PERIODIC ORBITS AND SLIDING BIFURCATIONS IN DISCONTINUOUS DRY FRICTION OSCILLATORS

M. Lima and J. Cassiano
UFABC, Santo Andre, Brazil

First Joint Meeting Brazil-Spain in Mathematics

Fortaleza, Dec. 7-10, 2015

Discontinuous oscillations

In Control theory the discontinuous oscillations phenomena have been modeled by the introduction of discontinuities in semi-linear vector fields, using sign functions.

Discontinuous oscillations

In Control theory the discontinuous oscillations phenomena have been modeled by the introduction of discontinuities in semi-linear vector fields, using sign functions.

These systems are called relay systems and they have gain great importance to understand the discontinuous behaviour, based on the escaping, sliding and sewing regions and the flow on the discontinuity set using the Filippov convention.

Discontinuous oscillations

In Control theory the discontinuous oscillations phenomena have been modeled by the introduction of discontinuities in semi-linear vector fields, using sign functions.

These systems are called relay systems and they have gain great importance to understand the discontinuous behaviour, based on the escaping, sliding and sewing regions and the flow on the discontinuity set using the Filippov convention.

It will be introduced some definitions and derived some results of a piecewise smooth mechanical oscillator.

Piecewise Continuous Vector Field-General Definitions

(i) $\Sigma \subset \mathbb{R}^n$ a C^∞ $(n-1)$-dimensional surface.

(ii) $p \in \Sigma$ and $h: (\mathbb{R}^n, \Sigma) \to (\mathbb{R}, 0)$ a C^∞ local implicit representation of Σ at p with $dh(p) \neq 0$.

(iii) $\Sigma^+ = \{x \in \mathbb{R}^n;\ h(x) > 0\}$ and $\Sigma^- = \{x \in \mathbb{R}^n;\ h(x) < 0\}$ (note that Σ represents the common boundary separating Σ^+ and Σ^-.)

(iv) $\mathcal{X}^r$ the set of all germs in p of C^r v.f. on $(\mathbb{R}^n, p)$ with the C^r-topology (r big enough).

(v) G^r the set of all germs in p of v.f. X on $\mathbb{R}^n$ satisfying

$$X(q) = \begin{cases} X_+(q) & q \in \Sigma^+, \\ X_-(q) & q \in \Sigma^-, \end{cases} \qquad X_+, X_- \in \mathcal{X}^r.$$

Define the functions

$$X_i h(p) = < X_i(p), \bigtriangledown h(p) >,$$

$$X_i^2 h(p) = < X_i(p), \bigtriangledown (X_i h)(p) >, ...$$

$$X_i^k h(p) = < X_i(p), \bigtriangledown (X_i^{k-1} h)(p) >, \; k \geq 2 \; \text{ for } i \in \{+, -\}.$$

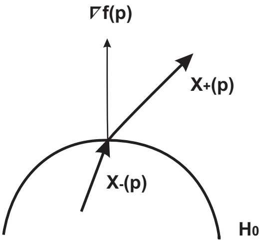
Consider $\mathcal{G}^r = \mathcal{X}^r \times \mathcal{X}^r$.

So given $X = (X_+, X_-) \in \mathcal{G}^r$ we distinguish the following regions in Σ

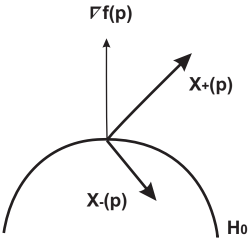
(a) *Sewing Region* Σ_c: if $X_+h(p)X_-h(p) > 0$.

(b) *Escaping Region* Σ_e: if $X_+h(p) > 0$ and $X_-h(p) < 0$.

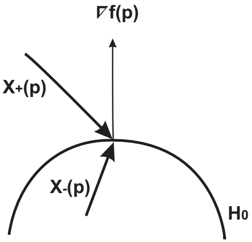
(c) *Sliding Region* Σ_s: if $X_+h(p) < 0$ and $X_-h(p) > 0$.



