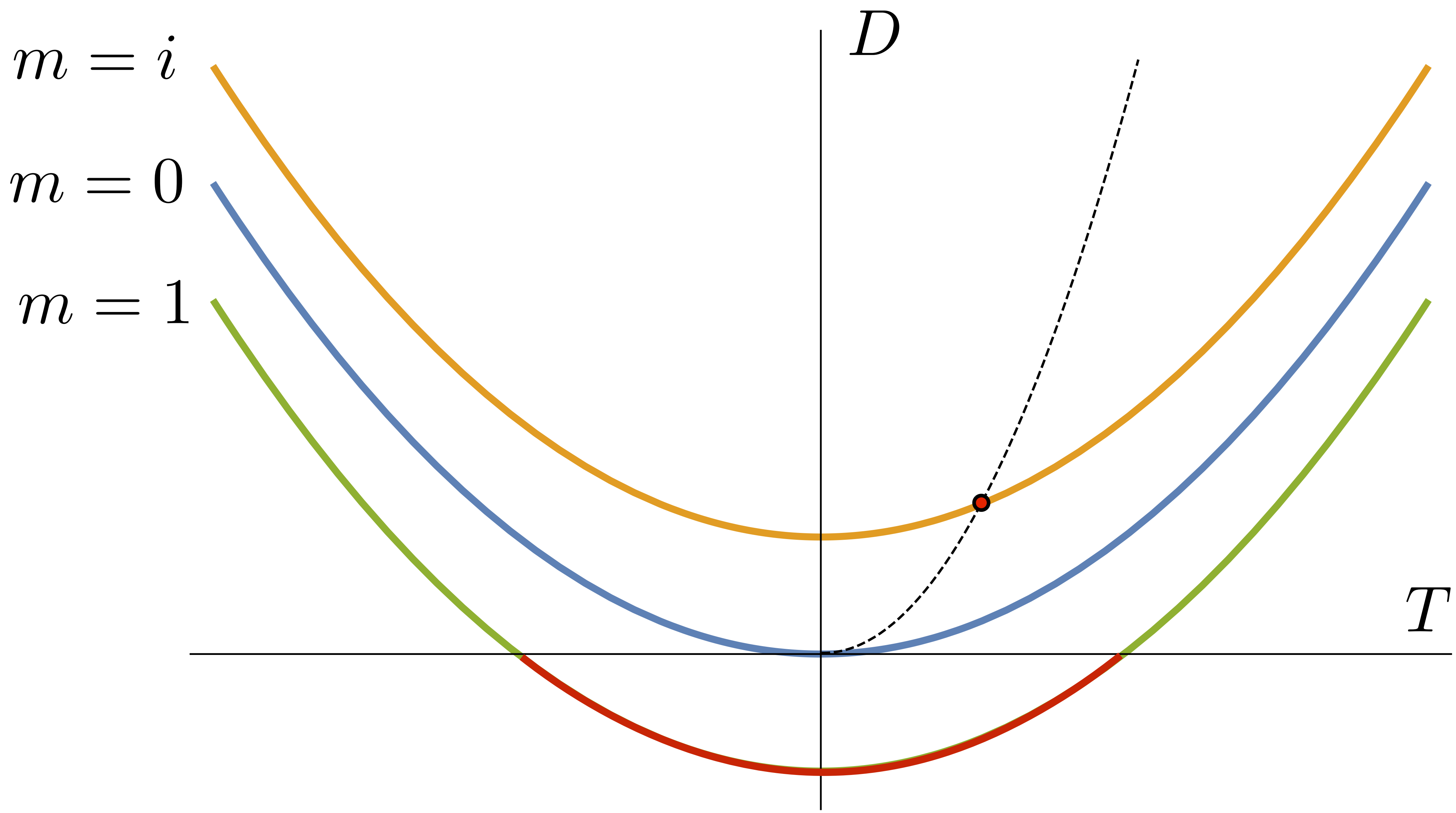
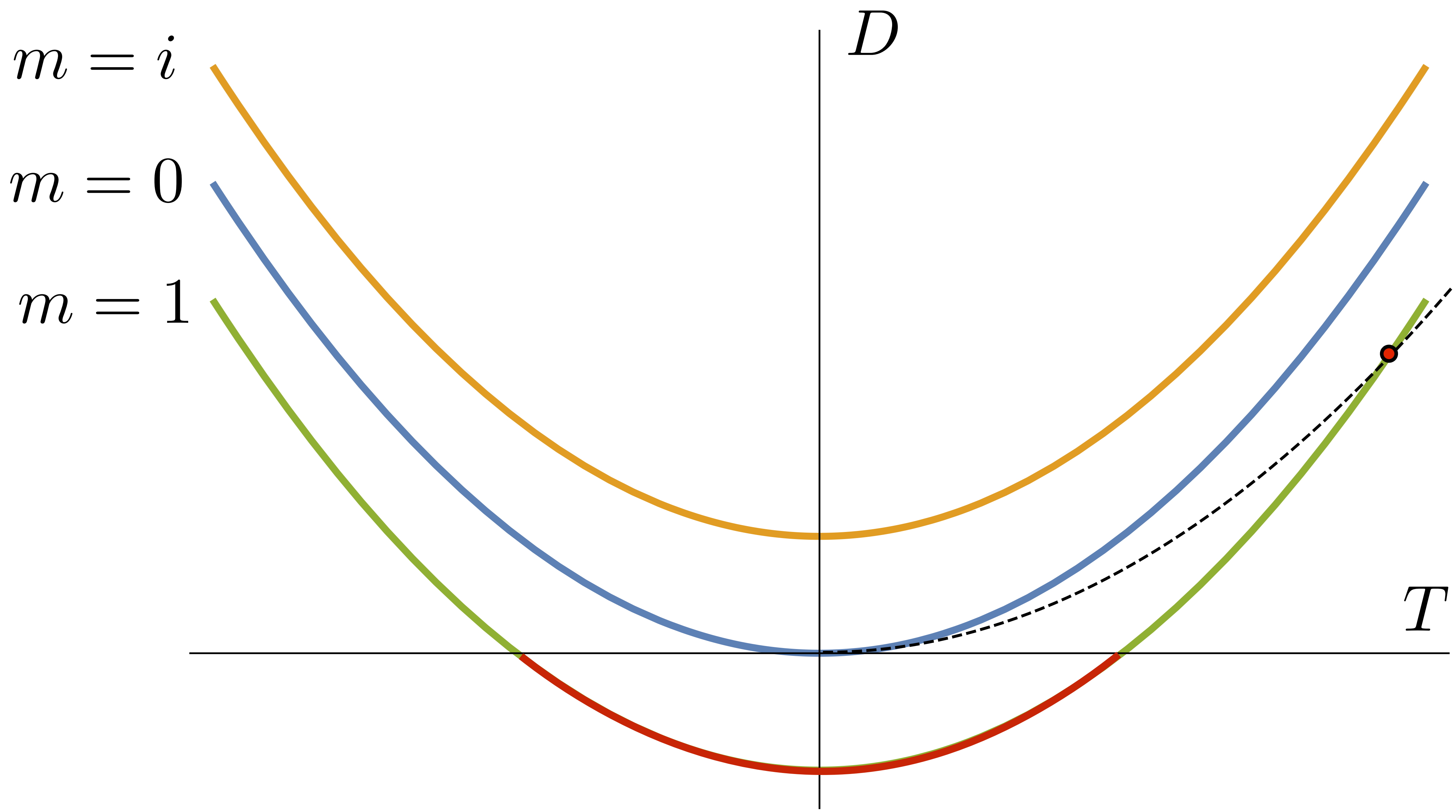


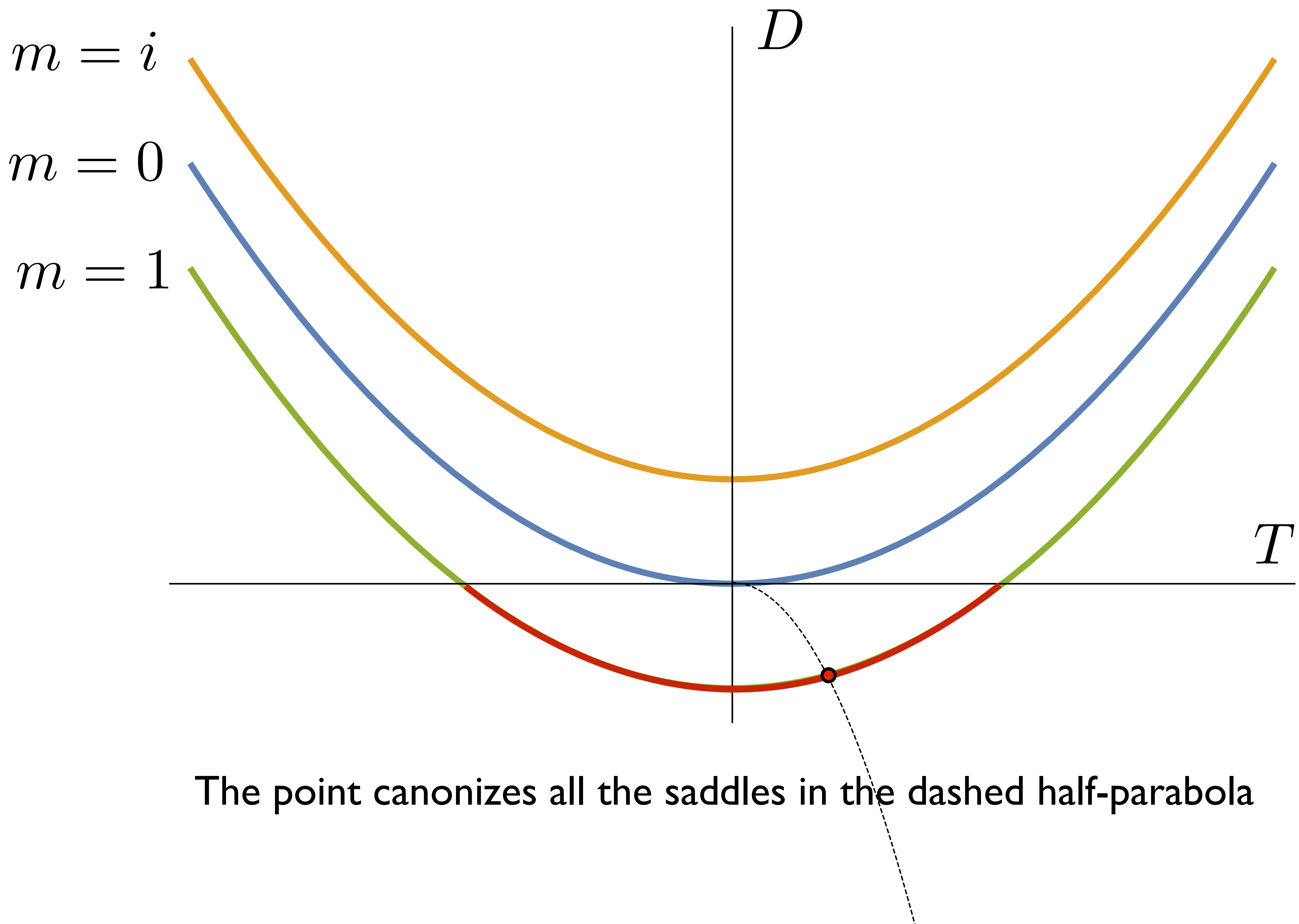
The point canonizes all the foci in the dashed half-parabola

The universal canonical form in the trace-determinant plane



The point canonizes all the nodes in the dashed half-parabola

The universal canonical form in the trace-determinant plane



The point canonizes all the saddles in the dashed half-parabola

The canonical form in the focus-saddle case

$$\dot{\mathbf{x}} = \begin{cases} A^-\mathbf{x} + \mathbf{b}^-, & \text{if } \mathbf{x} \in S^-, \\ A^+\mathbf{x} + \mathbf{b}^+, & \text{if } \mathbf{x} \in S^+, \end{cases} \qquad (1)$$

Proposition If in system (1) $a_{12}^+ < 0$, $a_{12}^- < 0$, $4\det A^- - \mathrm{tr}(A^-)^2 > 0$, and $\det A^+ < 0$, then after some continuous change of variables we arrive at the normalized canonical form

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_L & -1 \\ \gamma_L^2 + 1 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ a_L \end{pmatrix}, \quad \text{if } \mathbf{x} \in S^-,$$

$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - 1 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} -b \\ a_R \end{pmatrix}, \quad \text{if } \mathbf{x} \in S^+,$$

where $|\gamma_R| < 1$.

The canonical form in the focus-saddle case

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$$\dot{\mathbf{x}} = \begin{pmatrix} 2\gamma_R & -1 \\ \gamma_R^2 - 1 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \boxed{-b} \\ a_R \end{pmatrix}, \quad \text{if } \mathbf{x} \in S^+,$$

where $|\gamma_R| < 1$.

The canonical form in the focus-saddle case

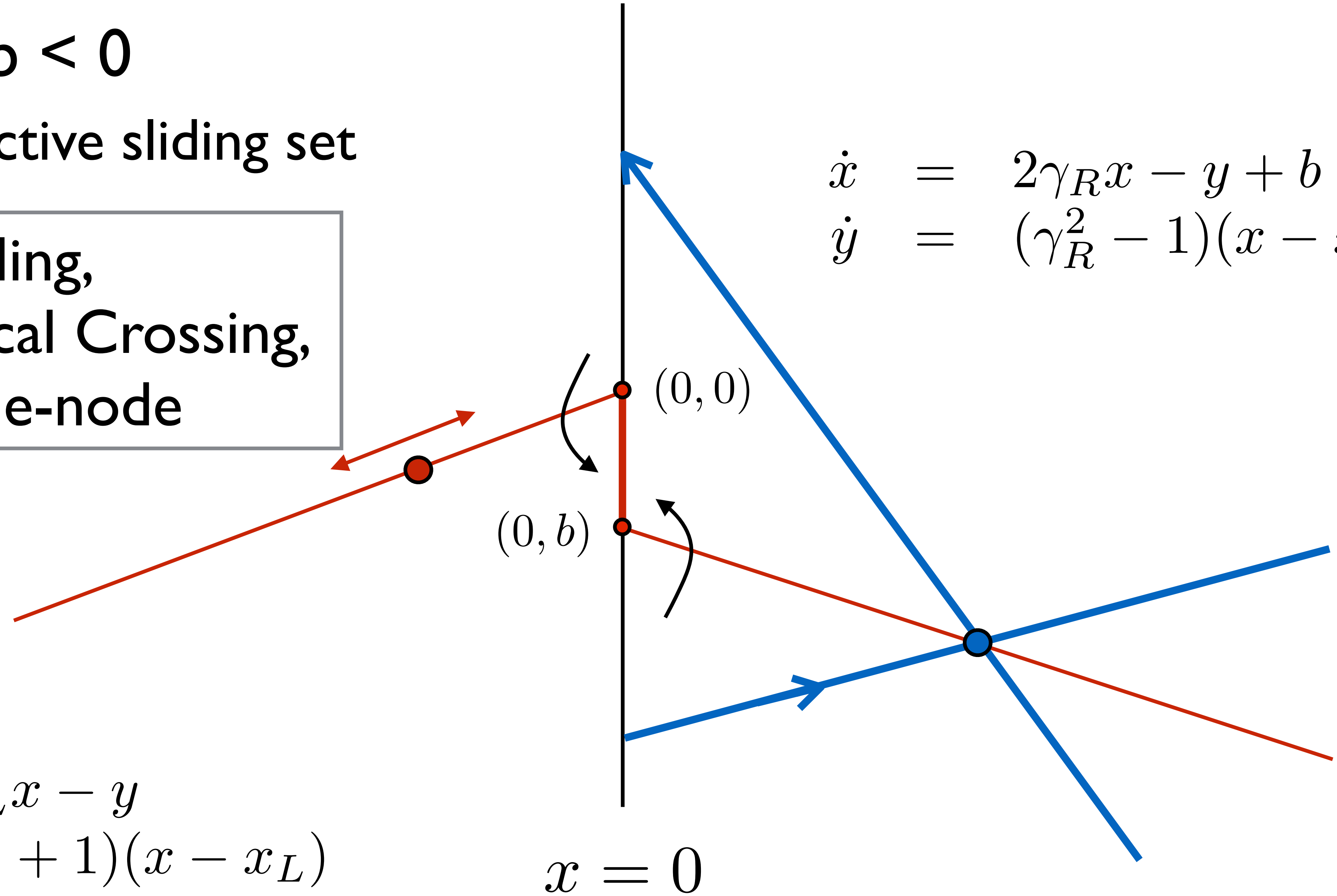
$$\boxed{-1 < \gamma_R < 0, \quad \gamma_L > 0}$$

