PERSISTENCE OF PERIODIC SOLUTIONS IN NONSMOOTH SYSTEMS

1^{st} joint meeting Brazil–Spain in mathematics

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setting the problem

THE UNPERTURBED SYSTEM

Consider the following n–dimensional system

$$Z_0(\mathbf{x}, z) = \begin{cases} X_0^+(\mathbf{x}, z), & \text{if} \quad z > 0 \\ X_0^-(\mathbf{x}, z), & \text{if} \quad z < 0, \end{cases} \tag{1}$$

where $(\mathbf{x}, z) \in D \subset \mathbb{R}^{n-1} \times \mathbb{R}$, $X_0^\pm = (X_{0,1}^\pm, X_{0,2}^\pm, \ldots, X_{0,n}^\pm)$, and $\Sigma = \{z = 0\}$ is the switching hyperplane.

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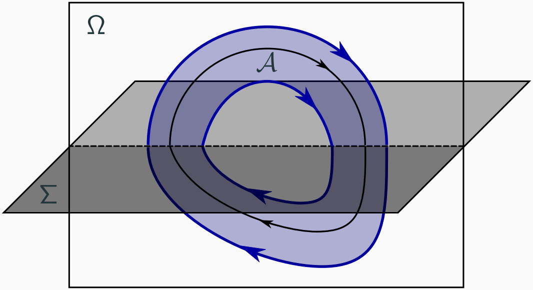

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HYPOTHESIS (H): The hyperplane $\Omega=\{x=0\}$ has a period annulus $\mathcal{A}=\{(y,z)\in\Omega:\ r_0\leq|(\mathbf{y},z)|\leq r_1\}$ surrounding the origin and fulfilled by crossing periodic solutions of the unperturbed system $(\dot{x},\dot{y},\dot{z})^T=Z(x,y,z;0)$.

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NOTATION:
$\varphi^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \left(\varphi_1^{\pm}(t, x, \mathbf{y}, z; \varepsilon), \varphi_2^{\pm}(t, x, \mathbf{y}, z; \varepsilon), \ldots, \varphi_n^{\pm}(t, x, \mathbf{y}, z; \varepsilon)\right)$
is the solutions of the systems $(\dot{x}, \dot{\mathbf{y}}, \dot{z})^T = Z^{\pm}(x, \mathbf{y}, z; \varepsilon)$ such that
$\varphi^{\pm}(0, x, \mathbf{y}, z; \varepsilon) = (x, \mathbf{y}, z)$.

PREVIOUS WORKS

- Z. Du, Y. Li and W. Zhang, *Bifurcation of periodic orbits in a class of planar Filippov systems*, Nonlinear Analysis **69** (2008), 3610–3628.
- X. Liu and M. Han, *Poincaré bifurcation of a three-dimensional system*, Chaos Soliton Fract **23** (2005), 1385–1398.

the melnikov–like function

THE MELNIKOV–LIKE FUNCTION

From hypotheses we find a neighbourhood $U \subset \mathbb{R}^n$ of $\mathcal{A}$ such that, for $|\varepsilon| \neq 0$ small enough, there exists a time $t^{\pm}(x, \mathbf{y}; \varepsilon) \lesseqgtr 0$ such that an orbit of (2) starting in $(x, \mathbf{y}, 0) \in U \cap \Sigma$ returns to Σ, that is $\varphi_n^{\pm}(t^{\pm}(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon) = 0$.

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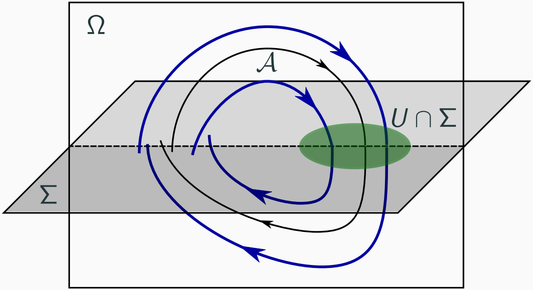

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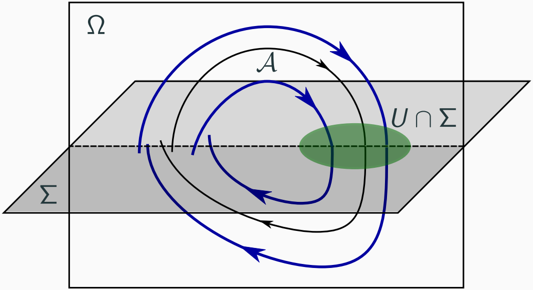


We may write

$$\varphi_i^{\pm}(t, x, \mathbf{y}, z; \varepsilon) = \psi_{0,i}^{\pm}(t, x, \mathbf{y}, z) + \varepsilon \psi_{1,i}^{\pm}(t, x, \mathbf{y}, z) + \mathcal{O}(\varepsilon^2), \quad i = 1, 2, \ldots, n,$$

$$t^{\pm}(x, \mathbf{y}; \varepsilon) = \tau_0^{\pm}(x, \mathbf{y}) + \varepsilon \tau_1^{\pm}(x, \mathbf{y}) + \mathcal{O}(\varepsilon^2).$$

$\psi_0^{\pm}(t, x, \mathbf{y}, z)$: solutions of $(\dot{x}, \dot{\mathbf{y}}, \dot{z})^T = Z^{\pm}(x, \mathbf{y}, z; 0)$.