Sewing



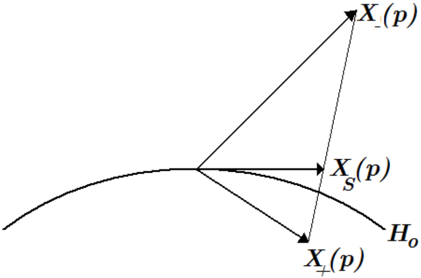
Escaping



Sliding

The solution of X through $p \in \Sigma_s \subset \Sigma$ follows the orbit of the v.f. $X_s = X_s(X_+, X_-)$ (sliding vector field). This v.f. is defined following the Fillipov convention (see [Filippov]).

X_s is tangent to Σ and is defined at $p \in \Sigma$ by the vector $X_s(p) = m - p$ where m is the point where the segment joining $p + X_+(p)$ and $p + X_-(p)$ is tangent to Σ.



Observe that if $X_+(p)$ and $X_-(p)$ are linear dependent then p is a critical point of X_s (pseudo-equilibrium of X).

Remark

All "curves" in Σ separating the above-named regions are constituted by points where X_+ or X_- are tangent to Σ (singularities of X).

(i) $p \in \Sigma$ is an Σ-singular (resp. Σ-regular) point of $X_+ \in \mathcal{X}^r$ if $X_+h(p) = 0$ (resp. $X_+h(p) \neq 0$). We denote $\partial\Sigma_c^+$ the set of singular points of X_+.

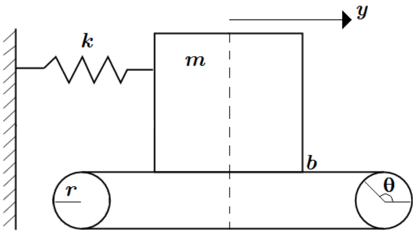
(ii) $p \in \Sigma$ is a fold (resp. cusp) point of X_+ if $X_+h(p) = 0$ and $X_+^2h(p) \neq 0$ (resp. $X_+h(p) = 0 = X_+^2h(p)$ and $\{dh(p), d(Xh)(p), d(X_+^2h(p))\}$) are linearly independent.

A fold singularity $p \in \Sigma$ is visible (resp. invisible) when $X_+^2h(p) > 0$, (resp. $X_+^2h(p) < 0$.)

Similar definitions are derived for the v.f. X_-.

Schematic Representation of the problem

Consider a block connected to a fixed linear-elastic spring on a moving conveyor belt, subject to Coulomb friction (Coulomb friction) and a periodic external force.



The system equations

This system is represented by the differential equations:

$$my'' + ky = -\mu N \; sgn\,(y' - r\theta')\,,$$
$$J\theta'' + b\theta' = \mu r N \; sgn\,(y' - r\theta') + \mu N \sin(\Omega\tau),$$

parameters

m: mass of block; k: elastic constant; r: turn radius; b: viscous friction in the conveyor belt-engine system ; M: engine torque; J: system inertia moment; N: normal force; and μ: Coulomb friction constant.

variables

y: block displacement ; θ: rotation angle.

$'$: denotes derivative with respect to the time τ.

After the change of variables

$$x_1 = \frac{y}{r}, \quad x_2 = \left(\frac{y'}{r} - \theta'\right)\sqrt{\frac{m}{k}}, \quad x_3 = \theta, \quad x_4 = \theta'\sqrt{\frac{m}{k}}$$

and the time scaling $t = \tau\sqrt{\frac{k}{m}}$, we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 + x_4, \\
\dot{x}_2 &= -x_1 + \varsigma x_4 - \eta sgn(x_2) - \sin(s), \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= -\varsigma x_4 + sgn(x_2) + \sin(s), \\
\dot{s} &= \omega,
\end{aligned} \tag{1}$$

where $\eta = 1 + \frac{J}{r^2 m}$; $\varsigma = \frac{b}{J}\sqrt{\frac{m}{k}}$; $\omega, \varsigma > 0$.