b < 0

Attractive sliding set

- -Buckling,
- -Critical Crossing,
- -Saddle-node

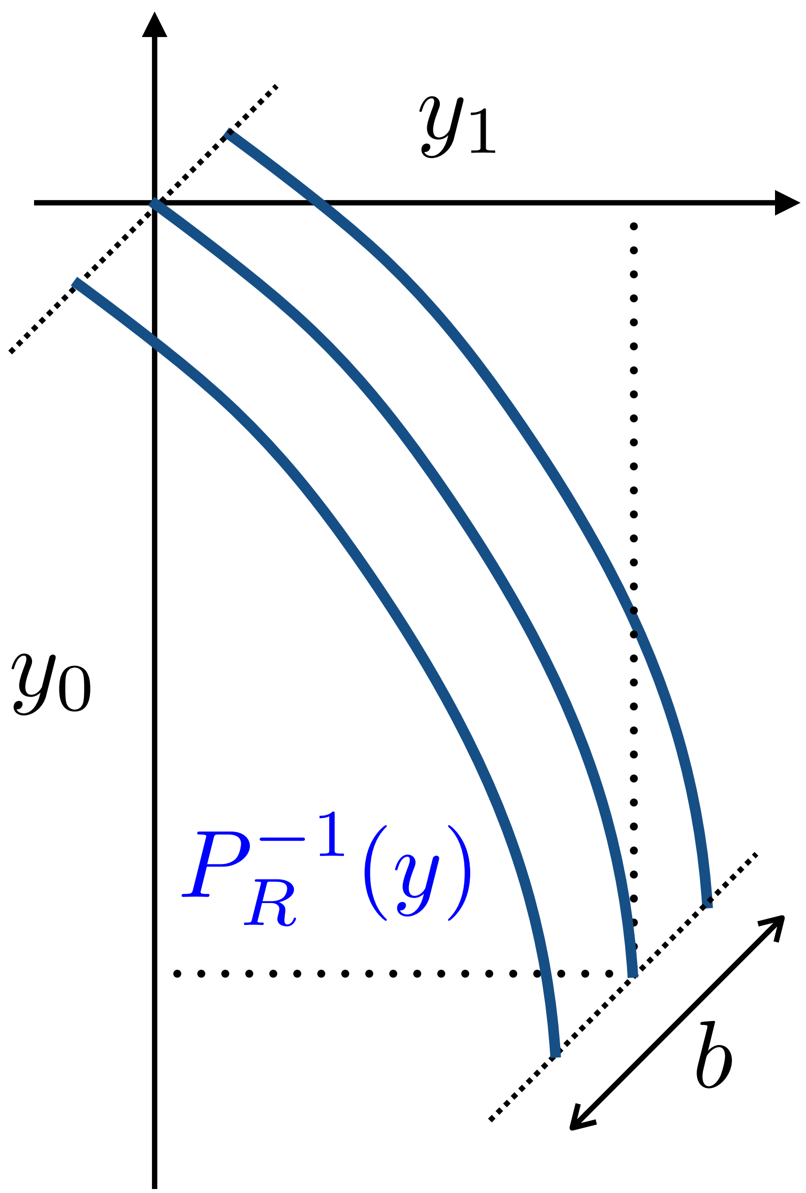
$$\begin{aligned}\dot{x} &= 2\gamma_R x - y + b\\ \dot{y} &= (\gamma_R^2 - 1)(x - x_R)\end{aligned}$$

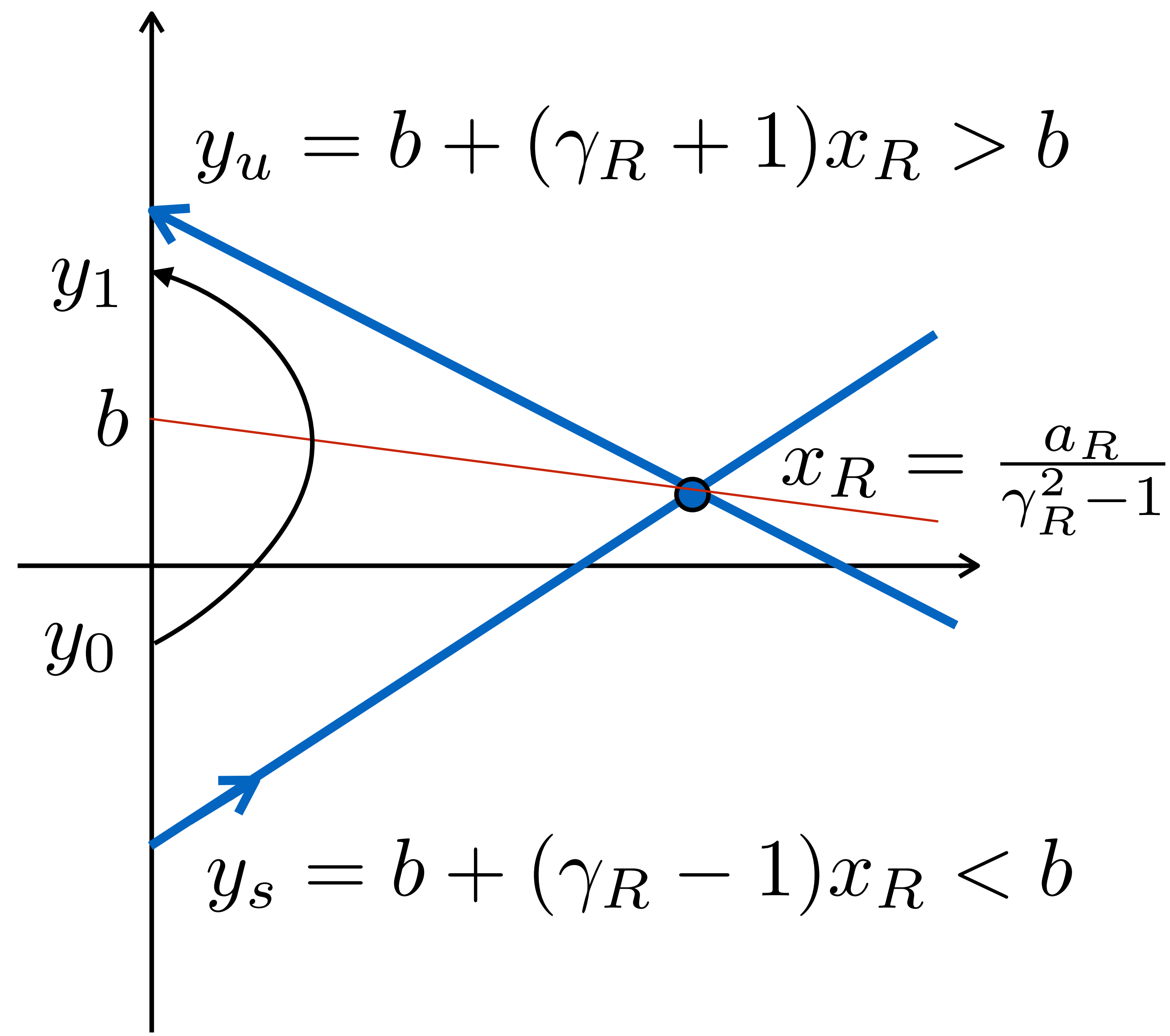


$$\begin{aligned}\dot{x} &= 2\gamma_L x - y\\ \dot{y} &= (\gamma_L^2 + 1)(x - x_L)\end{aligned}$$

Computing the half-return maps: the right one

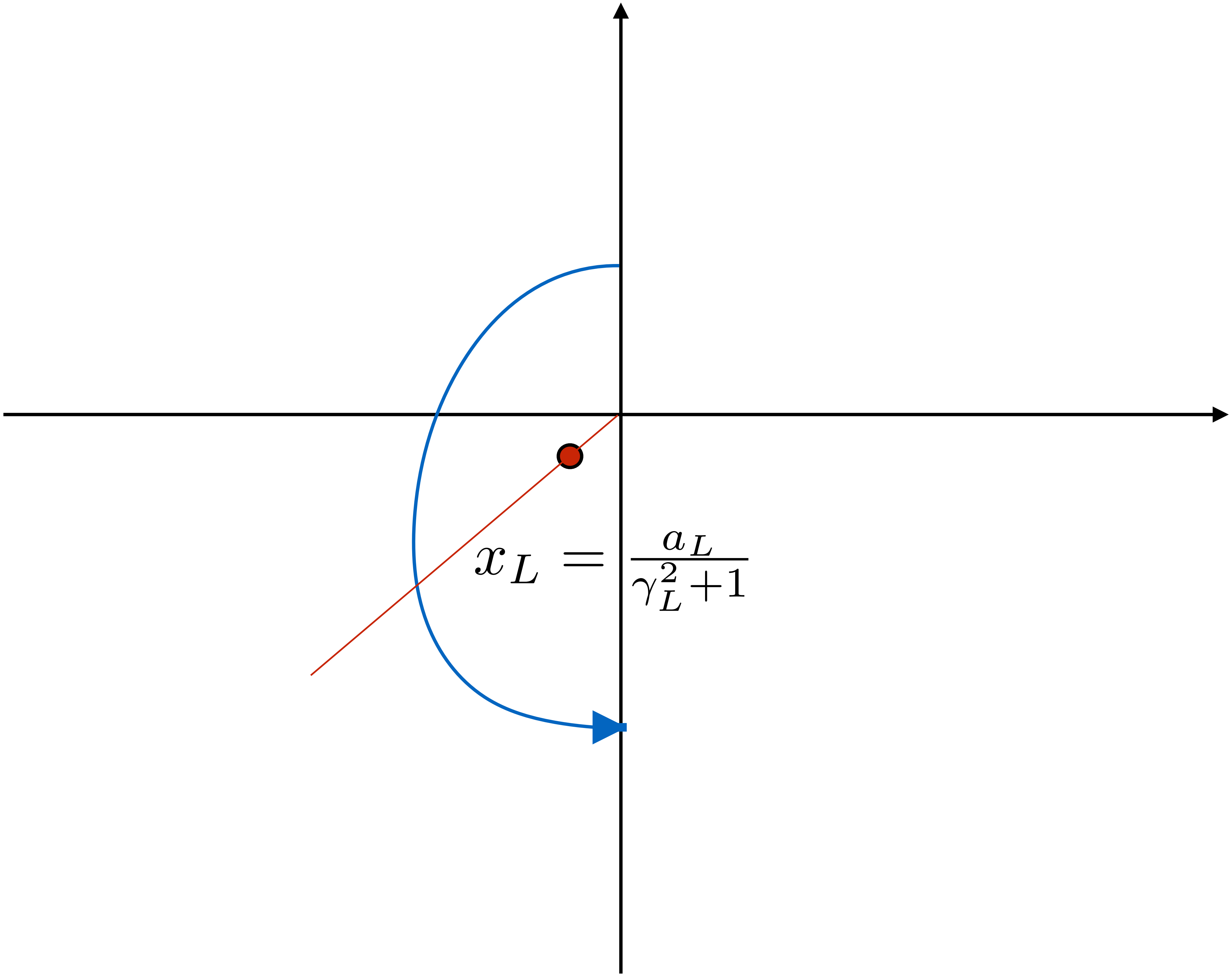
$$\boxed{-1 < \gamma_R < 0, \quad \gamma_L > 0}$$





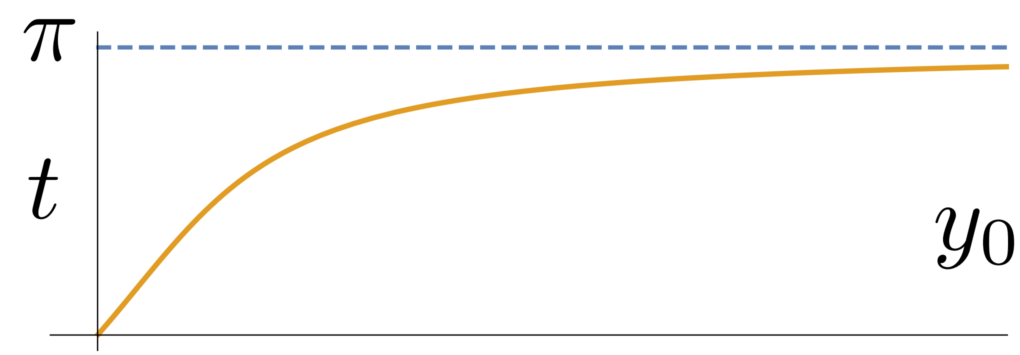
Computing the half-return maps: the left one

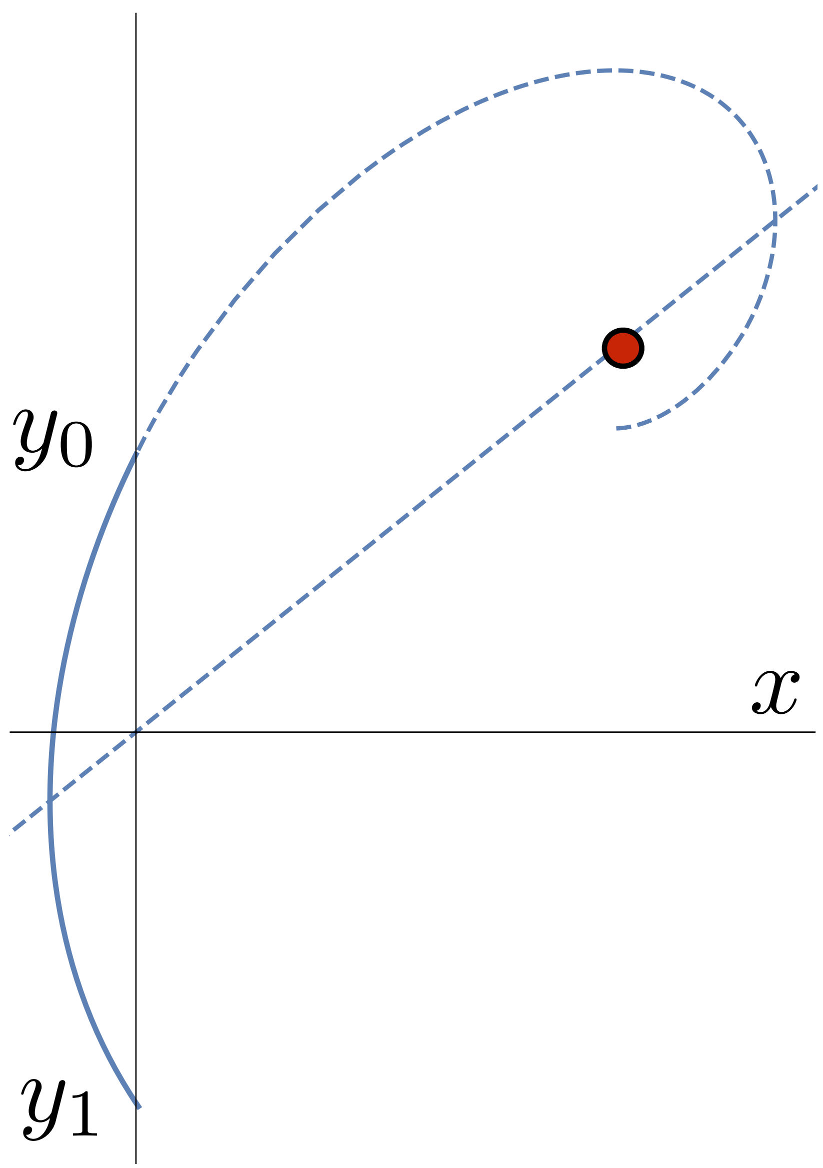
$$\boxed{-1 < \gamma_R < 0, \quad \gamma_L > 0}$$