THE MELNIKOV–LIKE FUNCTION

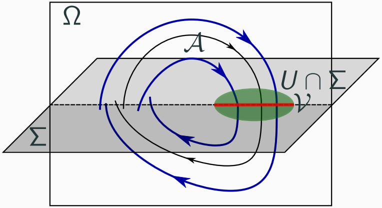
Proposition: *Let* $p = (x, \mathbf{y}, z) \in \mathbb{R} \times \mathbb{R}^{n-2} \times \mathbb{R}$*. Denote* $\sigma_j^{\pm}(\mathbf{y}) = \tau_j^{\pm}(0, \mathbf{y})$*,* $j = 0, 1$*, and* $Y^{\pm}(t, p_0) = D_p \psi_0^{\pm}(t, p_0)$ *the derivative of* $\psi_0^{\pm}(t, p)$ *with respect to the initial condition* p *evaluated at* $p_0 = (x_0, \mathbf{y}_0, z_0)$*. Then the following equalities hold:*

$$\psi_1^{\pm}(t, p) = Y^{\pm}(t, p) \int_0^t Y^{\pm}(s, p)^{-1} X_1^{\pm}\left(\psi_0^{\pm}(s, p)\right) ds,$$

$$\sigma_1^{\pm}(\mathbf{y}) = \tau^{\pm}(0, \mathbf{y}) = -\frac{\psi_{1,n}^{\pm}(\sigma_0^{\pm}(\mathbf{y}), 0, \mathbf{y}, 0)}{X_{0,n}^{\pm}\left(\psi_0^{\pm}(\sigma_0^{\pm}(\mathbf{y}), 0, \mathbf{y}, 0)\right)}.$$

THE MELNIKOV–LIKE FUNCTION

$\mathcal{V} = \{\nu \in \mathbb{R}^{n-2} : \; p_\nu = (0, \nu, 0) \in U\}.$



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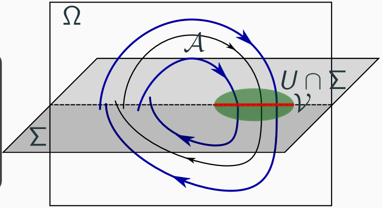
$$\mathcal{M} : \mathcal{V} \to \mathbb{R}^{n-2}$$
$$\mathcal{M}(\nu) = \Lambda(\nu) - \frac{\lambda_1(\nu)}{\omega_1(\nu)}\Omega(\nu),$$



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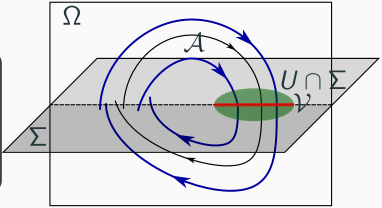

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$$\begin{aligned}\lambda_i(\nu) = \quad & X^+_{0,i}(\psi^+_0(\sigma^+_0(\nu), p_\nu))\sigma^+_1(\nu) - X^-_{0,i}(\psi^-_0(\sigma^-_0(\nu), p_\nu))\sigma^-_1(\nu) \\ & + \psi^+_{1,i}(\sigma^+_0(\nu), p_\nu) - \psi^-_{1,i}(\sigma^-_0(\nu), p_\nu),\end{aligned}$$

$$\begin{aligned}\omega_i(\nu) = \quad & \frac{\partial \psi^+_{0,i}}{\partial x}(\sigma^+_0(\nu), p_\nu) - \frac{X^+_{0,i}\big(\psi^+_0(\sigma^+_0(\nu), p_\nu)\big)}{X^+_{0,n}(\psi^+_0(\sigma^+_0(\nu), p_\nu))}\frac{\partial \psi^+_{0,n}}{\partial x}(\sigma^+_0(\nu), p_\nu) \\ & - \frac{\partial \psi^-_{0,i}}{\partial x}(\sigma^-_0(\nu), p_\nu) + \frac{X^-_{0,i}\big(\psi^-_0(\sigma^+_0(\nu), p_\nu)\big)}{X^-_{0,n}(\psi^-_0(\sigma^-_0(\nu), p_\nu))}\frac{\partial \psi^-_{0,n}}{\partial x}(\sigma^-_0(\nu), p_\nu),\end{aligned}$$

for $i = 1, 2, \ldots, n-1$, and

$$\Lambda(\nu) = \big(\lambda_2(\nu), \lambda_3(\nu), \ldots, \lambda_{n-1}(\nu)\big),$$

$$\Omega(\nu) = \big(\omega_2(\nu), \omega_3(\nu), \ldots, \omega_{n-1}(\nu)\big).$$

THE MELNIKOV–LIKE FUNCTION

Here $J_{\mathcal{M}}(\nu)$ denote the determinant of the Jacobian matrix of $\mathcal{M}$ evaluated at ν.