As the x_3-variable does not appear in the first, second, fourth and fifth equations of system (1), the dynamics of this system can be easily obtained from the dynamics of the four dimensional reduced system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_4, \\ \dot{x}_2 &= -x_1 + \varsigma x_4 - \eta sgn\left(x_2\right) - \sin(s), \\ \dot{x}_4 &= -\varsigma x_4 + sgn\left(x_2\right) + \sin(s), \\ \dot{s} &= \omega, \end{aligned} \tag{2}$$

Theorem 1

Consider system (2) with parameter values $(\varsigma, \eta, \omega)$ satisfying $\varsigma,\ \eta,\ \omega > 0$. Then

i) For $\varsigma = \cot(\pi/2\omega)$ system (2) with η given by

$$\eta(\omega) = \frac{\omega^2}{\omega^2 + \csc^2\left(\frac{\pi}{2\omega}\right) - 1} - \tanh\left(\frac{\pi\cot\left(\frac{\pi}{2\omega}\right)}{2\omega}\right),$$

and $\omega \in \left(\bigcup_{n\in\mathbb{N}} (1/(2n+1), \omega_n)\right) \cup (1, \omega^*)$ admits an one-turn crossing periodic orbit through the point $(0, 0, z_0, 3\pi/2) \in \Sigma_c^+$ with

$$z_0 = -\frac{\cot\left(\frac{\pi}{2\omega}\right)}{\omega^2 + \cot^2\left(\frac{\pi}{2\omega}\right)} - \tan\left(\frac{\pi}{2\omega}\right)\tanh\left(\frac{\pi\cot\left(\frac{\pi}{2\omega}\right)}{2\omega}\right), \tag{3}$$

if condition (18) is satisfied. Moreover, in this case, the point $(0, 0, z_0, 3\pi/2)$ is a visible fold for system (2) and this system performs a codimension-one sliding bifurcation in such a way that a small perturbation of ς such that $\varsigma > \cot(\pi/2\omega)$ the associated system admits a crossing-sliding periodic orbit.

ii) For $\varsigma < \cot(\pi/2\omega)$ system (2) with

$$\eta(s_0,\varsigma,\omega) = \frac{-\sin\left(\frac{\pi}{\omega}\right)\left(\frac{\varsigma\sin(s_0)-\omega\cos(s_0)}{\varsigma^2+\omega^2} - \frac{\tanh\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma}\right) + \cos\left(\frac{\pi}{\omega}\right) - 1}{\cos\left(\frac{\pi}{\omega}\right) - 1}, \quad (4)$$

where $s_0 \in (\bar{s}_0, \hat{s}_0)$ is such that, $3\pi/2 \in (\bar{s}_0, \hat{s}_0)$ and $\eta > 0$, admits an one-turn crossing periodic orbit through the point $(0, 0, z_0, s_0)$ with

$$z_0 = \frac{\varsigma\sin(s_0) - \omega\cos(s_0)}{\varsigma^2+\omega^2} - \frac{\tanh\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma}, \quad (5)$$

if condition (18) is satisfied. Moreover, for any $\bar{s}_0$ and for $\hat{s}_0$ sufficiently close to $3\pi/2$ system (2) performs a codimension-one sliding bifurcation for $s_0 = \bar{s}_0$ and $s_0 = \hat{s}_0$, where

$$\bar{s}_0 = 2\pi - \arccos\left(-\frac{a_+ b_+ + \sqrt{c_+^2\left(-a_+^2 + b_+^2 + c_+^2\right)}}{b_+^2 + c_+^2}\right),$$
$$\hat{s}_0 = 2\pi - \arccos\left(\frac{-a_+ b_+ + \sqrt{c_+^2\left(-a_+^2 + b_+^2 + c_+^2\right)}}{b_+^2 + c_+^2}\right), \qquad (6)$$

with

$$a_+ = \left(\frac{\cot\left(\frac{\pi}{2\omega}\right)}{\varsigma} - 1\right)\tanh\left(\frac{\pi\varsigma}{2\omega}\right) - 1, \qquad b_+ = \frac{\omega\left(\cot\left(\frac{\pi}{2\omega}\right) - \varsigma\right)}{\varsigma^2 + \omega^2}, \qquad c_+ = -\frac{\varsigma\cot\left(\frac{\pi}{2\omega}\right) + \omega^2}{\varsigma^2 + \omega^2}.$$

This bifurcation is such that for a small perturbation of ς doing $\varsigma > \cot(\pi/2\omega)$ the associated system admits a crossing-sliding periodic orbit.

iii) For $\varsigma > \cot(\pi/2\omega)$ system (2) does not admit one-turn crossing periodic orbits.

iv) For $\eta \geq 1$ system (2) does not admit one-turn crossing periodic orbits.

