On the existence of limit cycles and invariant surfaces of sewing piecewise linear differential systems on $\mathbb{R}^3$

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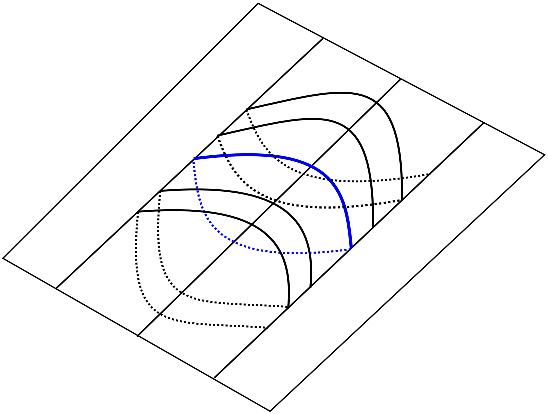
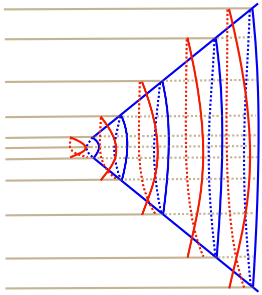
In this work, we consider a class of discontinuous piecewise linear differential systems in $\mathbb{R}^3$ with two pieces separated by a plane and we investigate the existence of limit cycles and invariant surfaces.

In this way, we give conditions for the existence of differential systems having:

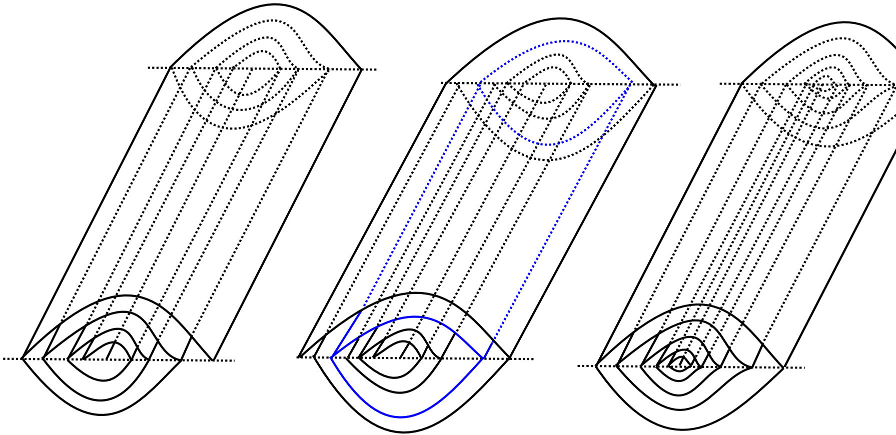
A unique limit cycle.



A unique one-parameter family of periodic orbits.



Scrolls, a unique invariant cylinder or infinitely many invariant cylinders.



Notes

We note that the existence of the one-parameter family of periodic orbits is an analogous result to the Lyapunov Center Theorem related to smooth vector fields. See, for instance (Buzzi and Lamb, [2]), (Gouzé and Sari, [8]), (Jacquemard and Teixeira [9]) and, (Li and Shi, [11]).

This work is an extension of (Medrado and Torregrosa, [14]) and in our approach we use essentially the Theorem of Rolle for dynamical systems (see (Khovanskii, [10])) to address the problem of to show the existence of limit cycles or invariant surfaces to find zeroes of intersection of algebraic curves.

About this thema I

The discontinuous piecewise linear differential systems plays an important role inside the theory of nonlinear dynamical systems. In the models of physical problems or processes is natural to use the piecewise-smooth dynamical systems when their motion is characterized by smooth flow and eventually, interrupted by instantaneous events (see (Brogliato, [1]), (Jong and Gouzé,[4]), (Gouzé and Tari, [7])).

About this thema II

There are many non-smooth processes in this context, for example, impact, switching, sliding and other discrete state transitions. They are used also in nonlinear engineering models, where certain devices are accurately modeled by them, see for instance (di Bernardo et al, [5]), (Makarenkov and Lamb, [13]), (Ponce, Ros and Vela, [17]) and, references quoted in these.

About this thema III

In (Ponce et al, [3]), (Ponce, Ros and Vela, [15] and [16]), the authors consider a family of continuous piecewise linear systems in $\mathbb{R}^3$ and characterize limit cycles and cones foliated by periodic orbits. In (Lima and Llibre, [12]) is proved the existence of limit cycles and invariant cylinders for a class of discontinuous vector field in dimension $2n$.