Theorem: *In addition to hypothesis* (H) *we assume that* $\omega_1(\nu) \neq 0$ *for every* $\nu \in \mathcal{V}$. *Then for each* $\nu^* \in \mathcal{V}$ *such that* $\mathcal{M}(\nu^*) = 0$ *and* $J_{\mathcal{M}}(\nu^*) \neq 0$, *there exists a unique crossing periodic solution* $\phi(t, \varepsilon)$ *of system* (2) *such that* $\phi(0, \varepsilon) \to (0, \nu^*, 0) \in \mathcal{A}$ *when* $\varepsilon \to 0$. *Moreover if* $\mathcal{M}(\nu) \neq 0$ *for every* $\nu \in \mathcal{V}$ *then there are no crossing periodic solutions bifurcating from* $\mathcal{A}$ *for* $|\varepsilon| \neq 0$ *sufficiently small.*

sketch of the proof

LYAPUNOV–SCHMIDT REDUCTION

Lemma: *Assume that $k \leq d$ are positive integers and denote by $\xi : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^k$ and $\xi^{\perp} : \mathbb{R}^k \times \mathbb{R}^{d-k} \to \mathbb{R}^{d-k}$ the projections onto the first k coordinates and onto the last $d-k$ coordinates, respectively. Let $\mathcal{D}$ and V be open bounded subsets of $\mathbb{R}^d$ and $\mathbb{R}^k$, respectively. Let g_0, g_1 and $\beta : \overline{V} \to \mathbb{R}^{d-k}$ be $\mathcal{C}^2$ functions, consider $g : \mathcal{D} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ as*

$$g(\zeta, \varepsilon) = g_0(\zeta) + \varepsilon g_1(\zeta) + \mathcal{O}(\varepsilon^2),$$

and take $\mathcal{Z} = \{\zeta_\nu = (\nu, \beta(\nu)) : \nu \in \overline{V}\} \subset \mathcal{D}$. We denote by Γ_ν the upper right corner $k \times (d-k)$ matrix of $D\, g_0(\zeta_\nu)$, and by Δ_ν the lower right corner $(d-k) \times (d-k)$ matrix of $D\, g_0(\zeta_\nu)$. Assume that for each $\zeta_\nu \in \mathcal{Z}$, $\det(\Delta_\nu) \neq 0$ and $g_0(\zeta_\nu) = 0$. We define the bifurcation function $f_1 : \overline{V} \to \mathbb{R}^k$ as

$$f_1(\nu) = -\Gamma_\nu \Delta_\nu^{-1} \xi^{\perp} g_1(\zeta_\nu) + \xi g_1(\zeta_\nu)$$

If there exists $\nu^* \in V$ such that $f_1(\nu^*) = 0$ and $J_{f_1}(\nu^*) \neq 0$, then there exists ν_ε such that $g(\zeta_{\nu_\varepsilon}, \varepsilon) = 0$ and $\zeta_{\nu_\varepsilon} \to \zeta_{\nu^*}$ when $\varepsilon \to 0$.

CLOSING MAP

For $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{n-2}$ such that $(x, \mathbf{y}, 0) \in U$ define

$$\delta(x, \mathbf{y}; \varepsilon) = \big(\delta_1(x, \mathbf{y}; \varepsilon),\ \delta_2(x, \mathbf{y}; \varepsilon), \ldots, \delta_{n-1}(x, \mathbf{y}; \varepsilon)\big) \text{ as}$$
$$\delta_i(x, \mathbf{y}; \varepsilon) = \varphi_i^+(t^+(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon) - \varphi_i^-(t^-(x, \mathbf{y}; \varepsilon), x, \mathbf{y}, 0; \varepsilon).$$

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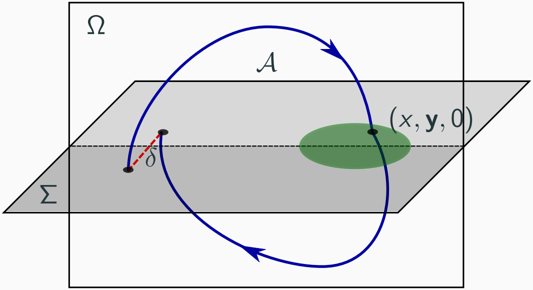

CLOSING MAP

$\delta(x_\varepsilon, \mathbf{y}_\varepsilon, \varepsilon) = 0$ for some $(x_\varepsilon, \mathbf{y}_\varepsilon, 0) \in U$ if and only if the solution of system (2) passing through $(x_\varepsilon, \mathbf{y}_\varepsilon, 0)$ is periodic.