Theorem 2

Consider system (2) with $\varsigma,\ \eta,\ \omega > 0$. Under the conditions

$$\begin{aligned} (\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 \geq 0 \quad \eta \in (0,1), \\ (\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 \leq 0 \quad \eta > 1, \end{aligned} \tag{7}$$

this system admits a sliding periodic orbit

$$\varphi_s(t) = (x_1(t), 0, x_4(t), \omega t + s_0), \tag{8}$$

where

$$x_1(t) = -\frac{(\eta-1)\big(\varsigma\omega(\eta-1)\cos(\omega t+s_0)+(\eta\omega^2-1)\sin(\omega t+s_0)\big)}{\varsigma^2\omega^2(\eta-1)^2+(\omega^2\eta-1)^2},$$

$$x_4(t) = \frac{(\eta-1)\omega\big(\varsigma\omega(\eta-1)\omega\sin(\omega t+s_0)+(\eta\omega^2-1)\cos(\omega t+s_0)\big)}{\varsigma^2\omega^2(\eta-1)^2+(\omega^2\eta-1)^2}.$$

In both cases, when $(\eta+1)\omega^2+(\eta-1)\varsigma^2-2=0$ the periodic orbit is tangent to the boundary of the sliding region Σ_s at the points $\varphi_s(t_+)\in\partial\Sigma_+$ and $\varphi_s(t_-)\in\Sigma_-$ where t_+ and t_- are

$$\begin{aligned} t_+ &= \arccos\left(-\frac{a\cos(s_0)+c\sin(s_0)}{\sqrt{a^2+b^2}}\right), \\ t_- &= \arccos\left(\frac{a\cos(s_0)+c\sin(s_0)}{\sqrt{a^2+b^2}}\right), \end{aligned}$$

with $a=\varsigma(\eta-1)\omega(\omega^2-1)$, $b=\varsigma^2(\eta-1)^2\omega^2+(\eta\omega^2-1)^2>0$ and $c=(\omega^2-1)(\eta\omega^2-1)$.

In both cases, when $(\eta+1)\omega^2+(\eta-1)\varsigma^2-2=0$ the periodic orbit is tangent to the boundary of the sliding region Σ_s at the points $\varphi_s(t_+)\in\partial\Sigma_+$ and $\varphi_s(t_-)\in\Sigma_-$ where t_+ and t_- are

$$\begin{aligned} t_+ &= \arccos\left(-\frac{a\cos(s_0)+c\sin(s_0)}{\sqrt{a^2+b^2}}\right), \\ t_- &= \arccos\left(\frac{a\cos(s_0)+c\sin(s_0)}{\sqrt{a^2+b^2}}\right), \end{aligned}$$

with $a=\varsigma(\eta-1)\omega(\omega^2-1)$, $b=\varsigma^2(\eta-1)^2\omega^2+(\eta\omega^2-1)^2>0$ and $c=(\omega^2-1)(\eta\omega^2-1)$.

Moreover, system (2) performs an adding-sliding bifurcation in such a way that under small perturbations of the parameters such that $(\eta-1)\left((\eta+1)\omega^2+(\eta-1)\varsigma^2-2\right)>0$ system (2) admits an adding-sliding periodic orbit. It is an orbit visiting Σ_s, Σ_+ and Σ_- with small portion in $\Sigma_\pm$.

Properties

Proposition: System (2) admits:

i) a symmetry given by $R(x_1, x_2, x_4, s) = (-x_1, -x_2, -x_4, s + \pi)$;

ii) a local first integral given by

$$G(x_1, x_2, x_4, s) = \begin{cases} G_+(x_1, x_2, x_4, s) = (x_1 - (1-\eta))^2 + (x_2 + x_4)^2 & \text{if } x_2 > 0, \\ G_-(x_1, x_2, x_4, s) = (x_1 + (1-\eta))^2 + (x_2 + x_4)^2 & \text{if } x_2 < 0. \end{cases}$$

Observe that system (2) is analytic restricted to each one of the sets

$$\Sigma^{\pm} = \{x = (x_1, x_2, x_4, s) \in \mathbb{R}^3 \times \mathbb{T}^1;\ \pm x_2 > 0\}. \tag{9}$$

Moreover it has a unique discontinuity surface (called switching manifold) given by

$$\Sigma = \{x = (x_1, x_2, x_4, s) \in \mathbb{R}^3 \times \mathbb{T}^1;\ x_2 = 0\} = h^{-1}(0), \tag{10}$$

where h is the real function defined in $\mathbb{R}^3 \times \mathbb{T}^1$ by $h(x_1, x_2, x_4, s) = x_2$.