In this work, we deal with piecewise linear vector field

$$Z(\mathrm{x}) = \begin{cases} X^+(x) = A^+(\mathrm{x}) + B^+, & \text{if} \quad x \in \Sigma^+ \subset \mathbb{R}^3, \\ X^-(x) = A^-(\mathrm{x}) + B^-, & \text{if} \quad x \in \Sigma^- \subset \mathbb{R}^3, \end{cases} \tag{1}$$

where $A^{\pm}, B^{\pm}$ are matrices, $\Sigma = h^{-1}(0)$ with $h(x_1, x_2, x_3) = x_3$ and $\Sigma^{\pm} = \pm h > 0$. We observe that $\mathbb{R}^3 = \Sigma \cup \Sigma^+ \cup \Sigma^-$.

Sewing Piecewise Linear Differential Systems

We consider $\Sigma = \Sigma_S \cup \Sigma_T$ where

$$\text{Sewing set} : \Sigma_S = \{p \in \Sigma; X^+h(p)X^-h(p) > 0, \}$$

$$\text{Tangency set} : \Sigma_T = \{p \in \Sigma; X^{\pm}h(p) = 0 \text{ and } \pm(X^{\pm})^2 h(p) < 0\}$$

Tangency straight lines

Lemma

Let $Z = (X^+, X^-)$ be defined in (1) with $L_{X^\pm}$ the tangency straight lines of $X^\pm$. If $X^+h(p)X^-h(p) \geq 0$, for all $p \in \Sigma$ then the tangency straight lines are the same, i.e., $L_{X^+} \equiv L_{X^-}$.

Canonical form of $Z = (X^+, X^-)$

$$\begin{aligned} X^+ &= (a^+x + b^+z, c^+y + d^+z - 1, y), \\ X^- &= (a^-x + b^-z + m, c^-y + d^-z + 1, y). \end{aligned} \tag{2}$$

The associated eigenvalues of X^+ and X^- are $\lambda_1^\pm = a^\pm$,

$\lambda_2^\pm = (c^\pm + \sqrt{(c^\pm)^2 + 4d^\pm})/2$ and, $\lambda_3^\pm = (c^\pm - \sqrt{(c^\pm)^2 + 4d^\pm})/2$.

We define seven types:

(i) Sa If $\lambda_2^\pm \lambda_3^\pm < 0$.

(ii) No If $\lambda_2^\pm, \lambda_3^\pm \in \mathbb{R}$ and $\lambda_2^\pm \lambda_3^\pm > 0$.

(iii) Nd If $\lambda_2^\pm = \lambda_3^\pm$.

(iv) Fo If $\lambda_2^\pm, \lambda_3^\pm \in \mathbb{C}$ and $\mathrm{Re}(\lambda_2^\pm)\,\mathrm{Im}(\lambda_2^\pm) \neq 0$.

(v) Ce If $\lambda_2^\pm, \lambda_3^\pm \in \mathbb{C}$, $\mathrm{Re}(\lambda_2^\pm) = 0$ and, $\mathrm{Im}(\lambda_2^\pm) \neq 0$.

(vi) D_1 If $\lambda_2^\pm \lambda_3^\pm = 0$ and $(\lambda_2^\pm)^2 + (\lambda_3^\pm)^2 \neq 0$.

(vii) D_2 If $(\lambda_2^\pm)^2 + (\lambda_3^\pm)^2 = 0$.

Definition

We say that the piecewise vector field $Z = (X^+, X^-)$ is of type (T^+, T^-) for $T^\pm \in \{\text{Sa, No, Nd, Fo, Ce, D}_1\text{, D}_2\}$, if $X^\pm$ is of type $T^\pm$.

We observe that the type (T^+, T^-) is equal to (T^-, T^+) i.e., there is an equivalence between (T^+, T^-) and (T^-, T^+), for details see [6].

Let $Z = (X^+, X^-)$ be a piecewise linear vector field of type (T^+, T^-). The following statements hold.

1. The vector field Z has scrolls if only if is true one of the following conditions:
   1. $T^+ =$ Sa and $T^- \in \{$Sa, No, Nd, Fo, Ce, D$_1\}$;
   2. $T^+ =$ No and $T^- \in \{$No, Nd, Fo, Ce, D$_1$, D$_2\}$;
   3. $T^+ =$ Nd and $T^- \in \{$Nd, Fo, D$_1$, D$_2\}$;
   4. $T^+ =$ Fo and $T^- =$ D$_1$;
   5. $T^+ =$ D$_1$ and $T^- \in \{$D$_1$,D$_2\}$;

   with $\kappa^2 + \lambda^2 \neq 0$ and $\kappa\lambda \geq 0$, or $1 + \alpha^2\lambda/\kappa \leq 0$ and $\kappa\lambda < 0$.