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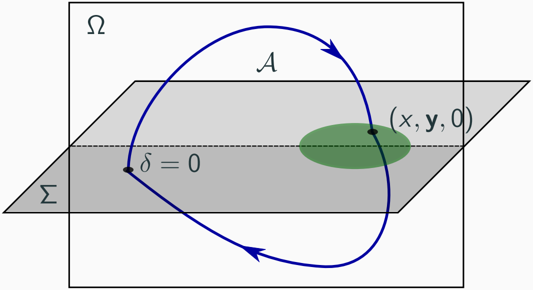

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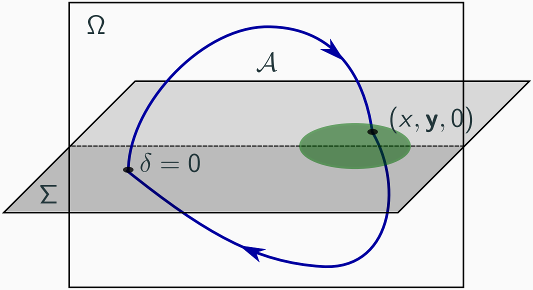
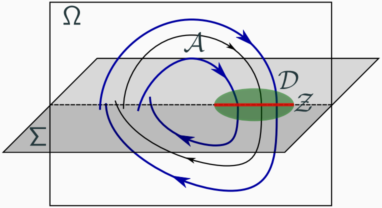


Note that $\delta_i(x, \mathbf{y}; \varepsilon) = \delta_i^0(x, \mathbf{y}) + \varepsilon\delta_i^1(x, \mathbf{y}) + \mathcal{O}(\varepsilon^2)$, where

$$\begin{aligned}
\delta_i^0(x, \mathbf{y}) = \quad & \psi_{0,i}^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0) - \psi_{0,i}^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0), \\
\delta_i^1(x, \mathbf{y}) = \quad & X_{0,i}^+(\psi_0^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0))\tau_1^+(x, \mathbf{y}) + \psi_{1,i}^+(\tau_0^+(x, \mathbf{y}), x, \mathbf{y}, 0) \\
& - X_{0,i}^-(\psi_0^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0))\tau_1^-(x, \mathbf{y}) - \psi_{1,i}^-(\tau_0^-(x, \mathbf{y}), x, \mathbf{y}, 0).
\end{aligned}$$

APPLYING THE LEMMA

$d = n - 1$, $k = n - 2$,
$\mathcal{D} = \{(\mathbf{a}, b) \in \mathbb{R}^{n-2} \times \mathbb{R} : (b, \mathbf{a}, 0) \in U\}$,
$V = \{\nu \in \mathcal{V} \subset \mathbb{R}^{n-2} :$
$r_0 < |\nu - \nu^*| < r_1\}$, $\beta = 0$, $\zeta_\nu = (\nu, 0)$;



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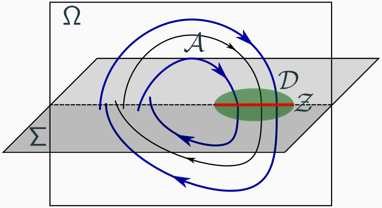

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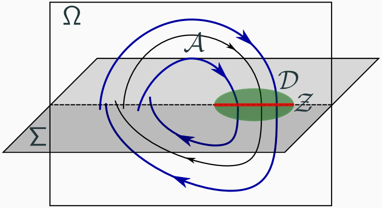

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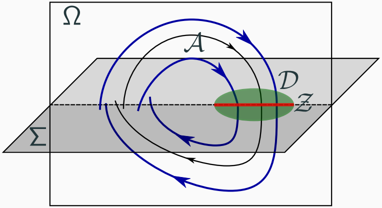

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$$g_1(\zeta_\nu) = \left(\Lambda(\nu), \lambda_1(\nu)\right)$$
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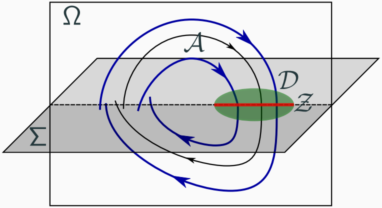

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$\underline{\delta}^j = (\delta_2^j, \delta_3^j, \dots, \delta_{n-1}^j)$ for $j = 0, 1$;

$$g_1(\zeta_\nu) = \big(\Lambda(\nu), \lambda_1(\nu)\big)$$
$$g_0(\zeta_\nu) = g_0(\nu, 0) = \big(\underline{\delta}^0(0, \nu), \delta_1^0(0, \nu)\big) = (0, 0).$$

Indeed the solution of the unperturbed system (2) passing through $(0, \nu, 0) \in \mathcal{A}$ is periodic, that is $\big(\delta_1^0(0, \nu), \underline{\delta}^0(0, \nu)\big) = \delta(0, \nu, 0) = (0, 0)$.