Σ can be split into the following way:

i) The sewing region given by $\Sigma_c = \{x \in \Sigma; \mid -x_1 + \varsigma x_4 - \sin(s) \mid > \eta\}$ with $\Sigma_c = \Sigma_c^+ \cup \Sigma_c^-$ where

- i.1) $\Sigma_c^+ = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) > \eta\}$ where the vector field points away from Σ in Σ^+ and toward Σ in Σ^-;
- i.2) $\Sigma_c^- = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) < -\eta\}$ where the vector field points toward Σ in Σ^+ and away from Σ in Σ^-;

Σ can be split into the following way:

i) The sewing region given by $\Sigma_c = \{x \in \Sigma; \mid -x_1 + \varsigma x_4 - \sin(s) \mid > \eta\}$ with $\Sigma_c = \Sigma_c^+ \cup \Sigma_c^-$ where
   - i.1) $\Sigma_c^+ = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) > \eta\}$ where the vector field points away from Σ in Σ^+ and toward Σ in Σ^-;
   - i.2) $\Sigma_c^- = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) < -\eta\}$ where the vector field points toward Σ in Σ^+ and away from Σ in Σ^-;

ii) The sliding region given by $\Sigma_s = \{x \in \Sigma; \mid -x_1 + \varsigma x_4 - \sin(s)\mid < \eta\}$ where the vector field points toward Σ in both Σ^+ and Σ^-.

Σ can be split into the following way:

i) The sewing region given by $\Sigma_c = \{x \in \Sigma; \mid -x_1 + \varsigma x_4 - \sin(s) \mid > \eta\}$ with $\Sigma_c = \Sigma_c^+ \cup \Sigma_c^-$ where

  i.1) $\Sigma_c^+ = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) > \eta\}$ where the vector field points away from Σ in Σ^+ and toward Σ in Σ^-;

  i.2) $\Sigma_c^- = \{x \in \Sigma; -x_1 + \varsigma x_4 - \sin(s) < -\eta\}$ where the vector field points toward Σ in Σ^+ and away from Σ in Σ^-;

ii) The sliding region given by $\Sigma_s = \{x \in \Sigma; | -x_1 + \varsigma x_4 - \sin(s)| < \eta\}$ where the vector field points toward Σ in both Σ^+ and Σ^-.

iii) For system (2) with $\eta > 0$ we have $X_- h > X_+ h$ so there is no escaping region.

Note that these regions are such that the common boundaries between Σ_c^+ and Σ_s (denoted by $\partial\Sigma_c^+$) and between Σ_c^- and Σ_s (denoted by $\partial\Sigma_c^-$) correspond to tangential contact between Σ and X_+ and X_-, respectively. In system (2), these sets are given by

$$
\begin{aligned}
\partial\Sigma_c^+ &= \{(x_1,0,x_4,s)\in\mathbb{R}^3\times\mathbb{T}^1;\ X_+h(x_1,0,x_4,s)=0\}\\
&= \{(x_1,0,x_4,s);\ -x_1+\varsigma x_4-\sin(s)=\eta\}\\
\partial\Sigma_c^- &= \{(x_1,0,x_4,s)\in\mathbb{R}^3\times\mathbb{T}^1;\ X_-h(x_1,0,x_4,s)=0\}\\
&= \{(x_1,0,x_4,s);\ -x_1+\varsigma x_4-\sin(s)=-\eta\}
\end{aligned}
\tag{11}
$$

where $X_ih(p)$ denotes the Lie derivative of h with respect to the vector field X_i in p.