2. The vector field Z has at most a unique invariant cylinder if only if is true one of the following conditions:
    1. $T^+ =$ Sa and $T^- \in \{$Sa, No, Nd, Fo, Ce, D_1, $D_2\}$;
    2. $T^+ =$ No and $T^- \in \{$No, Nd, Fo, Ce, $D_1\}$;
    3. $T^+ =$ Nd and $T^- \in \{$Nd, Fo, Ce, $D_1\}$;
    4. $T^+ =$ Fo and $T^- \in \{$Fo, Ce, D_1, $D_2\}$;
    5. $T^+ =$ Ce and $T^- = D_1$;

    with $\kappa\lambda < 0$ and $1 + \alpha^2\lambda/\kappa > 0$.

3. The vector field Z has infinitely many invariant cylinders if only if $\kappa = \lambda = 0$ and is true one of the following conditions:
   1. $T^+ =$ Sa and $T^- \in \{$Sa, No, Nd, Fo, Ce, $D_1\}$;
   2. $T^+ =$ No and $T^- \in \{$No, Nd, Fo, Ce, $D_1\}$;
   3. $T^+ =$ Nd and $T^- \in \{$Nd, Fo, $D_1\}$;
   4. $T^+ =$ Ce and $T^- \in \{$Ce, $D_2\}$;
   5. $T^+ = D_1$ and $T^- = D_1$;
   6. $T^+ = D_2$ and $T^- = D_2$.

The parameters κ and λ depend on the parameters $a^{\pm}, b^{\pm}, c^{\pm}, d^{\pm}$ and m of Z. These parameters are given in the Tables.

Main results – Theorem B.

Let $Z = (X^+, X^-)$ be a piecewise linear vector field with X^+ and X^- defined in (2). The following statements hold.

1. If $(a^+)^2 + (a^-)^2 = 0$ or if Z has no invariant cylinder then there are not limit cycles.
2. If $(a^+)^2 + (a^-)^2 \neq 0$ and Z has at most a unique invariant cylinder then Z has at most a unique limit cycle in this cylinder.
3. If $(a^+)^2 + (a^-)^2 \neq 0$, $a^+a^- \geq 0$ and Z has infinitely many invariant cylinders then there is an invariant surface formed of periodic orbits, where each periodic orbit is contained in an invariant cylinder.

Proposition

Consider the boundary value problem $\dot{\mathrm{x}} = Y^{\pm}(\mathrm{x}) = P\,\mathrm{x} + Q^{\pm}$ *with* $p_0 = (x(0), y(0), z(0)) = (x_0, y_0, 0)$,

$$P = \begin{pmatrix} \gamma & 0 & \delta \\ 0 & \sigma & \psi \\ 0 & 1 & 0 \end{pmatrix} \text{ and } Q^{\pm} = \begin{pmatrix} M \\ \pm 1 \\ 0 \end{pmatrix},$$

where $\gamma, \delta, \sigma, \psi, M \in \mathbb{R}$.
Let $\varphi^{\pm}(t, p_0)$ *be the solutions of* $\dot{\mathrm{x}} = Y^{\pm}(\mathrm{x})$ *and consider the straight line* $r_0 = \{(x, y, z) \in \mathbb{R}^3 : y = y_0, z = 0\}$. *Let* $\tau^{\pm} \in \mathbb{R}/\{0\}$ *such that* $z(\tau^{\pm}) = 0$ *then* $\varphi^{\pm}(\tau^{\pm}, r_0)$ *is a straight line parallel to* r_0 *given by* $r_1 = \{(x, y, z) \in \mathbb{R}^3 : y = y_1, z = 0\}$.

Proof.

The general solution $\varphi^{\pm}(t,(x_0,y_0,0))$ is

$$\mathrm{e}^{Pt}\,\mathrm{x}_0 + \mathrm{e}^{Pt}\int_0^t \mathrm{e}^{-P\eta}\,Q d\eta.$$