APPLYING THE LEMMA

Moreover

$$Dg_0(\nu,0)=\begin{pmatrix} \dfrac{\partial \underline{\delta}^0}{\partial y}(0,\nu) & \dfrac{\partial \underline{\delta}^0}{\partial x}(0,\nu) \\ \dfrac{\partial \delta_1^0}{\partial y}(0,\nu) & \dfrac{\partial \delta_1^0}{\partial x}(0,\nu) \end{pmatrix} = \begin{pmatrix} * & \Gamma_\nu \\ * & \Delta_\nu \end{pmatrix} = \begin{pmatrix} * & \Omega(\nu) \\ * & \omega_1(\nu) \end{pmatrix}$$

Indeed

$$\begin{aligned}\frac{\partial \delta_i^0}{\partial x}(0,\nu) = \quad & X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu),p_\nu)\big)\frac{\tau_0^+}{\partial x}(0,\nu) - X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu),p_\nu)\big)\frac{\tau_0^-}{\partial x}(0,\nu)\\ & + \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu),p_\nu) - \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu),p_\nu).\end{aligned}$$

APPLYING THE LEMMA

Moreover

$$Dg_0(\nu,0)=\begin{pmatrix}\dfrac{\partial\underline{\delta}^0}{\partial y}(0,\nu) & \dfrac{\partial\underline{\delta}^0}{\partial x}(0,\nu)\\[2ex] \dfrac{\partial\delta_1^0}{\partial y}(0,\nu) & \dfrac{\partial\delta_1^0}{\partial x}(0,\nu)\end{pmatrix}=\begin{pmatrix} * & \Gamma_\nu\\ * & \Delta_\nu\end{pmatrix}=\begin{pmatrix} * & \Omega(\nu)\\ * & \omega_1(\nu)\end{pmatrix}$$

Indeed

$$\begin{aligned}\frac{\partial\delta_i^0}{\partial x}(0,\nu)= \quad & X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu),p_\nu)\big)\frac{\tau_0^+}{\partial x}(0,\nu)-X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu),p_\nu)\big)\frac{\tau_0^-}{\partial x}(0,\nu)\\ &+\frac{\partial\psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu),p_\nu)-\frac{\partial\psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu),p_\nu).\end{aligned}$$

Computing $(\partial/\partial\varepsilon)\big(\varphi_n^\pm(t^\pm(x,y;\,\varepsilon),x,y,0;\varepsilon)\big)=0$ for $\varepsilon=0$ we get

$$\frac{\partial\tau_0^\pm}{\partial x}(0,\nu)=-\frac{\dfrac{\partial\psi_{0,n}^\pm}{\partial x}(\tau_0^\pm(0,\nu),p_\nu)}{X_{0,n}^\pm(\psi_0^\pm(\tau_0^\pm(0,\nu),p_\nu))}.$$

APPLYING THE LEMMA

Therefore

$$\begin{aligned}
\frac{\partial \delta_i^0}{\partial x}(0,\nu) = \quad & \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \frac{X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu), p_\nu)\big)}{X_{0,n}^+(\psi_0^+(\sigma_0^+(\nu), p_\nu))} \frac{\partial \psi_{0,n}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \\
& \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) + \frac{X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu), p_\nu, 0)\big)}{X_{0,n}^-(\psi_0^-(\sigma_0^-(\nu), p_\nu))} \frac{\partial \psi_{0,n}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) \\
= \quad & \omega_i(\nu), \quad \text{for} i = 1, 2, \ldots, n.
\end{aligned}$$

APPLYING THE LEMMA

Therefore

$$\begin{aligned}\frac{\partial \delta_i^0}{\partial x}(0,\nu) = \quad & \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu),p_\nu) - \frac{X_{0,i}^+\big(\psi_0^+(\sigma_0^+(\nu),p_\nu)\big)}{X_{0,n}^+(\psi_0^+(\sigma_0^+(\nu),p_\nu))}\frac{\partial \psi_{0,n}^+}{\partial x}(\sigma_0^+(\nu),p_\nu) - \\ & \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu),p_\nu) + \frac{X_{0,i}^-\big(\psi_0^-(\sigma_0^+(\nu),p_\nu,0)\big)}{X_{0,n}^-(\psi_0^-(\sigma_0^-(\nu),p_\nu))}\frac{\partial \psi_{0,n}^-}{\partial x}(\sigma_0^-(\nu),p_\nu) \\ = \quad & \omega_i(\nu), \quad \text{for} i = 1,2,\ldots,n.\end{aligned}$$

Now we compute the function f_1 of the Lemma:

APPLYING THE LEMMA

Therefore

$$
\begin{aligned}
\frac{\partial \delta_i^0}{\partial x}(0,\nu) = \quad & \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \frac{X_{0,i}^+(\psi_0^+(\sigma_0^+(\nu), p_\nu))}{X_{0,n}^+(\psi_0^+(\sigma_0^+(\nu), p_\nu))}\frac{\partial \psi_{0,n}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \\
& \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) + \frac{X_{0,i}^-(\psi_0^-(\sigma_0^+(\nu), p_\nu, 0))}{X_{0,n}^-(\psi_0^-(\sigma_0^-(\nu), p_\nu))}\frac{\partial \psi_{0,n}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) \\
= \quad & \omega_i(\nu), \quad \text{for} i = 1, 2, \dots, n.
\end{aligned}
$$

Now we compute the function f_1 of the Lemma:

$$
\begin{aligned}
f_1(\nu) = \quad & -\Gamma_\nu \Delta_\nu^{-1} \xi^\perp g_1(\zeta_\nu) + \xi g_1(\zeta_\nu) \\
= \quad & -\frac{\lambda_1(\nu)}{\omega_1(\nu)}\Omega(\nu) + \Lambda(\nu) = \mathcal{M}(\nu).
\end{aligned}
$$

APPLYING THE LEMMA

Therefore

$$
\begin{aligned}
\frac{\partial \delta_i^0}{\partial x}(0,\nu) = \quad & \frac{\partial \psi_{0,i}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \frac{X_{0,i}^+(\psi_0^+(\sigma_0^+(\nu), p_\nu))}{X_{0,n}^+(\psi_0^+(\sigma_0^+(\nu), p_\nu))} \frac{\partial \psi_{0,n}^+}{\partial x}(\sigma_0^+(\nu), p_\nu) - \\
& \frac{\partial \psi_{0,i}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) + \frac{X_{0,i}^-(\psi_0^-(\sigma_0^+(\nu), p_\nu, 0))}{X_{0,n}^-(\psi_0^-(\sigma_0^-(\nu), p_\nu))} \frac{\partial \psi_{0,n}^-}{\partial x}(\sigma_0^-(\nu), p_\nu) \\
= \quad & \omega_i(\nu), \quad \text{for} i = 1, 2, \ldots, n.
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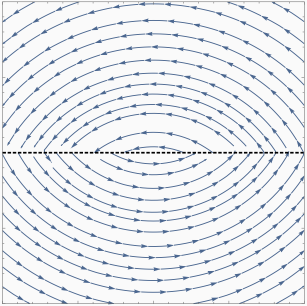
\end{aligned}
$$

Since $\omega_1(\nu) \neq 0$, the proof follows by applying the Lemma.

piecewise linear vector fields

CENTER–CENTER

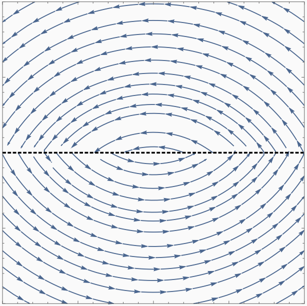
$(x, -z - \mathrm{sign}(z), x + y)$



$$\begin{cases} \dot{x} = x + \varepsilon\left(\alpha_0^+ + \alpha_2^+ y + \alpha_3^+ z\right) & \\ \dot{y} = -z - 1 + \varepsilon\beta_2^+ y & \text{if} \quad z > 0, \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^+ + \kappa_3^+ z\right) & \\ \dot{x} = x + \varepsilon\left(\alpha_0^- + \alpha_2^- y + \alpha_3^- z\right) & \\ \dot{y} = -z + 1 + \varepsilon\left(\beta_0^- + \beta_2^- y + \beta_3^- z\right) & \text{if} \quad z < 0. \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^- + \kappa_2^- y + \kappa_3^- z\right) & \end{cases}$$

CENTER–CENTER

$(x, -z - \mathrm{sign}(z), x + y)$



$$\begin{cases} \dot{x} = x + \varepsilon\left(\alpha_0^+ + \alpha_2^+ y + \alpha_3^+ z\right) \\ \dot{y} = -z - 1 + \varepsilon \beta_2^+ y & \text{if} \quad z > 0, \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^+ + \kappa_3^+ z\right) \\ \dot{x} = x + \varepsilon\left(\alpha_0^- + \alpha_2^- y + \alpha_3^- z\right) \\ \dot{y} = -z + 1 + \varepsilon\left(\beta_0^- + \beta_2^- y + \beta_3^- z\right) & \text{if} \quad z < 0. \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^- + \kappa_2^- y + \kappa_3^- z\right) \end{cases}$$

$$\psi_0^+(t, 0, \nu, 0) = \Big(0, \nu \cos t - \sin t, \nu \sin t + \cos t - 1\Big),$$

$$\psi_0^-(t, 0, \nu, 0) = \Big(0, \nu \cos t + \sin t, \nu \sin t - \cos t + 1\Big),$$