The cubic tangencies of system (2) occur at the subsets of $\partial\Sigma_c^{\pm}$ given by

$$\left(-\frac{\zeta^2(\eta-1)+\eta+\zeta\omega\cos(s)+\sin(s)}{\zeta^2+1}, 0, \frac{\zeta+\zeta\sin(s)-\omega\cos(s)}{\zeta^2+1}, s\right);\ 1-(\omega^2-1)\sin(s)\neq 0, \qquad \text{for } \partial\Sigma_c^{+},$$

and (12)

$$\left(\frac{\zeta^2(\eta-1)+\eta-\zeta\omega\cos(s)-\sin(s)}{\zeta^2+1}, 0, -\frac{\zeta+\zeta(-\sin(s))+\omega\cos(s)}{\zeta^2+1}, s\right);\ 1-(\omega^2-1)\sin(s)\neq 0, \quad \text{for } \partial\Sigma_c^{-}.$$

The sliding vector field $X_s(x_1, 0, x_4, s)$ defined in Σ_s following the Filippov′s convex method [Filippov] has the form

$$\begin{aligned} \dot{x}_1 &= x_4, \\ \dot{x}_4 &= -\frac{x_1 + \varsigma(\eta - 1)x_4 - (\eta - 1)\sin(s)}{\eta}, \\ \dot{s} &= \omega. \end{aligned} \tag{13}$$

The sliding vector field $X_s(x_1, 0, x_4, s)$ defined in Σ_s following the Filippov′s convex method [Filippov] has the form

$$\begin{aligned} \dot{x}_1 &= x_4, \\ \dot{x}_4 &= -\frac{x_1 + \varsigma(\eta - 1)x_4 - (\eta - 1)\sin(s)}{\eta}, \\ \dot{s} &= \omega. \end{aligned} \tag{13}$$

Remark: Observe that the distance between the boundary of the sliding region ($\partial\Sigma_c^+$ and $\partial\Sigma_c^-$) is η. However, if η could assume any real value then for $\eta = 0$ the surfaces $\partial\Sigma_c^+$ and $\partial\Sigma_c^-$ coincides and we have no sliding region. On the other hand, when $\eta < 0$ we have no sliding region and system (2) has an escaping region between $\partial\Sigma_c^+$ and $\partial\Sigma_c^-$.

Proof of Theorem 1

Taking as initial condition a point $(0, 0, z_0, s_0)$ satisfying condition

$$\varsigma z_0 - \eta - \sin(s_0) \geqslant 0$$

then $(0, 0, z_0, s_0) \in \Sigma_c^+ \cup \partial\Sigma_c^+$.

Proof of Theorem 1

Taking as initial condition a point $(0, 0, z_0, s_0)$ satisfying condition

$$\varsigma z_0 - \eta - \sin(s_0) \geqslant 0$$

then $(0, 0, z_0, s_0) \in \Sigma_c^+ \cup \partial\Sigma_c^+$.

As X_+ is a linear vector field we can solve system (2) and obtain

$$\begin{aligned}
x_1(t) = & \quad (\eta - 1)(\cos t - 1) + z_0 \sin t, \\
x_2(t) = & \quad -\frac{e^{-\varsigma t}(e^{\varsigma t} + \varsigma z_0 - 1)}{\varsigma} + \frac{e^{-\varsigma t}(\varsigma \sin(s_0) - \omega \cos(s_0))}{\varsigma^2 + \omega^2} \\
& + \frac{-\varsigma \sin(s_0 + t\omega) + \omega \cos(s_0 + t\omega) - (\eta - 1)\left(\varsigma^2 + \omega^2\right)\sin(t) + z_0\left(\varsigma^2 + \omega^2\right)\cos(t)}{\varsigma^2 + \omega^2}, \\
x_4(t) = & \quad \frac{e^{-\varsigma t}\left(\varsigma\left(-\varsigma \sin(s_0) + e^{\varsigma t}(\varsigma \sin(s_0 + t\omega) - \omega \cos(s_0 + t\omega)) + \omega \cos(s_0)\right) + \left(\varsigma^2 + \omega^2\right)\left(e^{\varsigma t} + \varsigma z_0 - 1\right)\right)}{\varsigma(\varsigma^2 + \omega^2)},
\end{aligned} \tag{14}$$