Observe that the matrix e^{Pt} has zeroes at positions $(2,1)$ and $(3,1)$. So, we can write the solution $\varphi^{\pm}(t,p_0)$ by

$$\begin{array}{rl} x^{\pm}(t) = & \mathrm{e}^{\gamma t}\,x_0 + f_{12}^{\pm}y_0 + f_{13}^{\pm}, \\ y^{\pm}(t) = & f_{22}^{\pm}y_0 + f_{23}^{\pm}, \\ z^{\pm}(t) = & f_{32}^{\pm}y_0 + f_{33}^{\pm}, \end{array} \tag{3}$$

where $f_{ij}^{\pm} = f_{ij}^{\pm}(t,\gamma,\delta,\sigma,\psi,M)$, for $i,j = 1,2,3$.
Now, as the Poincaré Application is well defined, there is a $\tau^{\pm}(y_0)$ such that $z^{\pm}(\tau^{\pm}(y_0)) = 0$. Then $y_1 = y^{\pm}(\tau^{\pm}(y_0))$ depends only of y_0. This implies that all orbits of $Y^{\pm}$ with origin at r_0 intersect $\Sigma = \{z = 0\}$ after time $\tau^{\pm}(y_0)$, i.e., $\varphi^{\pm}(\tau^{\pm}(y_0), r_0)$ is the straight line r_1. $\square$

Invariant cylinders I

Corollary

Consider the boundary value problems

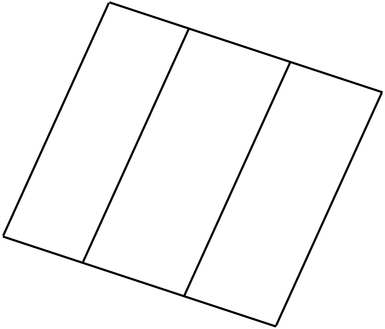
$$(A):\begin{cases}\dot{x} = X^{+}(x),\\ x(0) = (x_0, y_0, 0),\\ x(\tau) = (x_1, y_1, 0),\end{cases} \qquad (B):\begin{cases}\dot{x} = X^{-}(x),\\ x(0) = (\widetilde{x}_1, \widetilde{y}_1, 0),\\ x(\overline{\tau}) = (\widetilde{x}_0, \widetilde{y}_0, 0).\end{cases} \qquad (4)$$

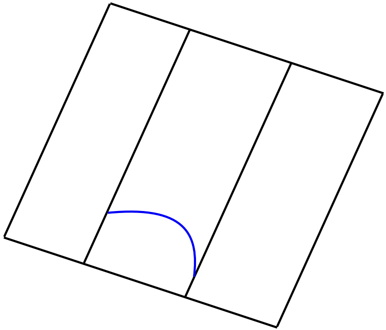
where X^{+} and X^{-} are given by (2)*. If $y_0 = \widetilde{y}_0$ and $y_1 = \widetilde{y}_1$ then there is an invariant cylinder for the vector field $Z = (X^{+}, X^{-})$.*

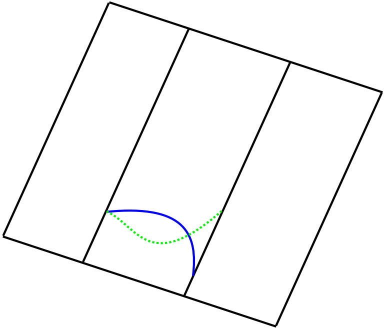
Invariant cylinders II

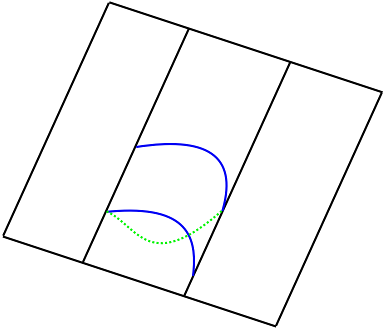
Proof.

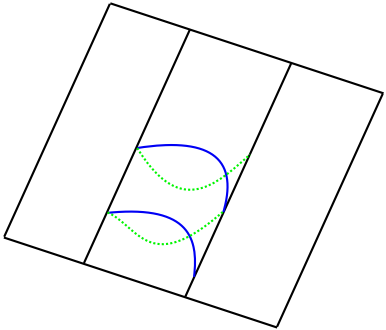
Let $\varphi^{\pm}(t,p)$ be the solutions of (A) and (B) respectively and the straight lines $r_0 = \{(x,y,z) \in \mathbb{R}^3 : y = y_0, z = 0\}$ and $r_1 = \{(x,y,z) \in \mathbb{R}^3 : y = y_1, z = 0\}$. From Proposition 0.1, we have that $\varphi^{+}(r_0,\tau) = r_1$ and $\varphi^{-}(r_1,\overline{\tau}) = r_0$. So, we obtain an invariant cylinder. $\square$