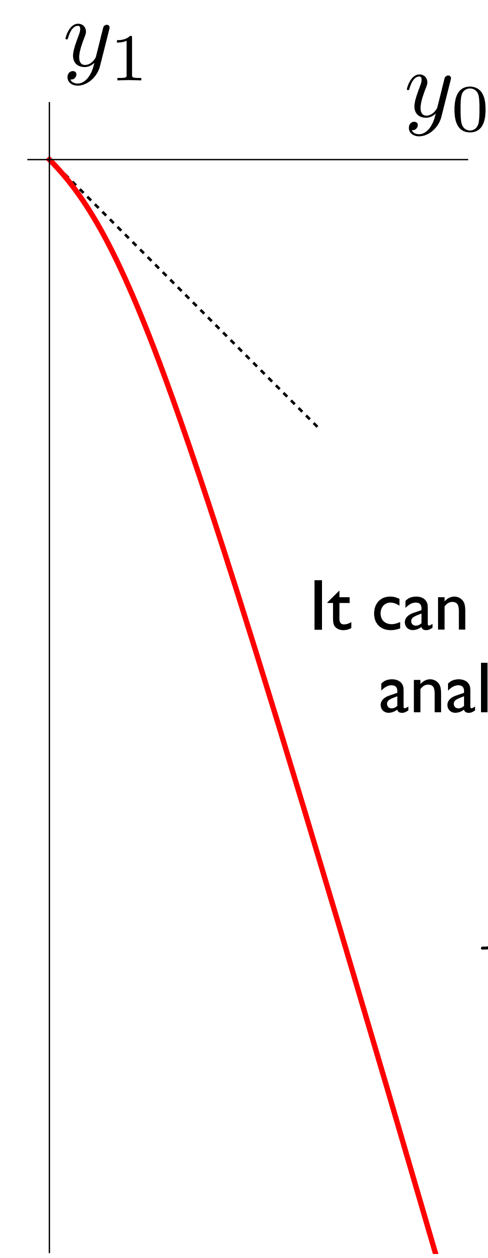


Computing the half-return maps: the left one

$a_L > 0$







It can be extended as an analytical involution

$P_L(y)$

$$P_L(y) = -y - \frac{4\gamma_L}{3a_L}y^2 - \frac{16\gamma_L^2}{9a_L^2}y^3 - \frac{4\gamma_L(9m_L^2 + 79\gamma_L^2)}{135a_L^3}y^4 + \cdots$$

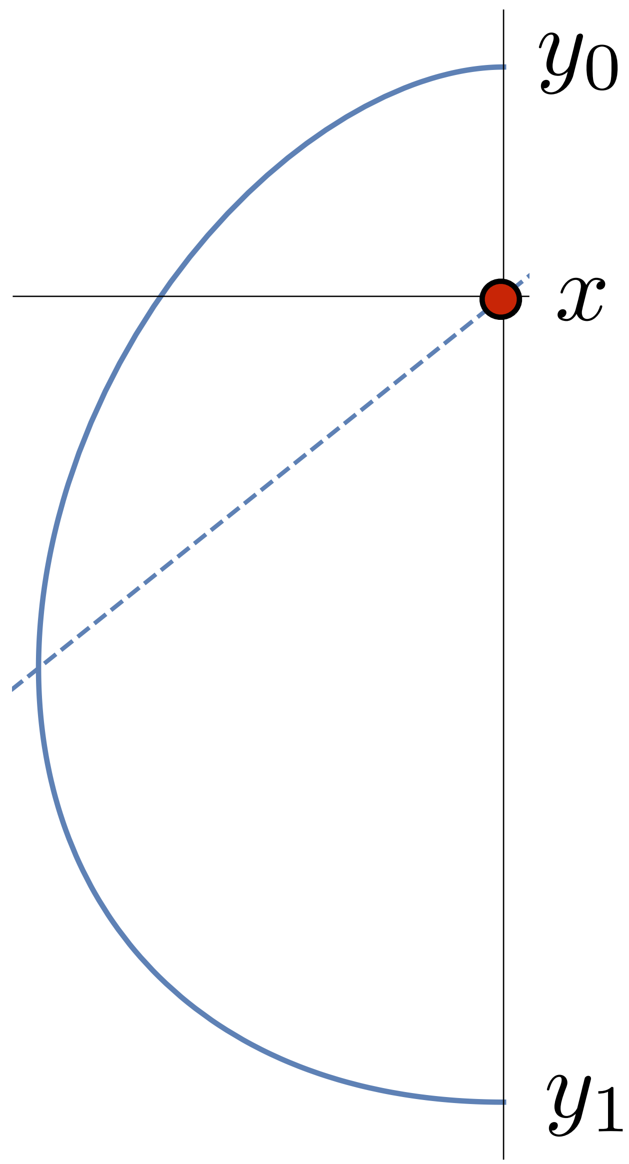
Computing the half-return maps: the left one

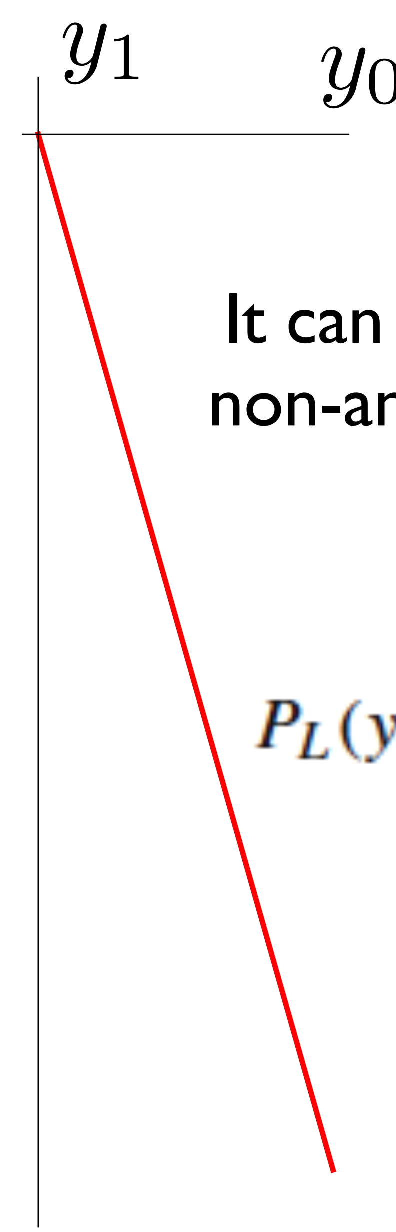
(a) When $a_L = 0$ *the left Poincaré map is defined for* $y \geqslant 0$ *as*

$$P_L(y) = -e^{\gamma_L \pi} y.$$

Computing the half-return maps: the left one

$$\boxed{a_L = 0}$$





It can be extended as a non-analytical involution

$$P_L(y) = -e^{\gamma\pi} y.$$

Computing the half-return maps: the left one

(b) Assuming $a_L < 0$ and $\gamma_L > 0$, the left Poincaré map P_L is well defined for $y \geqslant 0$ and is given by the expressions

$$y = a_L \frac{\varphi_{\gamma_L}(t)}{\varphi'_{\gamma_L}(t)} = a_L \frac{1 - e^{\gamma_L t}(\cos t - \gamma_L \sin t)}{(\gamma_L^2 + 1)e^{\gamma_L t} \sin t},$$

$$P_L(y) = a_L \frac{\varphi_{\gamma_L}(-t)}{\varphi'_{\gamma_L}(-t)} = -a_L \frac{1 - e^{-\gamma_L t}(\cos t + \gamma_L \sin t)}{(\gamma_L^2 + 1)e^{-\gamma_L t} \sin t},$$

where $\pi < t \leqslant \hat{t}$, being $\hat{t}$ the only value in $(\pi, 2\pi)$ such that $\varphi_{\gamma_L}(\hat{t}) = 0$.

$$\boxed{\varphi_\gamma(t) = 1 - e^{\gamma t}(\cos t - \gamma \sin t)}$$

Computing the half-return maps: the left one

In particular we have $P_L(0) = a_L\alpha_0$, *where*

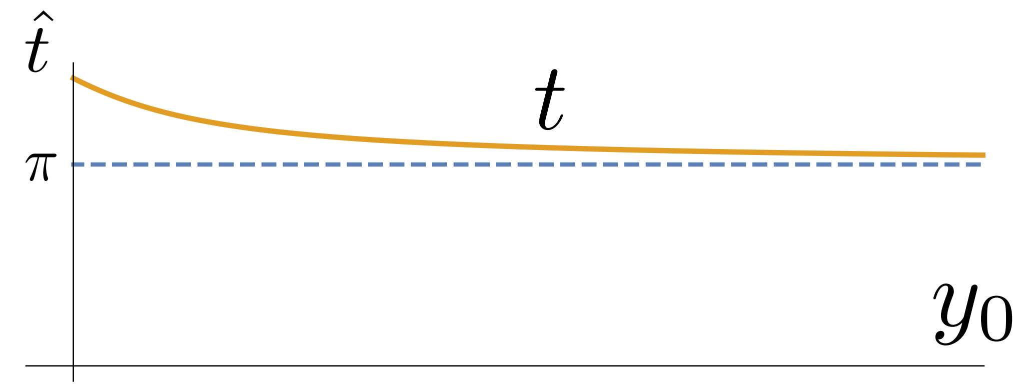
$$\alpha_0 = \frac{\cos\hat{t} + \gamma_L \sin\hat{t} - e^{\gamma_L\hat{t}}}{(\gamma_L^2+1)\sin\hat{t}} = \frac{2(\cos\hat{t} - \cosh\hat{t})}{(\gamma_L^2+1)\sin\hat{t}} > 0.$$

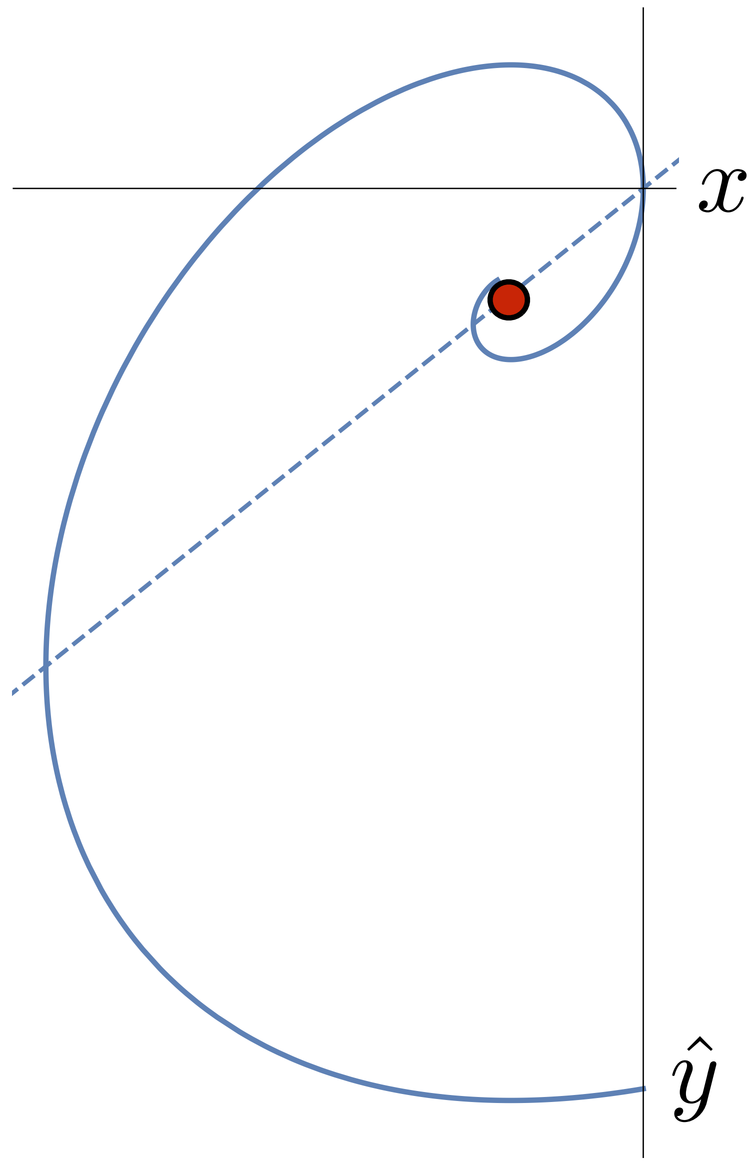
Furthermore, the two first derivatives of map P_L *satisfy*

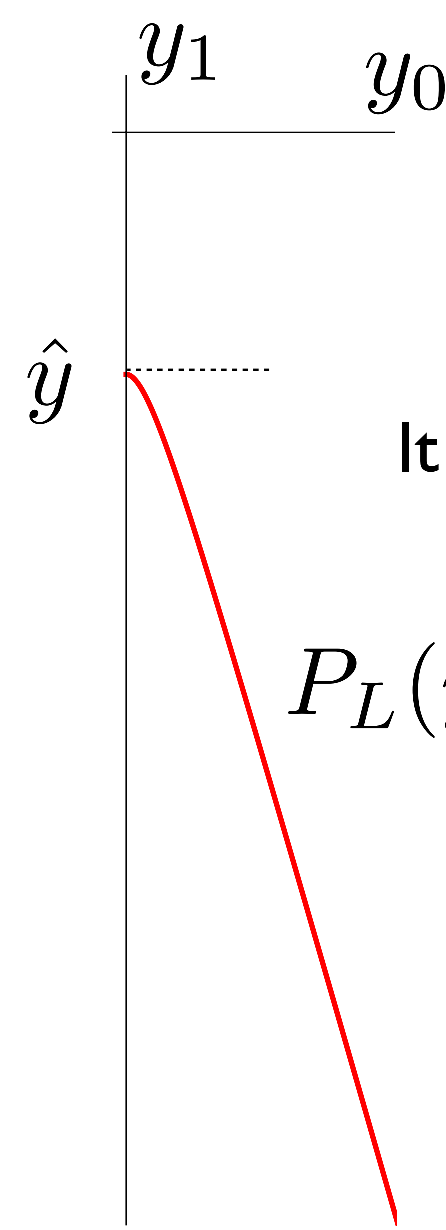
$$P_L'(0) = 0,\ P_L''(0) = \frac{e^{2\gamma_L\hat{t}}}{a_L\alpha_0},\ \textit{and}\ \lim_{y\to\infty} P_L'(y) = -e^{\gamma_L\pi}.$$