$$\sigma_0^\pm(t, 0, \nu, 0) = \pm 2 \arctan \nu$$

$$
\begin{aligned}
\omega_1(\nu) =& \, 2\sinh(2\arctan\nu),\\
\omega_2(\nu) =& \, \tfrac{1}{\nu} + \tfrac{e^{-2\arctan\nu}(\nu-1)}{2\nu} - \tfrac{e^{2\arctan\nu}(1+\nu)}{2\nu},\\
\lambda_1(\nu) =& \left(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(\nu-1)\right)\tfrac{e^{2\arctan\nu}}{2}\\
&+\left(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\right)\tfrac{e^{-2\arctan\nu}}{2}\\
&+\tfrac{1}{2}\left(-2\alpha_0^- - \alpha_0^- - \alpha_3^- - 2\alpha_0^+ + \alpha_2^+ + \alpha_3^+ + (\alpha_2^- + \alpha_3^- + \alpha_2^+ + \alpha_3^+)\nu\right),\\
\lambda_2(\nu) =& -\tfrac{\left(2\alpha_0^+ + (\alpha_2^+ + \alpha_3^+)(-1+\nu)\right)(1+\nu)}{4\nu}e^{2\arctan\nu}\\
&-\tfrac{(1+\nu)\left(2\alpha_0^- + (\alpha_2^- + \alpha_3^-)(1+\nu)\right)}{4\nu}e^{-2\arctan\nu}\\
&+\tfrac{1}{2\nu}\Big(-\alpha_2^- + \alpha_3^- - \alpha_2^+ + \alpha_3^+ - 2(\beta_2^- + \beta_2^+ + \kappa_3^- + \kappa_3^+)\\
&+2(\alpha_2^- + \alpha_3^- + 2\beta_3^- - 2\kappa_2^-)\nu\\
&+(\alpha_2^- - \alpha_3^- - \alpha_2^+ + \alpha_3^+ + 2(\beta_2^- - \beta_2^+ + \kappa_3^- - \kappa_3^+))\nu^2\Big)\arctan\nu\\
&+\tfrac{P(\nu)}{4(\nu+\nu^3)},
\end{aligned}
$$

where $P(\nu)$ is a polynomial of degree 4.

Now expanding the function $\mathcal{M}(\nu)$ in power series of ν we get

$$\mathcal{M}(\nu) = \sum_{i=0}^{8} C_i \nu^i + \mathcal{O}(\nu^9).$$

The parameters C_i, $i = 0, \dots 8$, are linear combinations of the parameters $\alpha_i^{\pm}$, β_i^{-}, κ_i^{-}, $i = 1, 2, 3$, β_2^{+}, κ_0^{+}, and κ_3^{+}. **Moreover they are linearly independent**.

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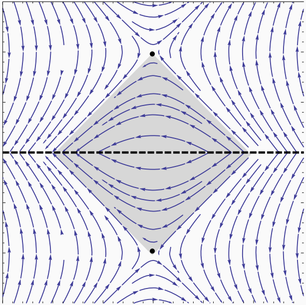
Proposition: *There exist parameters* $\alpha_i^{\pm}$, β_i^{-}, κ_i^{-}, $i = 1, 2, 3$, β_2^{+}, κ_0^{+}, *and* κ_3^{+} *such that the system admits, for* $|\varepsilon| \neq 0$ *small enough,* **at least** 8 **limit cycles** *converging, when* ε *goes to* 0*, to some of the periodic orbits contained in* $\{x = 0\}$.

SADDLE–SADDLE

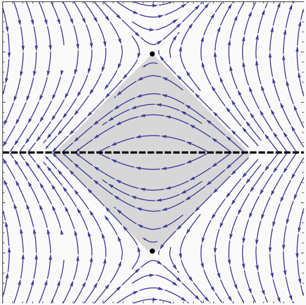
$(x, z - \mathrm{sign}(z), x + y)$



$$\begin{cases} \dot{x} = x + \varepsilon\left(\alpha_0^+ + \alpha_2^+ y + \alpha_3^+ z\right) & \\ \dot{y} = z - 1 + \varepsilon\beta_2^+ y & \text{if} \quad z > 0, \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^+ + \kappa_3^+ z\right) & \\ \dot{x} = x + \varepsilon\left(\alpha_0^- + \alpha_2^- y + \alpha_3^- z\right) & \\ \dot{y} = z + 1 + \varepsilon\left(\beta_0^- + \beta_2^- y + \beta_3^- z\right) & \text{if} \quad z < 0. \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^- + \kappa_2^- y + \kappa_3^- z\right) & \end{cases}$$

SADDLE–SADDLE

$(x, z - \operatorname{sign}(z), x + y)$



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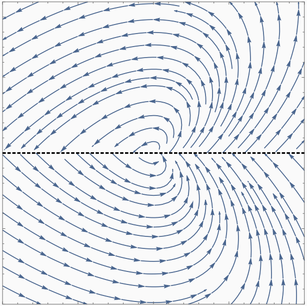
Proposition: *There exist parameters* $\alpha_i^\pm$, β_i^-, κ_i^-, $i = 1, 2, 3$, β_2^+, κ_0^+, *and* κ_3^+ *such that the system admits, for* $|\varepsilon| \neq 0$ *small enough,* **at least** 8 **limit cycles** *converging, when* ε *goes to* 0, *to some of the periodic orbits contained in* $\{x = 0\}$.