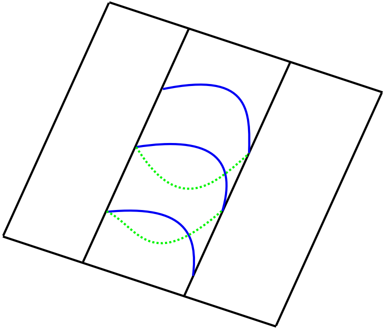


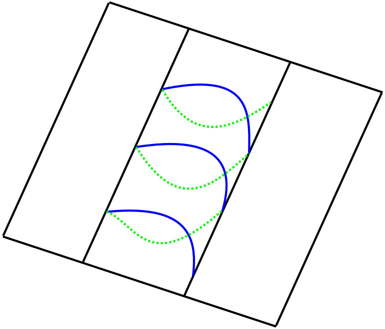


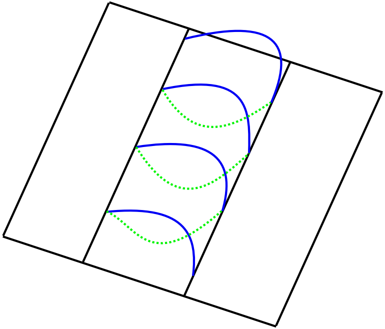


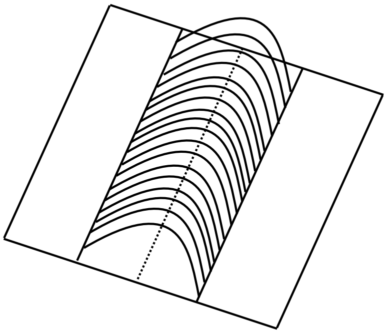


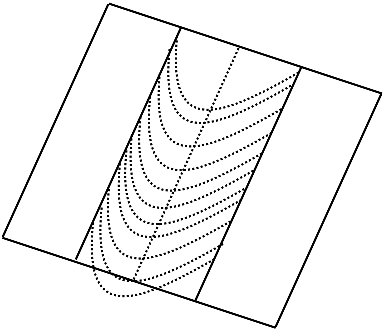












In order to prove these theorems, when the return applications is defined, we make a suitable substitution of variables and we address the proof to determine intersection points of curves which are associated to existence of invariant cylinders. For to determine the number of intersection points between these curves, we use also Theorem 0.3 proved by Kovanskii([10]).

Theorem A - Sketch of the proof for the case (Sa, Sa). II

Theorem

(Kovanskii, [10]) *Let X be a C^1 planar vector field without singular points in an open region $\Omega \subset \mathbb{R}^2$. If a C^1 curve, $\gamma \subset \Omega$, intersects an integral curve of X at two points then in between these points, there exists a point of tangency between γ and X.*

We consider X^+ of type Sa and we solve it using the conditions:

$$\mathrm{x}(0) = (x_0, y_0, 0) \text{ and } \mathrm{x}(\tau) = (x_1, y_1, 0).$$

Doing $(\rho, v, w) = (\mathrm{e}^{a^+\tau}, \mathrm{e}^{-(c^+ + s)\tau/2}, \mathrm{e}^{(c^+ - s)\tau/2})$ then we get $v = \rho^{\alpha_1}$, $w = \rho^{\alpha_2}$ and $w = v^{\alpha}$, where $\alpha_1 = -(c^+ + s)/2a^+$, $\alpha_2 = (c^+ - s)/2a^+$, $s = \sqrt{(c^+)^2 + 4d^+}$, $\alpha = (s - c^+)/(s + c^+)$. Moreover,

$$w = \frac{c^+ y_1 - s y_1 - 2}{c^+ y_0 - s y_0 - 2}, \quad v = \frac{c^+ y_0 + s y_0 - 2}{c^+ y_1 + s y_1 - 2},$$

$$\rho = \frac{4(a^+)^3 x_1 - 4(a^+)^2 c^+ x_1 + a^+ (c^+)^2 x_1 - a^+ s^2 x_1 + 4a^+ b^+ y_1 - 4b^+}{4(a^+)^3 x_0 - 4(a^+)^2 c^+ x_0 + a^+ (c^+)^2 x_0 - a^+ s^2 x_0 + 4a^+ b^+ y_0 - 4b^+}. \tag{5}$$