Computing the half-return maps: the left one

$$a_L < 0$$

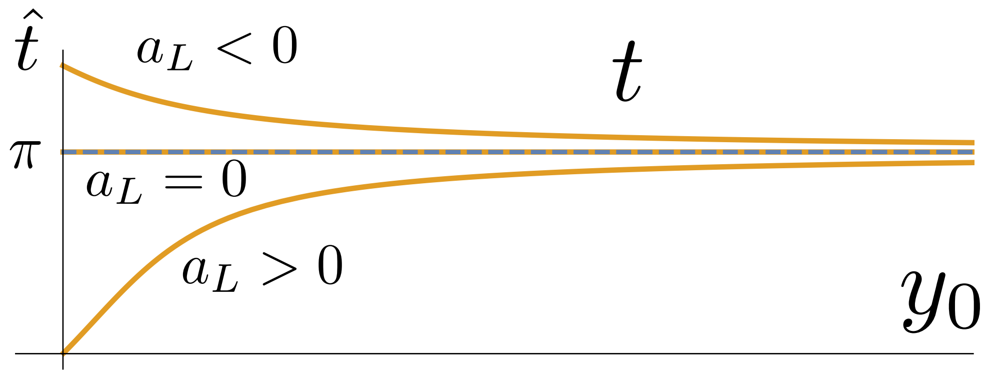


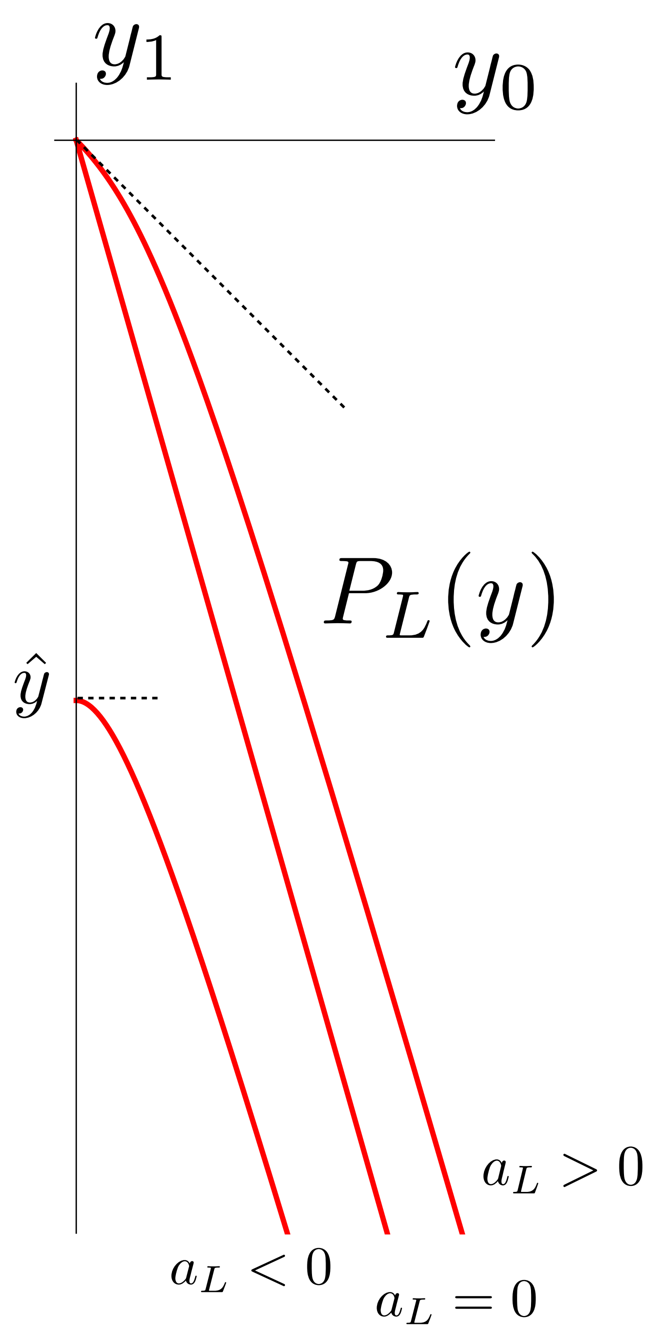




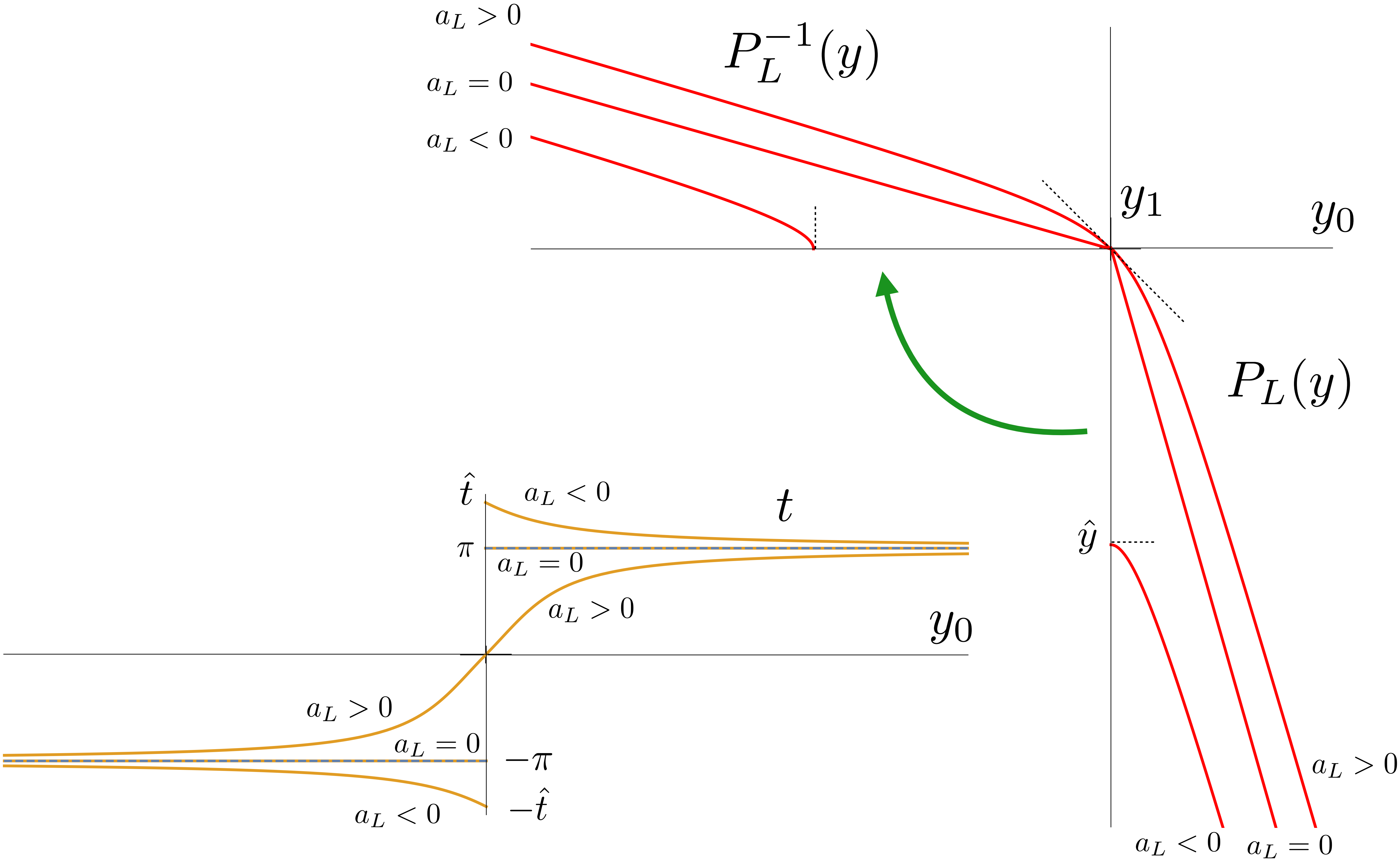
It cannot be extended as an involution

The left boundary focus transition





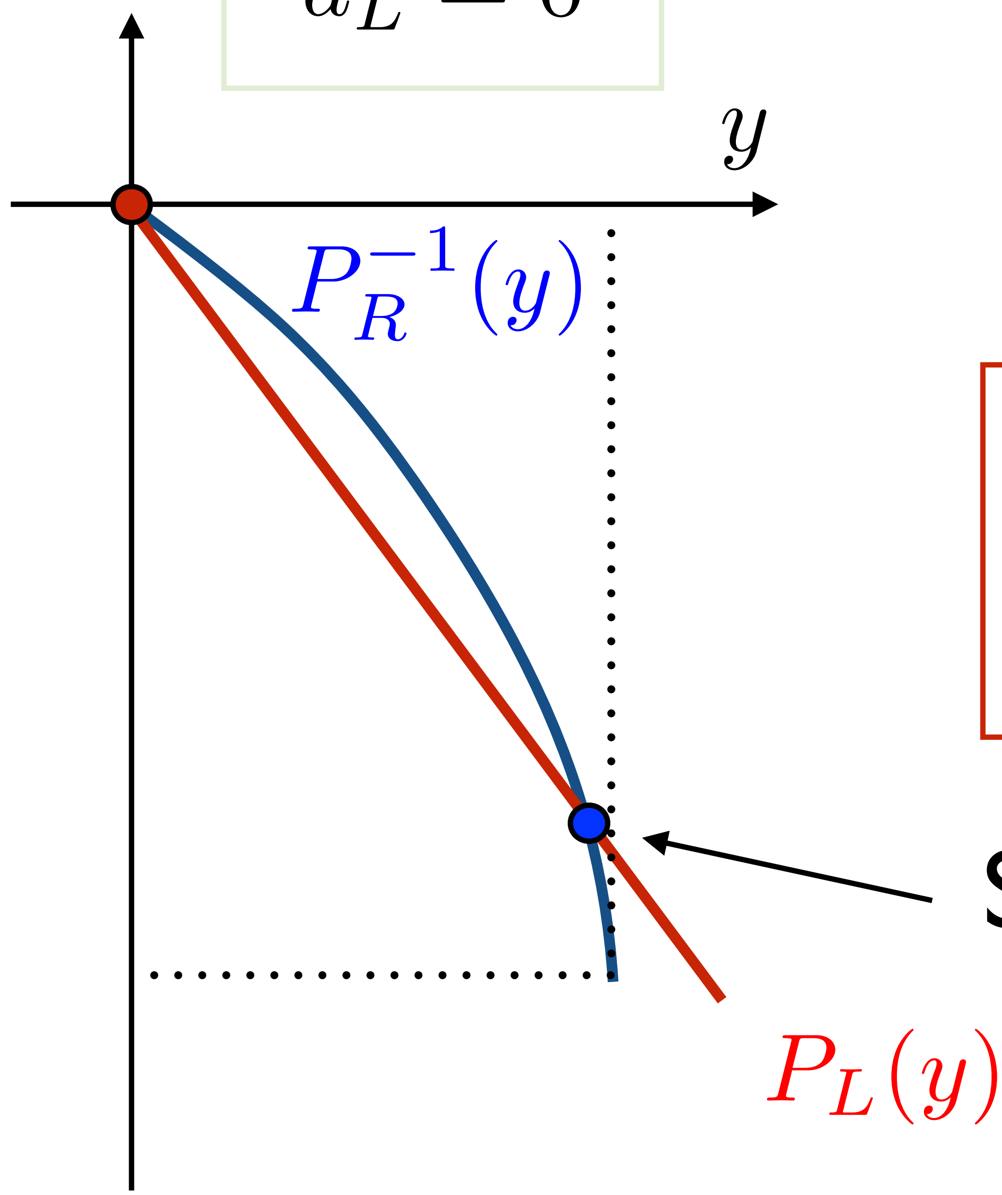
The left boundary focus transition



The boundary focus-saddle case

$$a_L = 0$$

$b = 0$

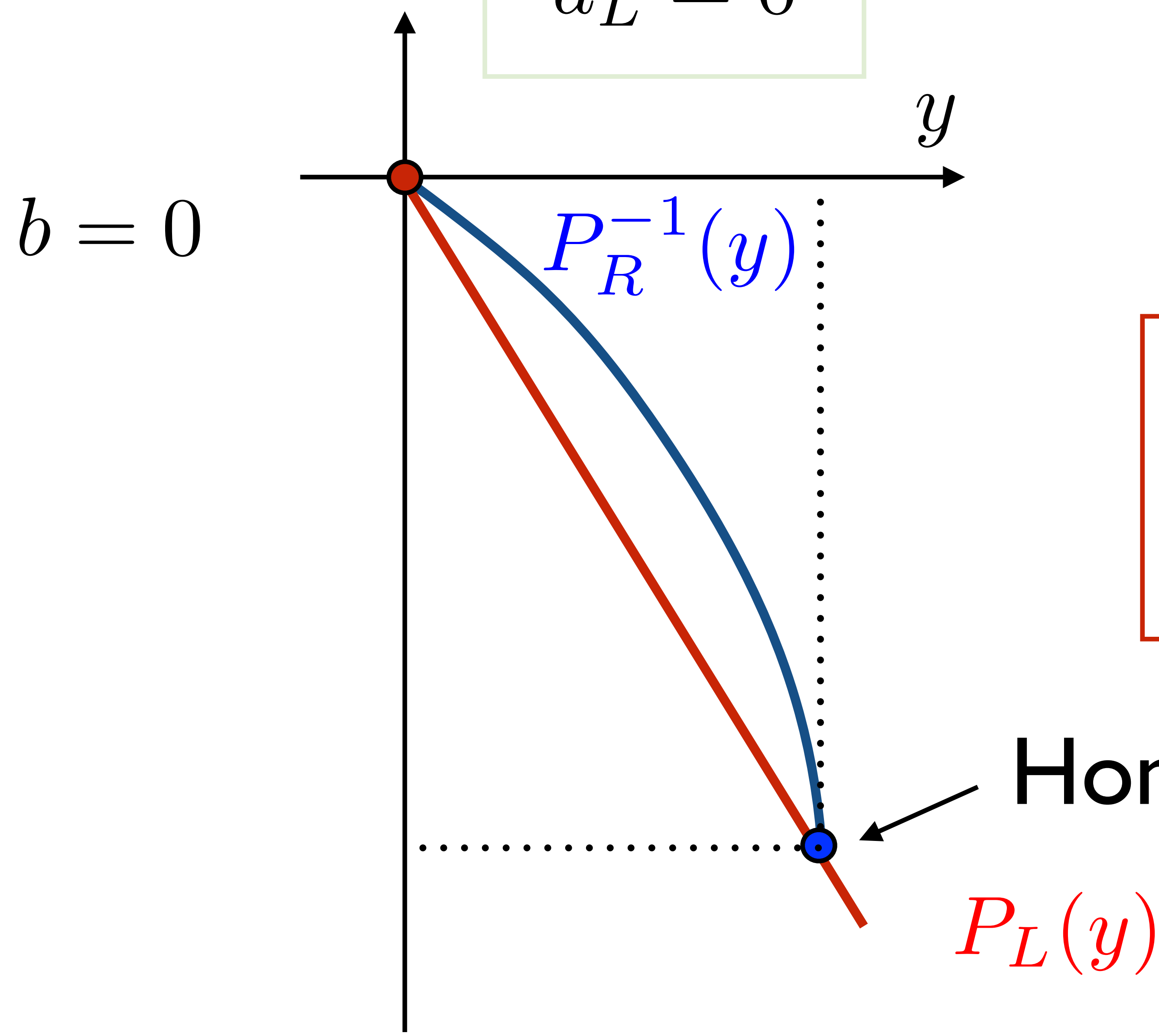


$$e^{\pi \gamma_L} < \frac{1 - \gamma_R}{1 + \gamma_R}$$

Stable limit cycle!