FOCUS–FOCUS

$$(x, -z + \mathrm{sign}(z)y, x + y)$$



$$\begin{cases} \dot{x} = x + \varepsilon\left(\alpha_0^+ + \alpha_1^+ x + \alpha_2^+ y + \alpha_3^+ z\right) \\ \dot{y} = -z + y + \varepsilon\left(\beta_0^+ + \beta_1^+ x + \beta_2^+ y + \beta_3^+ z\right) & \text{if} \quad z > 0, \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^+ + \kappa_1^+ x + \kappa_2^+ y + \kappa_3^+ z\right) \\ \\ \dot{x} = x + \varepsilon\left(\alpha_0^- + \alpha_1^- x + \alpha_2^- y + \alpha_3^- z\right) \\ \dot{y} = -z - y + \varepsilon\left(\beta_0^- + \beta_1^- x + \beta_2^- y + \beta_3^- z\right) & \text{if} \quad z < 0. \\ \dot{z} = x + y + \varepsilon\left(\kappa_0^- + \kappa_1^- x + \kappa_2^- y + \kappa_3^- z\right) \end{cases}$$

$$\psi_0^+(t,0,\nu,0) = \left(0, \nu e^{\frac{t}{2}}\left(\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right) + \cos\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right),$$

$$\psi_0^-(t,0,\nu,0) = \left(0, \nu e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{-\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right),$$

$$\sigma_0^\pm(t,0,\nu,0) = \pm\frac{2\pi}{\sqrt{3}}.$$

$$\psi_0^+(t,0,\nu,0) = \left(0, \nu e^{\frac{t}{2}}\left(\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right) + \cos\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right),$$

$$\psi_0^-(t,0,\nu,0) = \left(0, \nu e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}t}{2}\right) - \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}t}{2}\right)\right), \frac{2\nu e^{-\frac{t}{2}}}{\sqrt{3}}\sin\left(\frac{\sqrt{3}t}{2}\right)\right),$$

$$\sigma_0^\pm(t,0,\nu,0) = \pm\frac{2\pi}{\sqrt{3}}.$$

$$\boxed{\mathcal{M}(\nu) = \frac{1+e^{\frac{\pi}{\sqrt{3}}}}{2}A + \left(\frac{e^{\frac{\pi}{\sqrt{3}}}}{3\sqrt{3}}B\right)\nu}$$

$$\begin{aligned}
A \quad &= \alpha_0^-\left(3 + 4e^{-\frac{2\pi}{\sqrt{3}}}\left(e^{\frac{\pi}{\sqrt{3}}} - 1\right) - 2\,\mathrm{sech}\left(\frac{\pi}{\sqrt{3}}\right)\right) \\
&+\alpha_0^+ - 2\alpha_0^+\,\mathrm{sech}\left(\frac{\pi}{\sqrt{3}}\right) - 3(\beta_0^- + \beta_0^+ + 2\kappa_0^-) \quad \text{and} \\
B \quad &= \frac{2(\alpha_2^- + \alpha_3^- + 3(\alpha_2^+ + \alpha_3^+))}{\sqrt{3}\left(e^{\frac{\pi}{\sqrt{3}}} - 1\right)} + \frac{\left(\tanh\left(\frac{\pi}{\sqrt{3}}\right) - 1\right)(\alpha_2^- + \alpha_3^- + 3(\alpha_2^+ + \alpha_3^+))}{\sqrt{3}} \\
&- \frac{2e^{-\sqrt{3}\pi}(\alpha_2^- + \alpha_3^-)}{\sqrt{3}} + 2\pi\alpha_2^- - \pi\alpha_3^- + 2\sqrt{3}\alpha_2^+ - 2\pi\alpha_2^+ + 2\sqrt{3}\alpha_3^+ - \pi\alpha_3^+ \\
&+3\pi\beta_3^- - 2\pi\beta_2^+ - \pi\beta_3^+ - 3\pi\kappa_2^- + 3\pi\kappa_3^- + \pi\kappa_2^+ - \pi\kappa_3^+.
\end{aligned}$$

FOCUS–FOCUS

Proposition: *If $B \neq 0$ then, for $|\varepsilon| \neq 0$ small enough, there is a unique crossing periodic solution $\phi(t, \varepsilon)$ bifurcating from periodic orbits contained in $\{x = 0\}$, such that*

$$\phi(0, \varepsilon) \to \left(0\,,\, -\frac{\sqrt{3}\left(1 + e^{\frac{\pi}{\sqrt{3}}}\right) A}{2B}\,,\, 0\right), \quad \textit{when} \quad \varepsilon \to 0,$$

Moreover if $A \neq 0$ and $B = 0$ then there are no crossing periodic solutions bifurcating from periodic orbits contained in $\{x = 0\}$.

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Thank you!