Doing the same to X^-, we get that doing

$$(\xi, V, W) = (e^{a^- \overline{\tau}}, e^{-(c^- + S)\overline{\tau}/2}, e^{-(-c^- + S)\overline{\tau}/2})$$

then we get $V = \xi^{\beta_1}, W = \xi^{\beta_2}$ and $W = V^{\beta}$, where $\beta_1 = -(c^- + S)/2a^-$, $\beta_2 = (c^- - S)/2a^-, S = \sqrt{(c^-)^2 + 4d^-}$, $\beta = (S - c^-)/(S + c^-)$. Moreover,

$$W = \frac{S\widetilde{y}_0 - c^- \widetilde{y}_0 - 2}{S\widetilde{y}_1 - c^- \widetilde{y}_1 - 2}, \quad V = \frac{S\widetilde{y}_1 + c^- \widetilde{y}_1 + 2}{S\widetilde{y}_0 + c^- \widetilde{y}_0 + 2},$$

$$\xi = \frac{(S^2+4a^-(c^- - a^-))m + (S^2 + c^-(4a^- - c^-) - 4(a^-)^2)a^- \widetilde{x}_0 - 4b^-(a^- \widetilde{y}_0 + 1)}{(S^2+4a^-(c^- - a^-))m + (S^2 + c^-(4a^- - c^-) - 4(a^-)^2)a^- \widetilde{x}_1 - 4b^-(a^- \widetilde{y}_1 + 1)}. \tag{6}$$

From the boundary value problem, follows that

$$\alpha > 1, \;\; 0 < v, w < 1 \text{ and } w = v^{\alpha}.$$

Expliciting x_0, y_0, y_1 in (5) we get

$$y_0 = -\frac{(-1+\alpha)(\alpha v - vw - \alpha + v)}{c^+(vw-1)\alpha},$$

$$y_1 = -\frac{(-1+\alpha)(\alpha vw - \alpha w - w + 1)}{c^+(vw-1)\alpha},$$

and $x_1 = \rho x_0 + B$, where

$$B = \frac{4b^+(a^+\rho y_0 - a^+ y_1 - \rho + 1)}{a^+(4(a^+)^2 - 4a^+c^+ + (c^+)^2 - s^2)}.$$

For X^- expliciting $\widetilde{x}_1$ in (6), we get

$$W = \frac{S\widetilde{y}_0 - c^-\widetilde{y}_0 - 2}{S\widetilde{y}_1 - c^-\widetilde{y}_1 - 2},$$

$$V = \frac{S\widetilde{y}_1 + c^-\widetilde{y}_1 + 2}{S\widetilde{y}_0 + c^-\widetilde{y}_0 + 2} \text{ and } \widetilde{x}_1 = \frac{1}{\xi}\widetilde{x}_0 + C,$$

where C is

$$\frac{4a^-b^-(\xi\widetilde{y}_1 - \widetilde{y}_0) + (\xi - 1)m(4(a^-)^2 - 4c^+4b^- - a^- - (S^2 - (c^-)^2))}{a^-\xi(c^- - 2a^- + S)(-c^- + 2a^- + S)}.$$

Substituting $\widetilde{y}_0 = y_0$ and $\widetilde{y}_1 = y_1$ in the expressions of V, W, we consider in the region $\Delta = (0,1) \times (0,1)$ contained in the plane vw, the curves

$$\begin{aligned} C_f = & \quad \{(v,w) \in \Delta; f(v,w) = w - v^{\alpha} = 0\}, \\ C_F = & \quad \{(v,w) \in \Delta; F(v,w) = W - V^{\beta} = 0\}, \end{aligned} \tag{7}$$

with

$$V = \frac{\Gamma_1 vw + \Gamma_2 w + \Gamma_3}{\Gamma_4 vw + \Gamma_5 v + \Gamma_6} \text{ e } W = \frac{\Gamma_7 vw + \Gamma_8 v + \Gamma_9}{\Gamma_{10} vw + \Gamma_{11} w + \Gamma_{12}}$$

where $\Gamma_i = \Gamma_i(\alpha, \beta, c^+, c^-)$.

Theorem A - Sketch of the proof for the case (Sa, Sa). VIII

Thus, for each point of intersection of the curves C_f and C_F, the piecewise linear vector field Z has an invariant cylinder. Note that the curve C_F does not depend of a^+ or a^-. Consequently, when $a^+a^- = 0$ the number of invariant cilinders is the same.

Let $\widetilde{X} = (v, \alpha w)$ be a vector field defined in Δ. So, C_f is an integral curve of $\widetilde{X}$. Consider the following system

$$\{F(v,w) = 0, \nabla F(v,w) \cdot \widetilde{X} = 0\}. \tag{8}$$

We get $\nabla F(v,w) \cdot \widetilde{X} = f_1(v,w) f_2(v,w)$ where

$$f_1(v,w) = \frac{y_0 \widehat{y_0} y_1 \widehat{y_1} (\beta+1)(\alpha+1)\beta(c^-)^2}{\widehat{D}},$$

$$f_2(v,w) = \kappa w (v-1)^2 + \lambda v (w-1)^2,$$

with

$$\kappa = \alpha^2(\alpha c^- + \beta c^+ - c^+ - c^-)(\alpha\beta c^- - \beta c^+ - \beta c^- + c^+),$$

$$\lambda = (\alpha\beta c^+ - \alpha c^+ - \alpha c^- + c^-)(\alpha\beta c^+ + \alpha\beta c^- - \alpha c^+ - \beta c^-).$$