The boundary focus-saddle case

$$a_L = 0$$



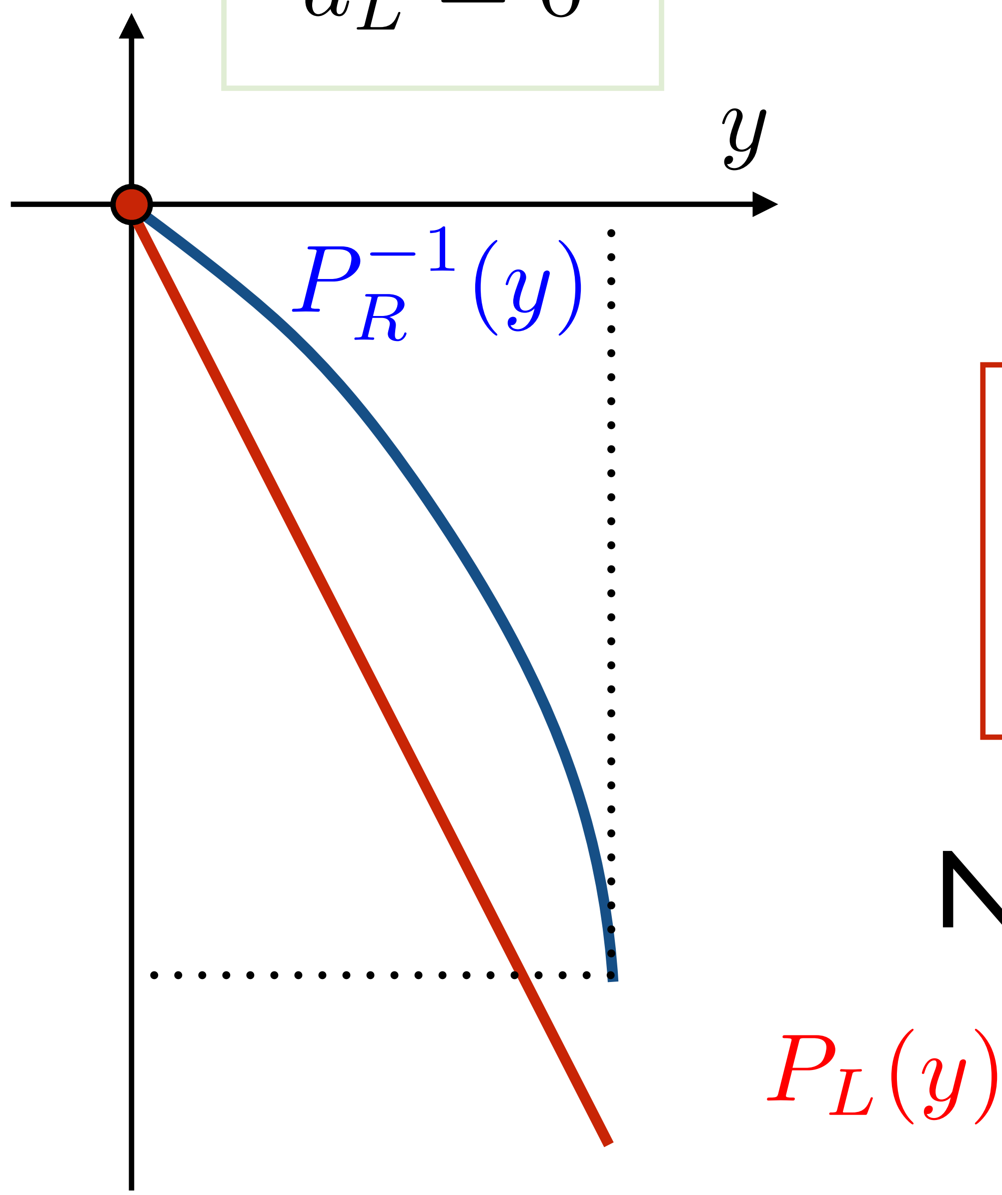
$$e^{\pi \gamma_L} = \frac{1 - \gamma_R}{1 + \gamma_R}$$

Homoclinic connection!

The boundary focus-saddle case

$$a_L = 0$$

$$b = 0$$



$$\boxed{e^{\pi\gamma_L} > \frac{1-\gamma_R}{1+\gamma_R}}$$

No periodic orbits!

The boundary focus-saddle case

Proposition Considering $\gamma_L > 0$, $-1 < \gamma_R < 0$, $a_L = 0$, $a_R < 0$, and $b = 0$, the following statements hold.

(a) If $e^{\pi\gamma_L} < (1-\gamma_R)/(1+\gamma_R)$, then there exists one stable limit cycle surrounding the unstable boundary focus at the origin and no homoclinic connections.

(b) If $e^{\pi\gamma_L} = (1-\gamma_R)/(1+\gamma_R)$, then there exists one homoclinic connection to the saddle and no periodic orbits surrounding the boundary focus.

(c) If $e^{\pi\gamma_L} > (1-\gamma_R)/(1+\gamma_R)$, then the system has no periodic orbits and no homoclinic connections.

The boundary focus-saddle case

Proposition Considering $\gamma_L > 0$, $-1 < \gamma_R < 0$, $a_L = 0$, $a_R < 0$, and $b = 0$, the following statements hold.

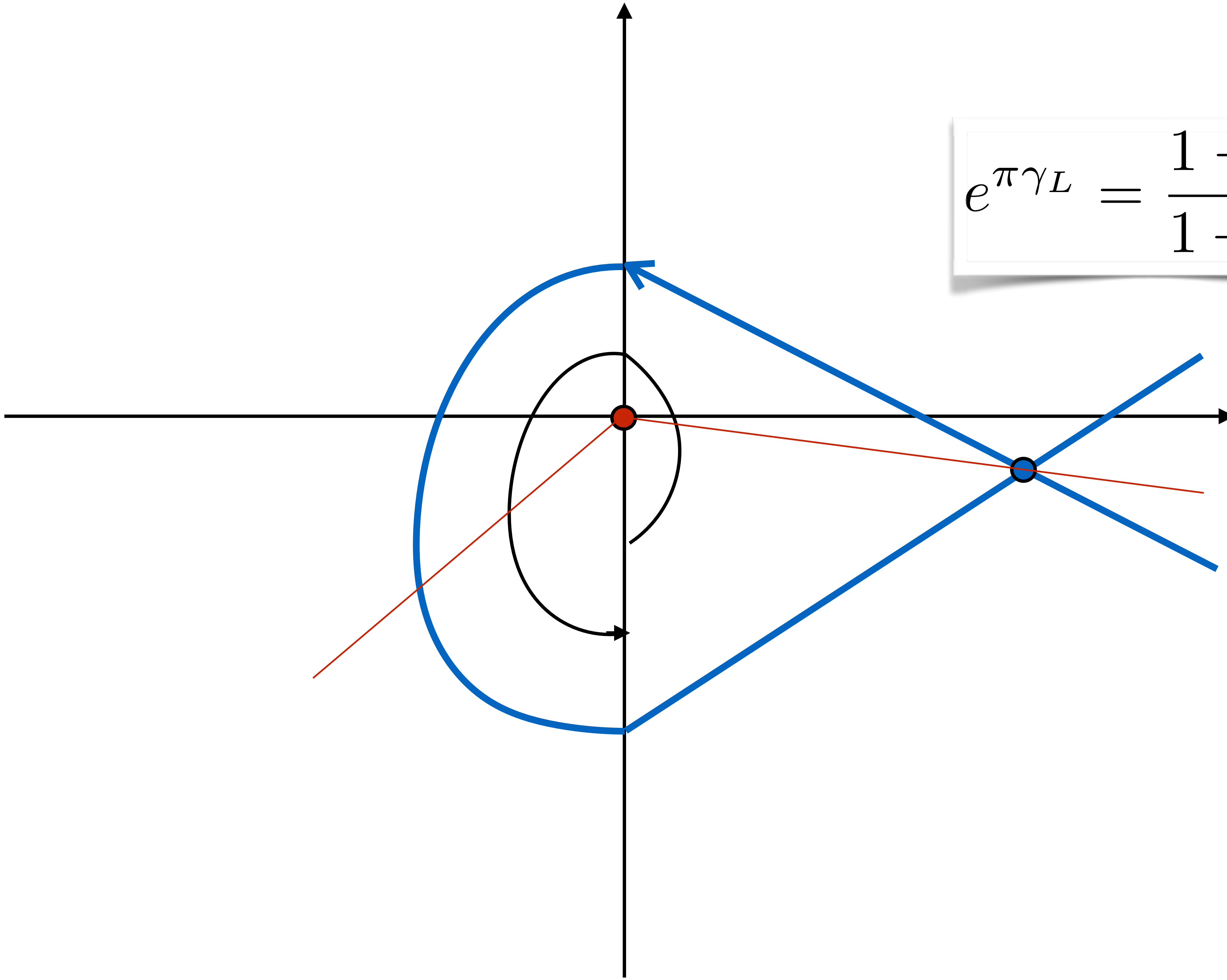
(a) If $e^{\pi\gamma_L} < (1 - \gamma_R)/(1 + \gamma_R)$, then there exists one stable limit cycle surrounding the unstable boundary focus at the origin and no homoclinic connections.

(b) If $e^{\pi\gamma_L} = (1 - \gamma_R)/(1 + \gamma_R)$, then there exists one homoclinic connection to the saddle and no periodic orbits surrounding the boundary focus.

(c) If $e^{\pi\gamma_L} > (1-\gamma_R)/(1+\gamma_R)$, then the system has no periodic orbits and no homoclinic connections.

Perturbing the critical boundary focus-saddle case

$$\boxed{-1 < \gamma_R < 0, \quad \gamma_L > 0}$$

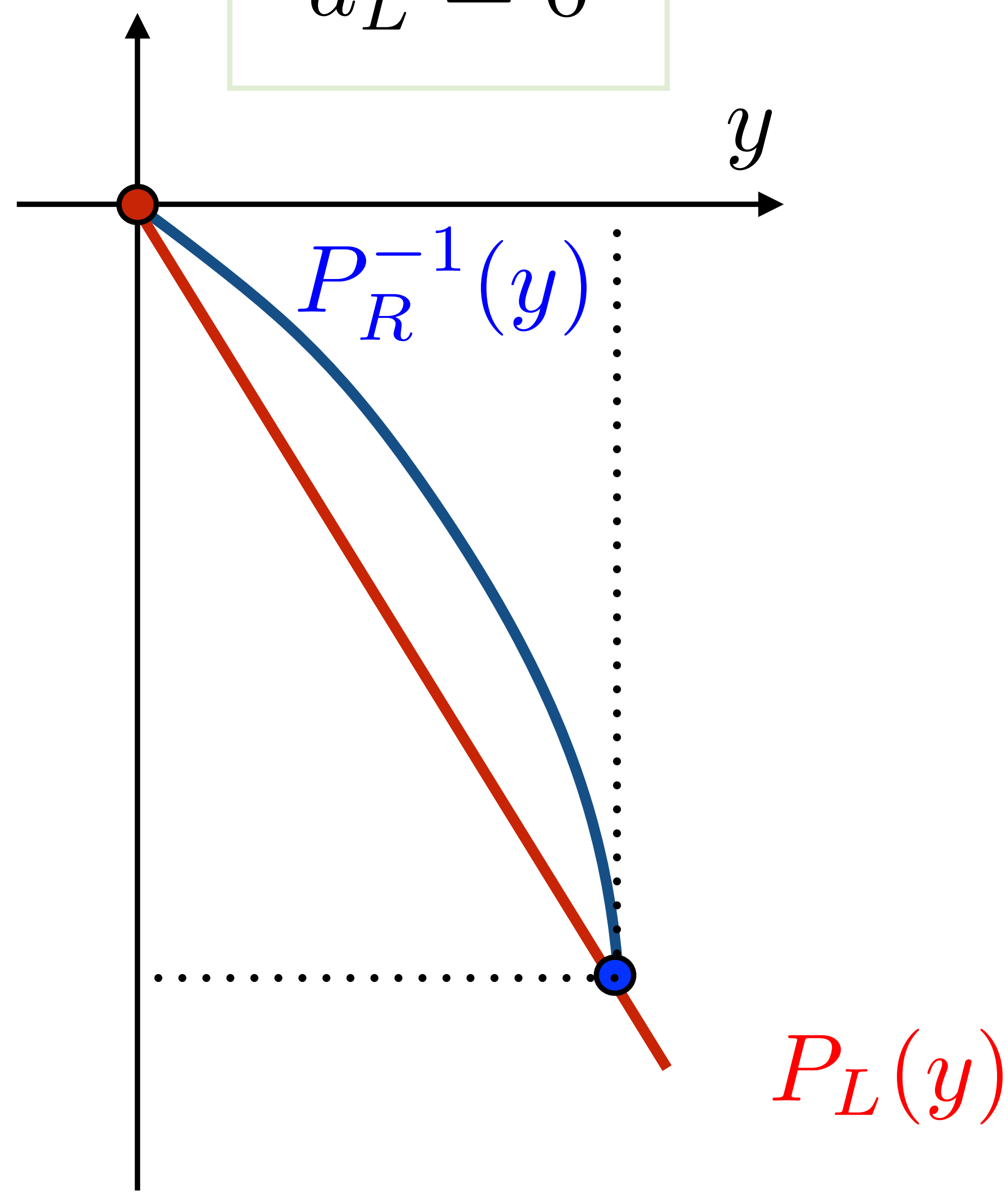


$$e^{\pi \gamma_L} = \frac{1-\gamma_R}{1+\gamma_R}$$

Perturbing the critical boundary focus-saddle case

$$a_L = 0$$

$$b = 0$$

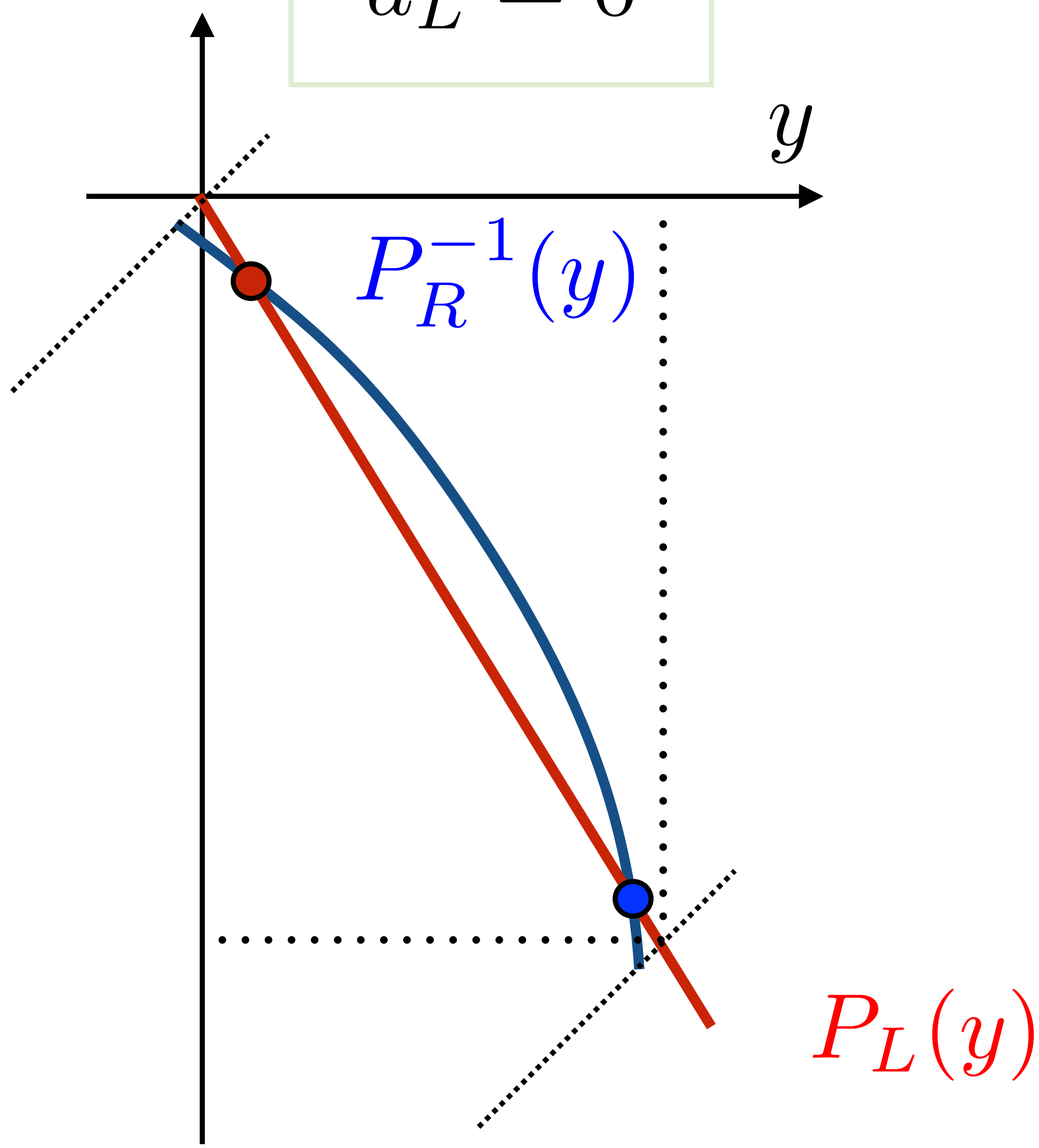


$$e^{\pi\gamma_L} = \frac{1-\gamma_R}{1+\gamma_R}$$

Perturbing the critical boundary focus-saddle case

$$a_L = 0$$

$b < 0$



$$e^{\pi\gamma_L} = \frac{1-\gamma_R}{1+\gamma_R}$$

Two limit cycles!

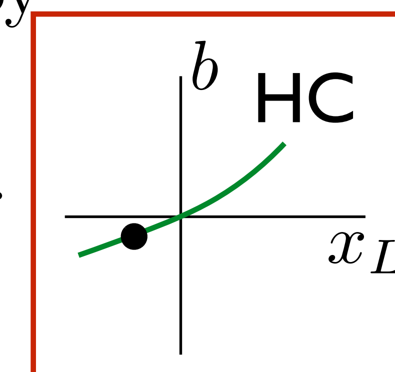
Persistence of homoclinic connections on a curve in the (a_L, b)-plane

Theorem Considering $-1 < \gamma_R < 0$, $\quad x_R > 0$, $\quad \gamma_L = \frac{1}{\pi}\ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right) > 0$, the following statements hold.

(a) For $x_L = 0$, $b = 0$ the origin is an unstable boundary focus surrounded by an homoclinic connection and there are no periodic orbits.

(b) The above homoclinic connection persists on the graph of a curve defined by $b = b_H(x_L)$ in a neighborhood of the origin in the parameter plane (x_L, b). The local expansion of the function $b_H(x_L)$ is given by

$$b = b_H(x_L) = 2\gamma_L x_L - \frac{(1+\gamma_L^2)\sinh(\pi\gamma_L)}{2x_R} x_L^2 + \cdots$$



(c) There exists $\delta^* > 0$ such that if $|x_L| < \delta^*$ then in passing from $b = b_H(x_L)$ to $b < b_H(x_L)$ we pass from the homoclinic orbit to a stable crossing periodic orbit.

$$x_L = \frac{a_L}{\gamma_L^2+1} \qquad x_R = \frac{a_R}{\gamma_R^2-1}$$

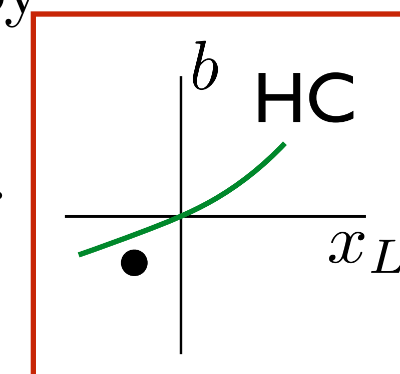
Persistence of homoclinic connections on a curve in the (a_L, b)-plane

Theorem Considering $-1 < \gamma_R < 0, \quad x_R > 0, \quad \gamma_L = \frac{1}{\pi}\ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right) > 0$, the following statements hold.

(a) For $x_L = 0$, $b = 0$ the origin is an unstable boundary focus surrounded by an homoclinic connection and there are no periodic orbits.

(b) The above homoclinic connection persists on the graph of a curve defined by $b = b_H(x_L)$ in a neighborhood of the origin in the parameter plane (x_L, b). The local expansion of the function $b_H(x_L)$ is given by

$$b = b_H(x_L) = 2\gamma_L x_L - \frac{(1+\gamma_L^2)\sinh(\pi\gamma_L)}{2x_R} x_L^2 + \cdots$$



(c) There exists $\delta^* > 0$ such that if $|x_L| < \delta^*$ then in passing from $b = b_H(x_L)$ to $b < b_H(x_L)$ we pass from the homoclinic orbit to a stable crossing periodic orbit.

A big limit cycle!

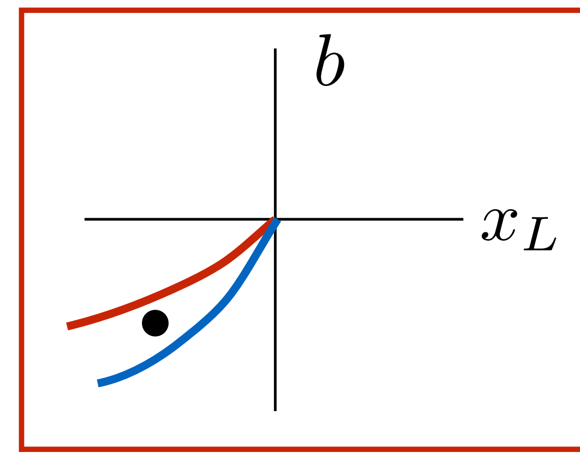
Two small limit cycles can bifurcate after some (a_L, b)-perturbation

Proposition Assume that $\gamma_L > 0$, $x_R > 0$, $\gamma_R < 0$. Then there exist $\delta > 0$ and two continuous functions $\beta_1(x_L)$ and $\beta_2(x_L)$ with $\beta_1(0) = \beta_2(0) = 0$ and satisfying

$$\beta_1(x_L) < \beta_2(x_L) < 0$$

for $-\delta < x_L < 0$, such that for the parameter sector defined by $\beta_1(x_L) < b < \beta_2(x_L)$ and $-\delta < x_L < 0$ the system has two small nested crossing periodic orbits. Moreover, we have

$$\beta_1(x_L) = -\frac{e^{\gamma_L \hat{t}} \sin \hat{t}}{2} x_L + O(x_L^2),$$

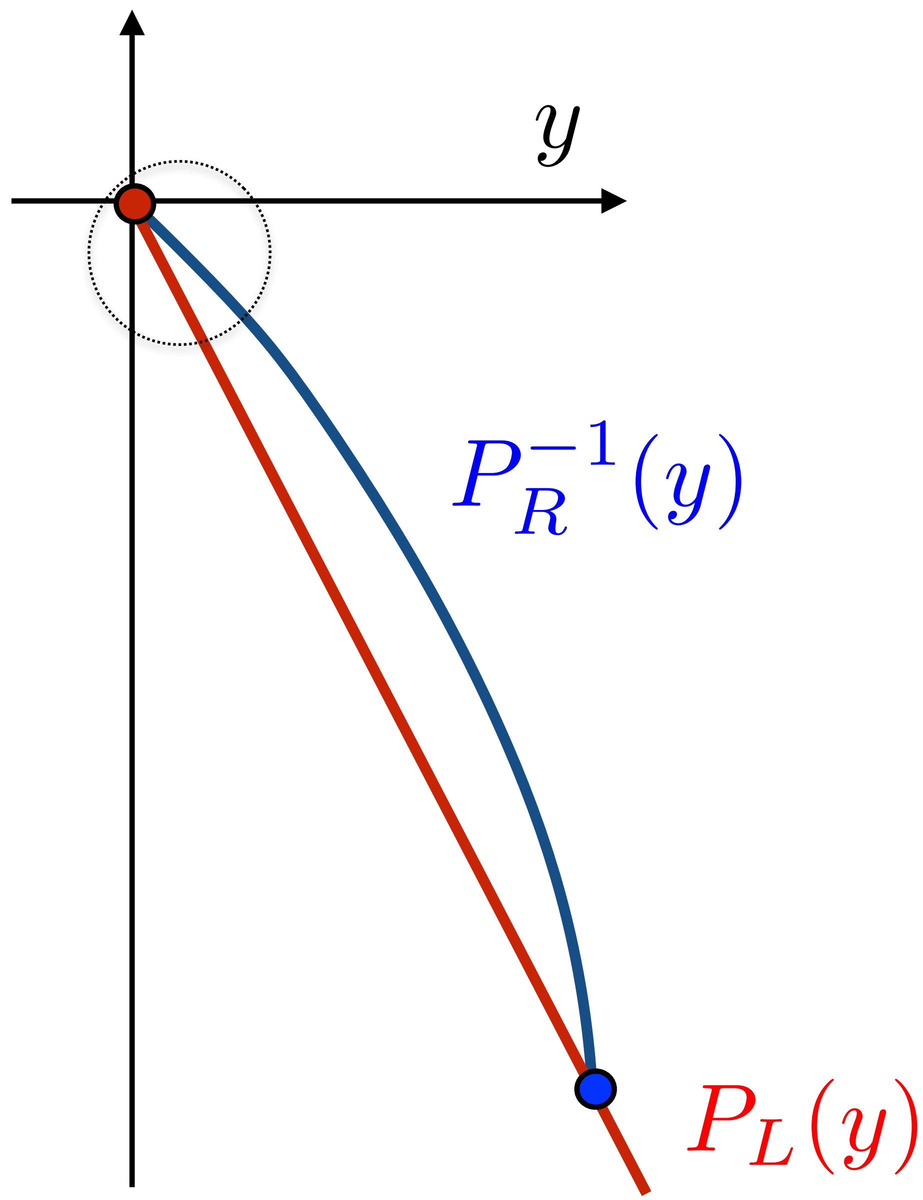


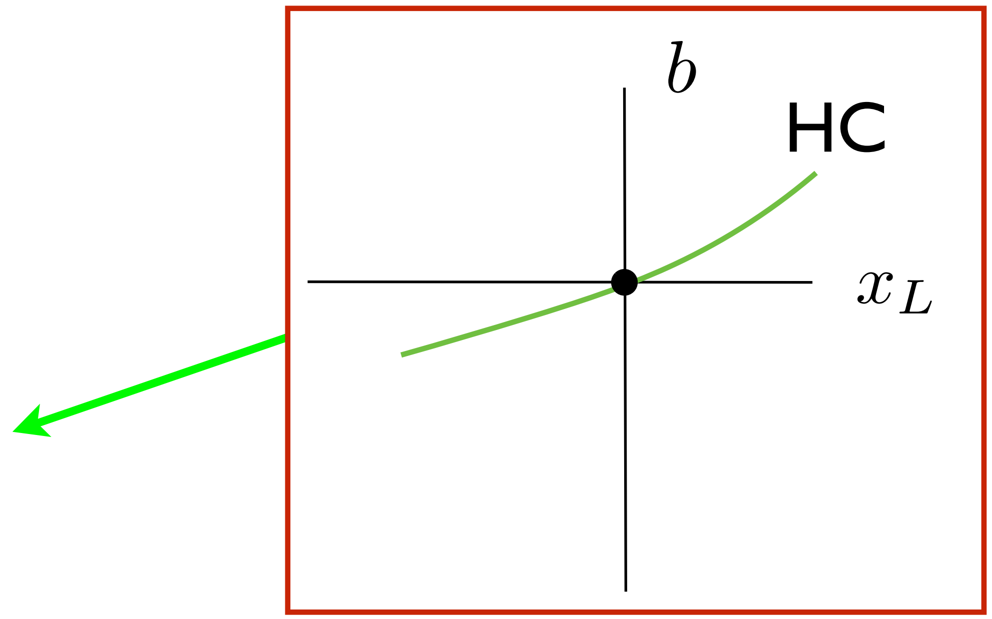
and

$$\beta_2(x_L) = \frac{\gamma_L}{\gamma_L^2 + 1}\left(1 - \frac{\sinh(\gamma_L t_M)}{\gamma_L \sin t_M}\right) x_L + O(x_L^2),$$

where t_M is the only solution of $\tanh(\gamma_L t) = \gamma_L \tan t$ in the interval $(\pi, 3\pi/2)$.