The system (8) is equivalent to $\{F(v,w)=0, f_2(v,w)=0\}$. We consider

$$C_{f_2} = \{(v,w) \in \Delta; f_2(v,w) = 0.\}$$

Thus $(v,w) \in \Delta \cap C_f \cap C_{f_2}$ if and only if (v,w) satisfies

$$\{w = v^\alpha, \kappa w(v-1)^2 + \lambda v(w-1)^2 = 0\},$$

or

$$\frac{\lambda v(v^\alpha - 1)^2}{\kappa v^\alpha (v-1)^2} + 1 = 0.$$

This equation admits one zero for $v \in (0,1)$ if $1 + \alpha^2\lambda/\kappa > 0$, otherwise it does not admit zeros in $(0,1)$.

Proof of statement (1) **of Theorem A**. $\kappa^2 + \lambda^2 \neq 0$ and $\kappa\lambda = 0$.

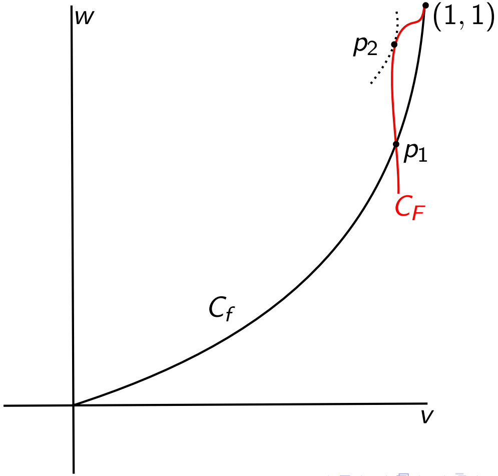
Suppose that $\kappa \neq 0$ and $\lambda = 0$. So,

$$f_2 = \kappa w(v-1)^2.$$

Assume that C_f and C_F intersect at a point p_1 in Δ. Follows from Khovanskii that there is $p_2 \in \Delta$ that is a solution of $\{F = 0, f_2 = 0\}$, but this is a contradiction since $f_2 \neq 0$ in Δ.

Therefore there are no invariant cylinders and consequently there are no periodic orbits, i.e., the differential system has a scroll.

Theorem A - Sketch of the proof for the case (Sa, Sa). XII



Proof of statement (2) of Theorem A: (Sa, Sa). I

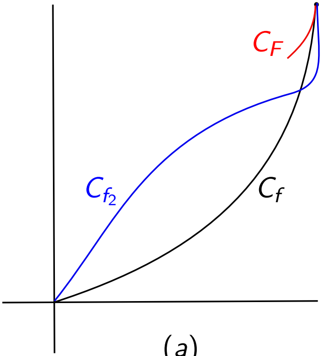
Now, $\kappa\lambda < 0$ and $1 + \alpha^2\lambda/\kappa > 0$.
Assume that C_f and C_F intersect at two points p_1 and p_2 in Δ.
From Khovanskii there are $p_3, p_4 \in \Delta$ which are solutions of

$$\{F = 0, f_2 = 0\},$$

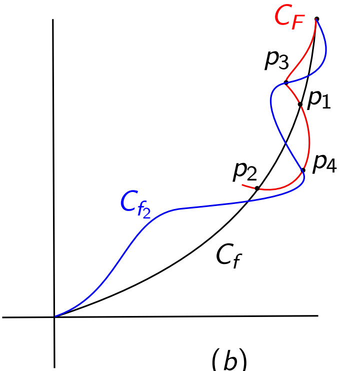
i.e., $p_3, p_4 \in C_{f_2}$, but this is a contradiction since that C_{f_2} intersects C_f at most at one point.

So, Z has at most a unique invariant cylinder.

Proof of statement (2) of Theorem A: $(\mathrm{Sa}, \mathrm{Sa})$. II



(a)



(b)

Finally, if $\kappa = \lambda = 0$, then

$$\nabla F(v, w) \cdot \widetilde{X} \equiv 0$$

and thus the curves C_f and C_F are coincident. In this case there is a continuous of invariant cylinders.