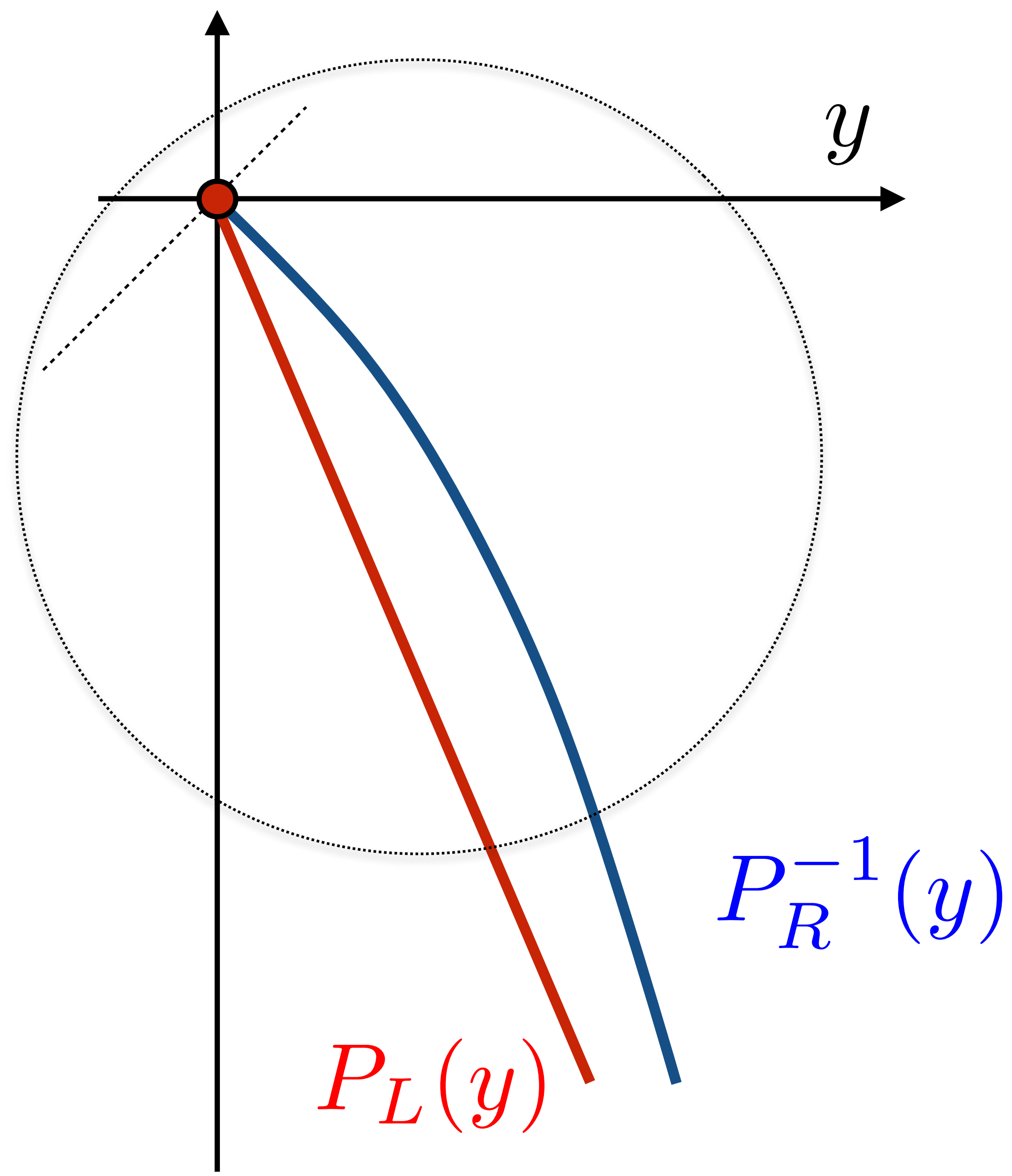
THREE limit cycles can bifurcate after some (a_L, b) -perturbation

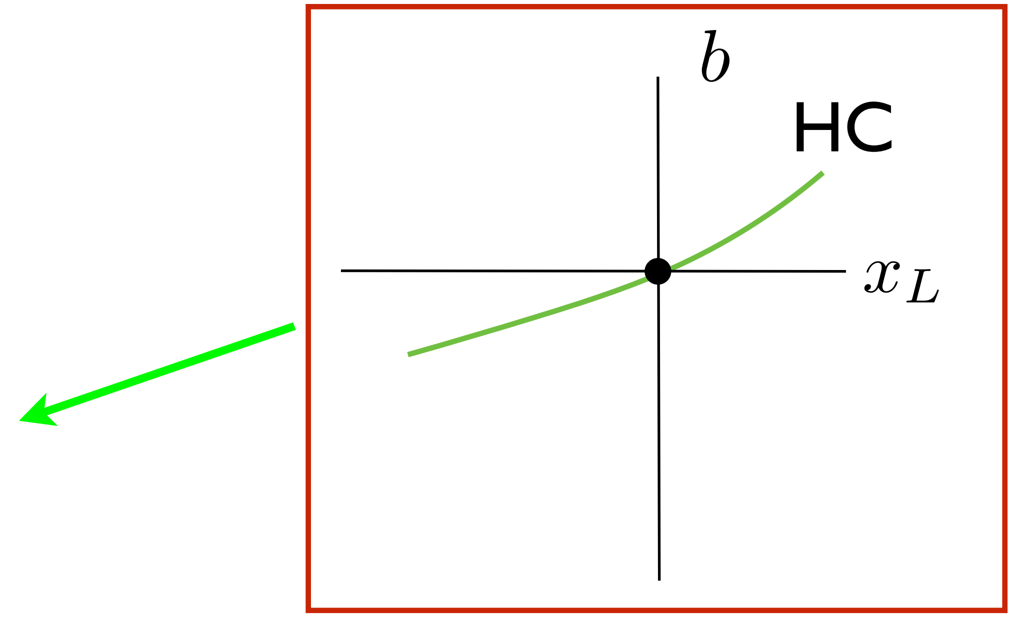




$$x_L = \frac{a_L}{\gamma_L^2+1}$$

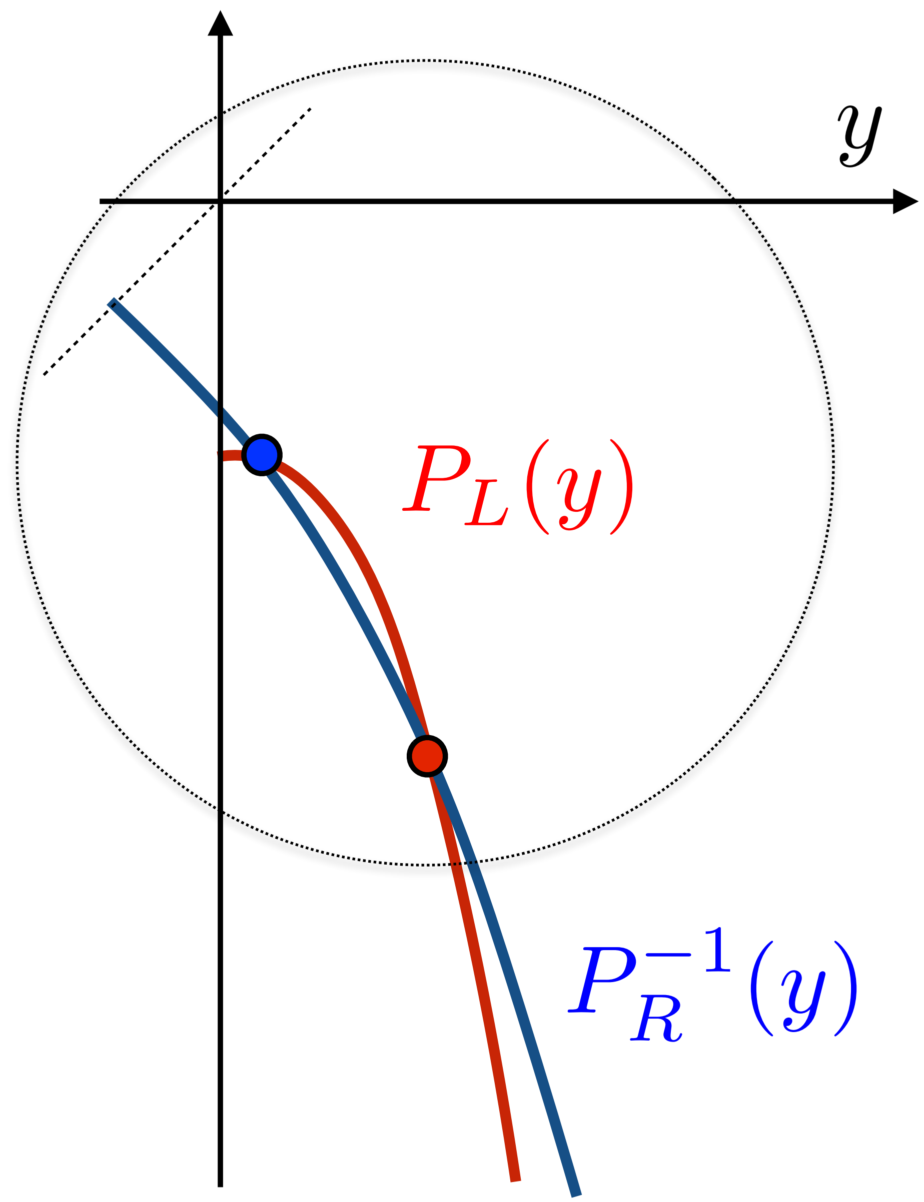
THREE limit cycles can bifurcate after some (a_L, b) -perturbation

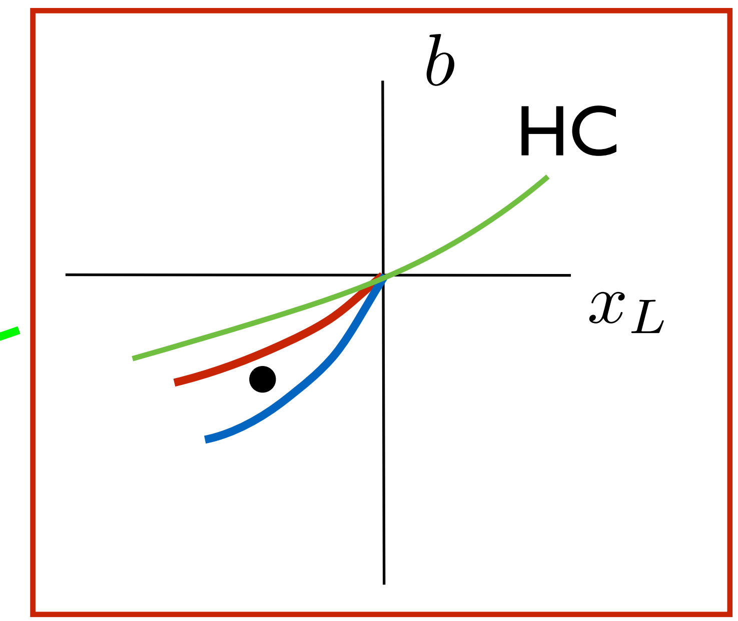




$$x_L = \frac{a_L}{\gamma_L^2+1}$$

THREE small limit cycles can bifurcate after some (a_L, b) -perturbation





Three limit cycles!

+ the big limit cycle from HC

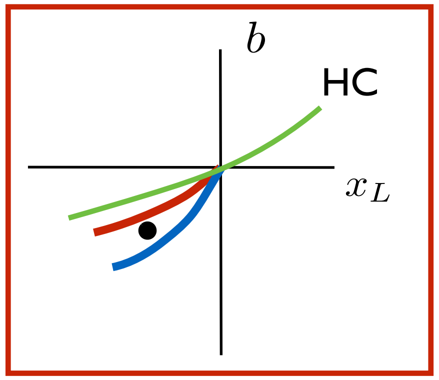
$$\boxed{x_L = \frac{a_L}{\gamma_L^2+1}}$$

THREE limit cycles can bifurcate after some (a_L, b)-perturbation

Final theorem Considering

$$-1 < \gamma_R < 0, \quad x_R > 0, \quad \gamma_L = \frac{1}{\pi} \ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right) > 0,$$

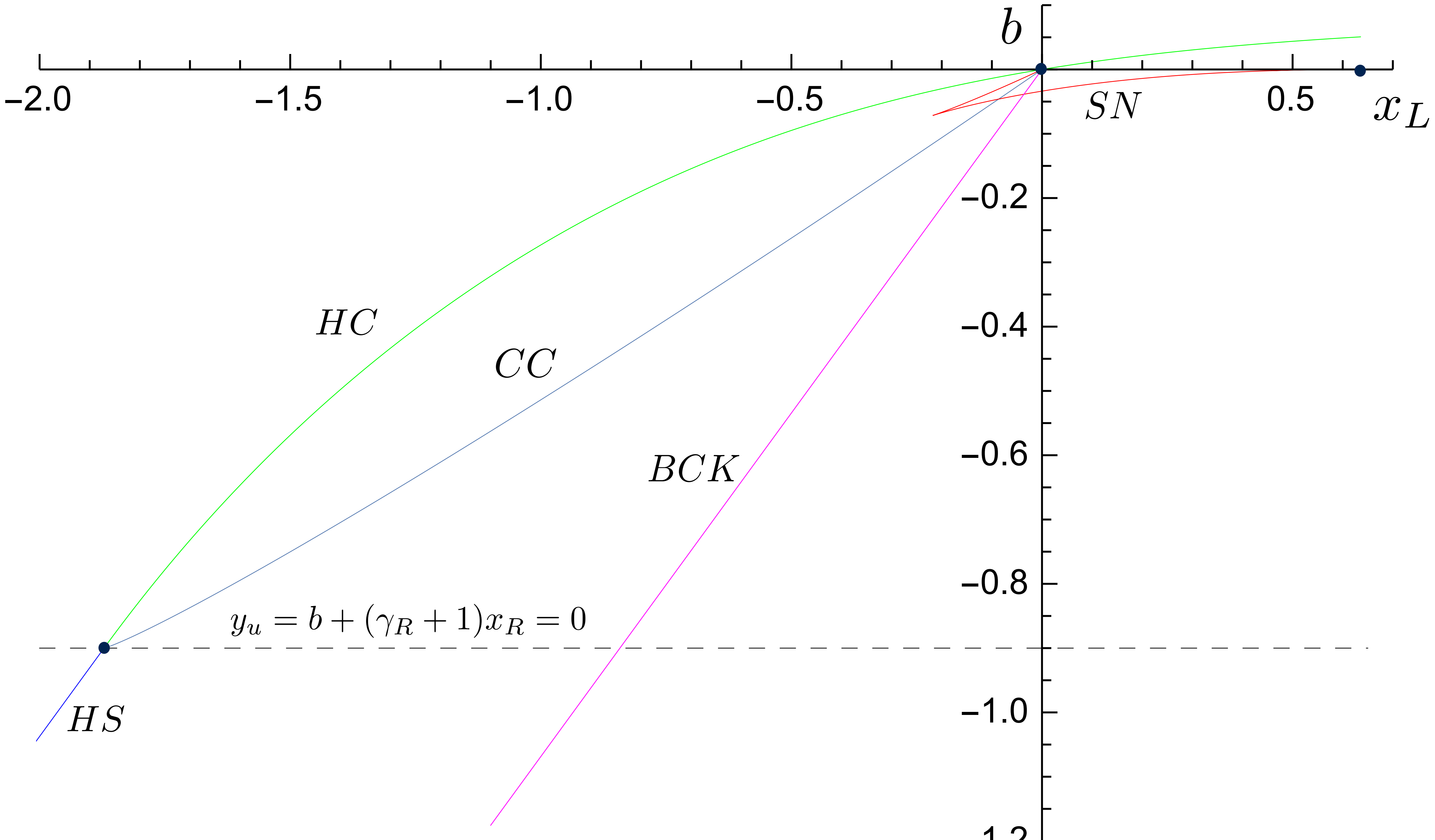
the following statement holds.
There exist $\delta > 0$ and two continuous functions $\beta_1(x_L)$ and $\beta_2(x_L)$ with $\beta_1(0) = \beta_2(0) = 0$ and satisfying



$$\beta_1(x_L) < \beta_2(x_L) < 0$$

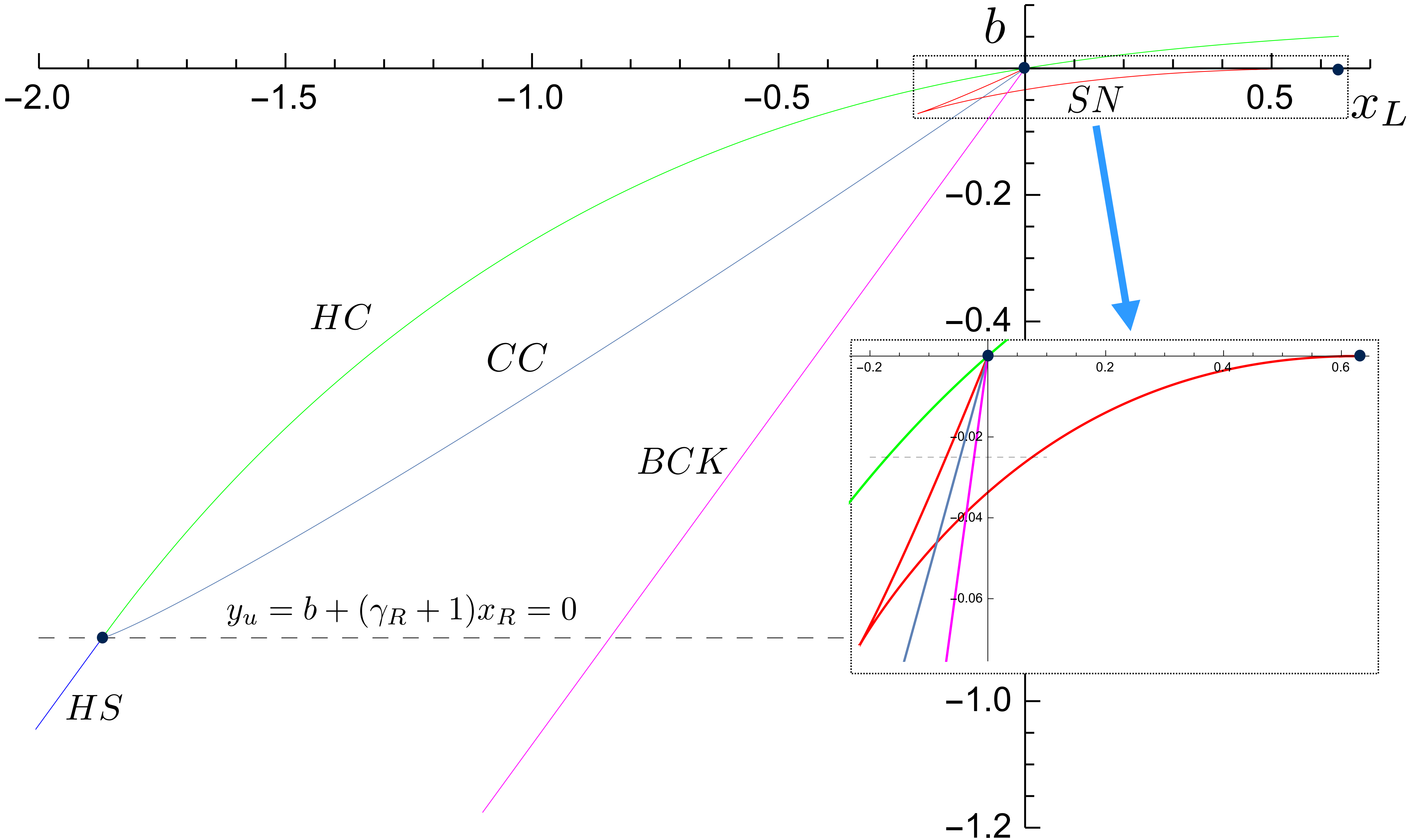
for $-\delta < x_L < 0$, such that for the parameter sector defined by $\beta_1(x_L) < b < \beta_2(x_L)$ and $-\delta < x_L < 0$ the system has three limit cycles: two small nested crossing periodic orbits surrounding the attractive sliding set, and a third big limit cycle surrounding them.

The parametric region with three limit cycles:



$$\boxed{\gamma_R = -0.1, \quad x_R = 1}$$

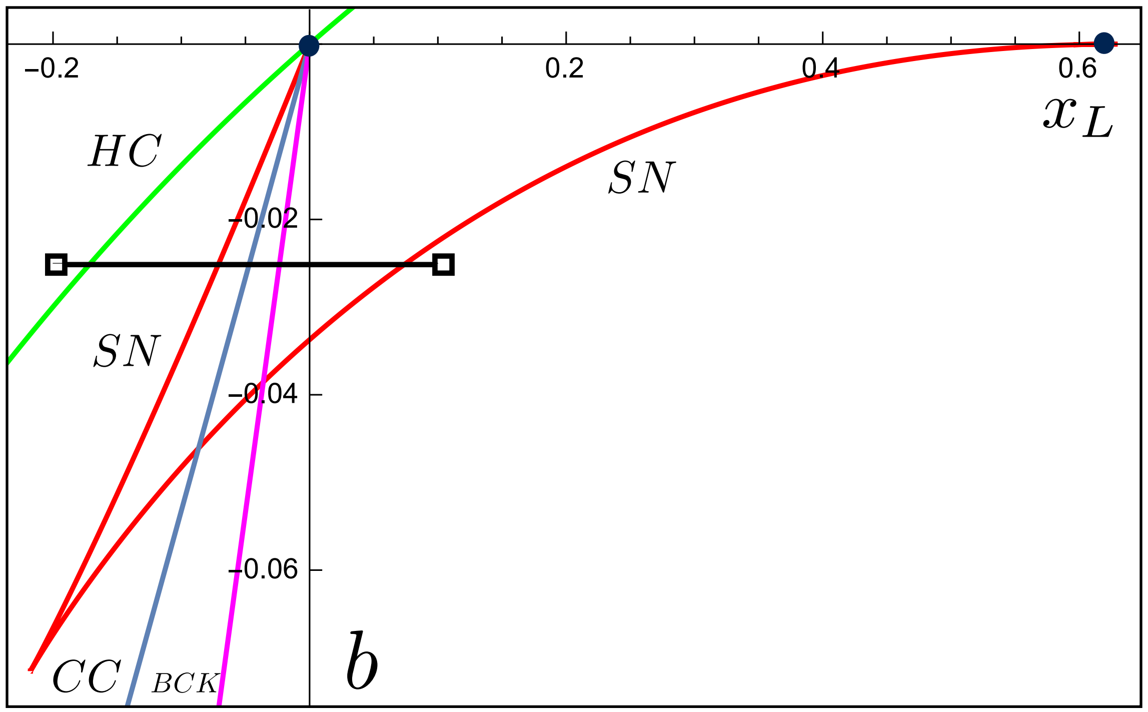
The parametric region with three limit cycles:

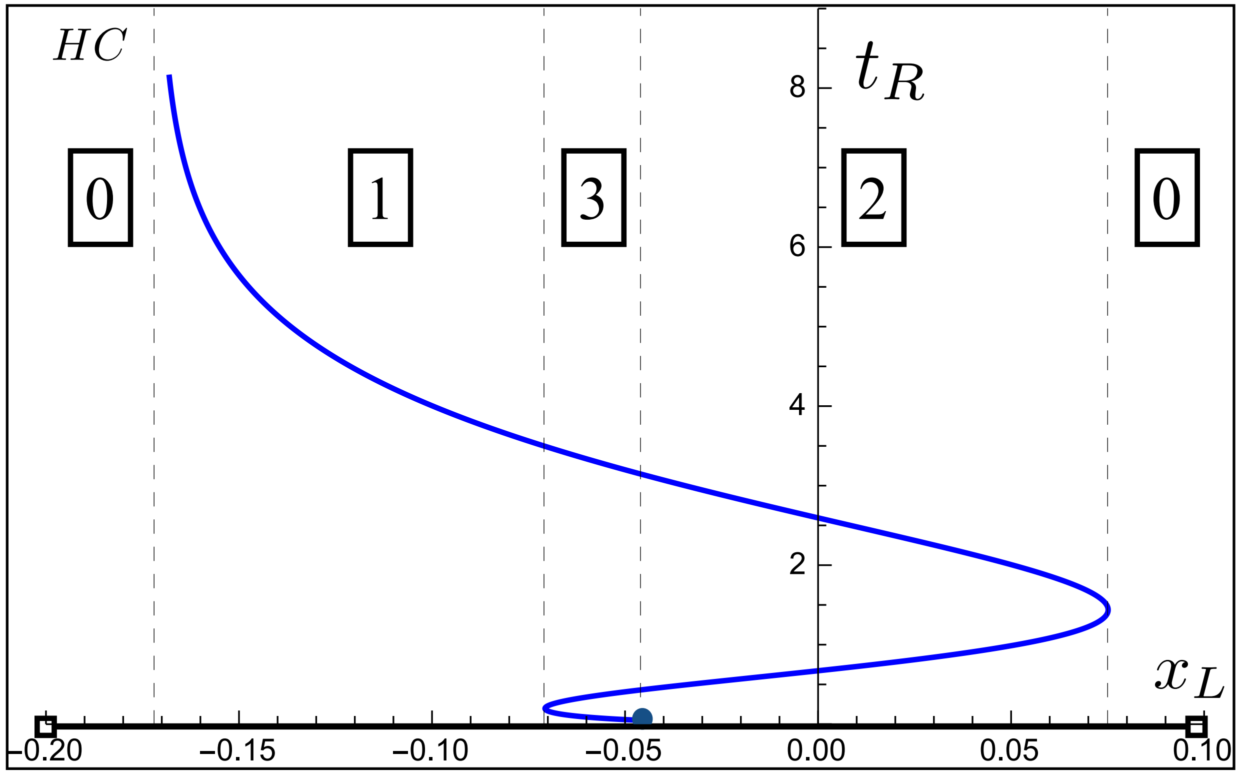


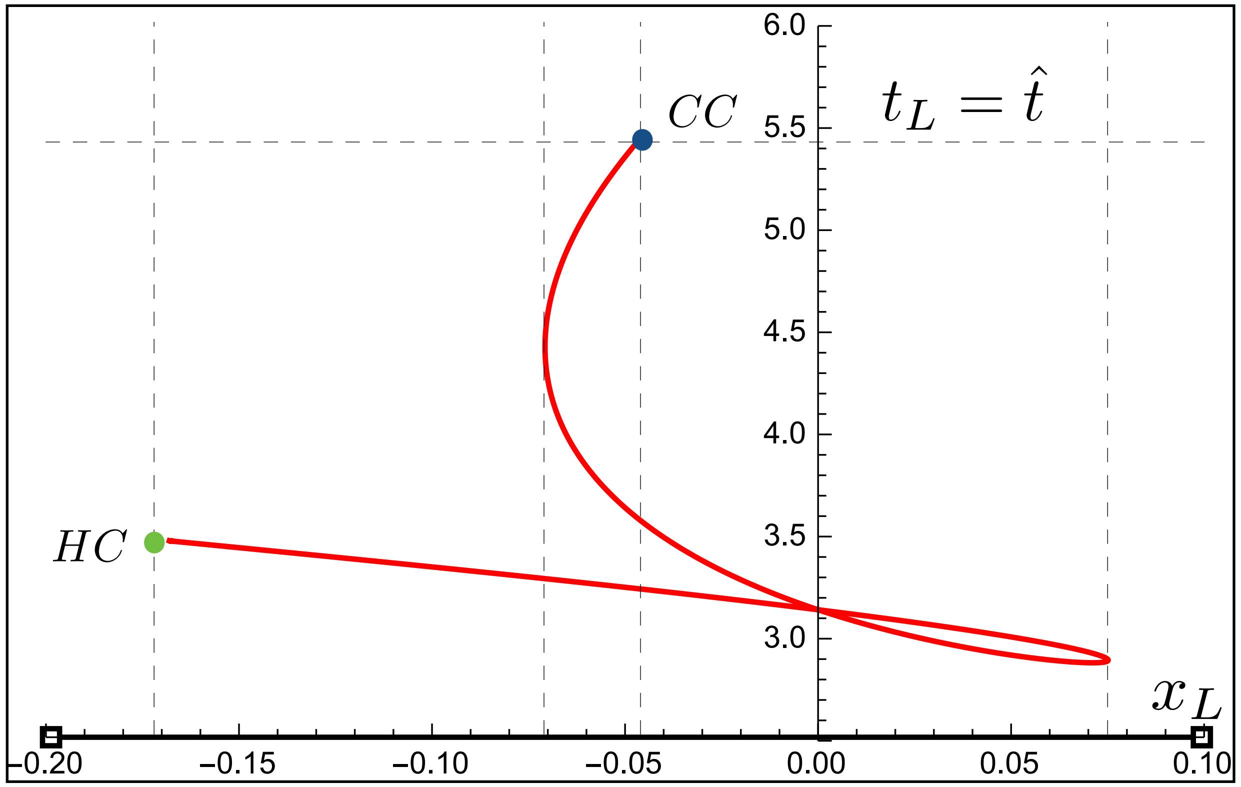
$$\boxed{\gamma_R = -0.1, \quad x_R = 1}$$

Flight times in a representative section of the bifurcation set

\# Crossing limit cycles







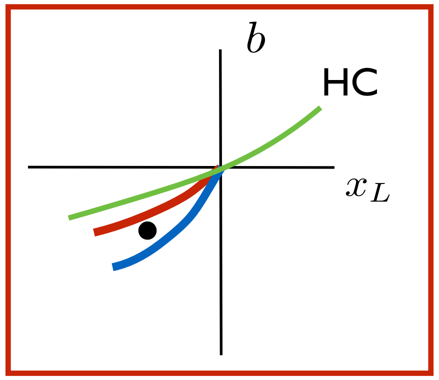
$$\gamma_R = -0.1, \quad x_R = 1$$

Three small limit cycles can bifurcate after some (a_L, b) -perturbation

Final theorem Considering

$$-1 < \gamma_R < 0, \quad x_R > 0, \quad \gamma_L = \frac{1}{\pi} \ln\left(\frac{1-\gamma_R}{1+\gamma_R}\right) > 0,$$

the following statement holds.
There exist $\delta > 0$ and two continuous functions $\beta_1(x_L)$ and $\beta_2(x_L)$ with $\beta_1(0) = \beta_2(0) = 0$ and satisfying



$$\beta_1(x_L) < \beta_2(x_L) < 0$$

for $-\delta < x_L < 0$, such that for the parameter sector defined by $\beta_1(x_L) < b < \beta_2(x_L)$ and $-\delta < x_L < 0$ the system has three limit cycles: two small nested crossing periodic orbits surrounding the attractive sliding set, and a third big limit cycle surrounding them.

Conclusions

- The study of all possible dynamics even in planar discontinuous PWL systems with two zones is a formidable challenge. Some headway is made in this problem thanks to a canonical form with fewer parameters.
- Thanks to the features of the boundary focus, we have shown how is possible to obtain three limit cycles, combining local and global bifurcations.
- A conjecture to prove or disprove: the maximum number of limit cycles to be found in the family is exactly three.

References

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- E. Freire, E. P. & F. Torres, A general mechanism to generate three limit cycles in planar Filippov systems with two zones, Nonlinear Dynamics 78 (2014) 251–263.
- E. Freire, E. P. & F. Torres, On the critical crossing cycle bifurcation in planar Filippov systems, J. Differential Equations 259 (2015) 7086–7107.

Superseding some previous work…

J. Math. Anal. Appl. 411 (2014) 83–94

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Concurrent homoclinic bifurcation and Hopf bifurcation for a class of planar Filippov systems ☆

Liping Li [a,*], Lihong Huang [b,c]

The planar Filippov system considered in this paper is modeled by

$$(\dot{u},\dot{v}) = \begin{cases} (a_0+\eta, b_1^+ u + b_2^+ u^2 + \varepsilon^+), & \text{if } v>0, \\ (-a_0 + a_1^- u + a_2^- v - \eta, b_1^- u + \varepsilon^-), & \text{if } v<0, \end{cases}$$

is shown that two limit cycles can appear from the two different kinds of bifurcation in planar Filippov systems.

Superseding some previous work…

Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **70**, 1–14; http://www.math.u-szeged.hu/ejqtde/

Three crossing limit cycles in planar piecewise linear systems with saddle-focus type

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Received 22 June 2014, appeared 4 January 2015

Abstract. This paper presents an analysis on the appearance of limit cycles in planar Filippov system with two linear subsystems separated by a straight line. Under the restriction that the orbits with points in the sliding and escaping regions are not considered, we provide firstly a topologically equivalent canonical form of saddle-focus dynamic with five parameters by using some convenient transformations of variables and parameters. Then, based on a very available fourth-order series expansion of the return map near an invisible parabolic type tangency point, we show that three crossing limit cycles surrounding the sliding set can be bifurcated from generic codimension-three singularities of planar discontinuous saddle-focus system. Our work improves and extends some existing results of other researchers.