Theorem B (Sa, Sa): Sketch of the proof. I

1. If Z has no invariant cylinder then do not exist limit cycles.
2. As C_F does not depend of a^+ or a^-, the number of invariant cylinders remains the same, independently of the configuration of a^+ and a^-.
3. From boundary value problems (4) (X^+ and X^-) and from (3) of the Proposition 0.1, we obtain respectively

$$x_1 = x_0 + \eta y_0 + \mu \text{ and } \widetilde{x}_1 = \widetilde{x}_0 + \widetilde{\eta}\widetilde{y}_1 + \widetilde{\mu}.$$

Here, we obtain (η, μ) and $(\widetilde{\eta}, \widetilde{\mu})$ directly from Proposition 0.1 replacing

$$(\gamma, \delta, M, \sigma, \psi, \pm 1) = (a^+, b^+, 0, c^+, d^+, -1) \text{ and}$$

$$(\gamma, \delta, M, \sigma, \psi, \pm 1) = (a^-, b^-, m, c^-, d^-, 1)$$

Suppose that $(a^+)^2 + (a^-)^2 = 0$.

Fixing an invariant cylinder $\widetilde{y}_0 = y_0$ and $\widetilde{y}_1 = y_1$, we get that the return times τ and $\overline{\tau}$ are also fixed. Thus, in this invariant cylinder, the number of limit cycles is given by the intersections of the straight lines $r^\pm$ given by

$$r^+ : x_1 = x_0 + \eta y_0 + \mu \quad \text{with} \quad r^- : \widetilde{x}_1 = \widetilde{x}_0 - (\widetilde{\eta} y_1 + \widetilde{\mu}),$$

where $\widetilde{x}_0 = x_0$ and $\widetilde{x}_1 = x_1$.
Doing

$$(\overline{B}, \overline{C}) = (\eta y_0 + \mu, -\widetilde{\eta} y_1 - \widetilde{\mu})$$

we obtain that either all solutions are closed in the cylinder, if $\overline{B} = \overline{C}$, or there is no closed solutions in this cylinder when $\overline{B} \neq \overline{C}$.

Theorem B (Sa, Sa): Sketch of the proof. III

Suppose that $(a^+)^2 + (a^-)^2 \neq 0$ and Z at most one invariant cylinder.
Fixed the invariant cylinder, the number of limit cycles is given by the intersections of

$$x_1 = \rho x_0 + B \text{ if } a^+ \neq 0 \text{ (or } x_1 = x_0 + \overline{B} \text{ if } a^+ = 0)$$

and

$$x_1 = x_0/\xi + C \text{ if } a^- \neq 0 \text{ (or } x_1 = x_0 + \overline{C} \text{ if } a^- = 0),$$

where $\rho = e^{a^+\tau}$, $\xi = e^{a^-\overline{\tau}}$, $(\overline{B}, \overline{C})$ are obtained as above and (B, C) obtained in the proof of Theorem A. Thus, there is at most one limit cycle.

Theorem B $(\mathrm{Sa}, \mathrm{Sa})$: Sketch of the proof. IV

Now, $(a^+)^2 + (a^-)^2 \neq 0, a^+a^- \geq 0$ and Z has infinitely many invariant cylinders.
Suppose initially $a^+a^- > 0$.
We will show that in each invariant cylinder there is a unique isolated periodic orbit.

Indeed, in each cylinder, the orbit periodic is given by the intersection of the straight lines $r^{\pm}$ given by

$$r^+ : x_1 = \rho x_0 + B \quad \text{and} \quad r^- : x_1 = \frac{1}{\xi} x_0 + C.$$

These straight lines has a unique intersection point provided that $\rho \neq 1/\xi$. Note that $\rho = 1/\xi \Leftrightarrow \tau = -a^- \overline{\tau}/a^+$, where τ and $\overline{\tau}$ are positives.

Theorem B (Sa, Sa): Sketch of the proof. V

With the hypothesis of that $a^+a^- > 0$, the relation $\tau = -a^-\overline{\tau}/a^+$ can not be satisfied and thus $\rho \neq 1/\xi$.

The intersection point in each cylinder is given by

$$x_0 = \frac{C - B}{\frac{1}{\xi} - \rho} \quad \text{e} \quad x_1 = \rho \frac{C - B}{\frac{1}{\xi} - \rho} + B.$$

Varying continuously the cylinders, the terms x_0 and x_1 also range continuously, and we obtain one invariant surface formed of periodic orbits, where each orbit is an invariant cylinder.

Theorem B (Sa, Sa): Sketch of the proof. VI

If $a^+ = 0$ and $a^- \neq 0$, the periodic orbit in each cylinder is given by intersection of straight lines

$$x_1 = x_0 + \overline{B} \quad \text{and} \quad x_1 = \frac{1}{\xi}x_0 + C.$$

The case $a^+ \neq 0$ and $a^- = 0$ follows analogously.

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