Now in order to this solution be an one-turn crossing periodic orbit we will use the symmetry R in the following sense:

a necessary condition for $\Gamma_{\varsigma,\eta,\omega}(t)$ being an one-turn periodic orbit is that

$$\Gamma_{\varsigma,\eta,\omega}(\pi/\omega) = (0, 0, -z_0, s_0 + \pi).$$

So, considering $\Gamma_{\varsigma,\eta,\omega}(t) = (x_1(t), x_2(t), x_4(t), s(t))$, we have to solve the system

$$x_1\left(\pi/\omega\right) = 0, \qquad x_2\left(\pi/\omega\right) = 0, \qquad x_4\left(\pi/\omega\right) = -z_0.$$

For $\omega \neq 1/2n$ we can solve the first equation of this system with respect to η to obtain

$$\eta = \frac{\cos\left(\frac{\pi}{\omega}\right) - z_0 \sin\left(\frac{\pi}{\omega}\right) - 1}{\cos\left(\frac{\pi}{\omega}\right) - 1}. \tag{15}$$

Substituting this η in the second and third equations of the previous system we obtain a single equation that can be solved for z_0 obtaining

$$z_0 = \frac{\varsigma \sin(s_0) - \omega \cos(s_0)}{\varsigma^2 + \omega^2} - \frac{\tanh\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma}.$$

Substituting this η in the second and third equations of the previous system we obtain a single equation that can be solved for z_0 obtaining

$$z_0 = \frac{\varsigma \sin(s_0) - \omega \cos(s_0)}{\varsigma^2 + \omega^2} - \frac{\tanh\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma}. \tag{16}$$

So for this choice of z_0 and η and since that $\omega \neq 1/2n$, $\Gamma_{\varsigma,\eta,\omega}(t)$, $t \in (0, \pi/\omega)$ and for $t \in (\pi/\omega, 2\pi/\omega)$ given by the symmetry R represents an one-turn periodic orbit of (2) if conditions

$$\varsigma z_0 - \eta - \sin(s_0) \geqslant 0, \tag{17}$$

and

$$x_2(t) > 0, \;\; \textit{for} \;\; t \in (0, \pi/\omega). \tag{18}$$

are simultaneously satisfied.

Now for having $(0,0,z_0,s_0) \in \Sigma_c^+$ it must satisfy

$$X_+h(0,0,z_0,s_0),\ X_-(0,0,z_0,s_0) > 0.$$

Moreover, as system (2) has no escaping region, it is sufficient that $X_+h(0,0,z_0,s_0) > 0$.

Now for having $(0,0,z_0,s_0) \in \Sigma_c^+$ it must satisfy

$$X_+h(0,0,z_0,s_0),\ X_-(0,0,z_0,s_0) > 0.$$

Moreover, as system (2) has no escaping region, it is sufficient that $X_+h(0,0,z_0,s_0) > 0$.

From the expressions of X_+h and X_-h and from the expression of z_0 in (16) we get

$$\begin{aligned} X_+h(0,0,z_0,s_0) &= a_+(\varsigma,\omega) + b_+(\varsigma,\omega)\cos(s_0) + c_+(\varsigma,\omega)\sin(s_0),\\ X_-h(0,0,z_0,s_0) &= a_-(\varsigma,\omega) + b_-(\varsigma,\omega)\cos(s_0) + c_-(\varsigma,\omega)\sin(s_0). \end{aligned} \tag{19}$$

with

$$a_+ = \left(\frac{\cot\left(\frac{\pi}{2\omega}\right)}{\varsigma} - 1\right)\tanh\left(\frac{\pi\varsigma}{2\omega}\right) - 1, \qquad b_+ = \frac{\omega\left(\cot\left(\frac{\pi}{2\omega}\right) - \varsigma\right)}{\varsigma^2+\omega^2}, \qquad c_+ = -\frac{\varsigma\cot\left(\frac{\pi}{2\omega}\right) + \omega^2}{\varsigma^2+\omega^2}.$$

Writing

$$\begin{aligned} f_+(s_0) &= X_+ h(0,0,z_0,s_0), \\ f_-(s_0) &= X_- h(0,0,z_0,s_0), \end{aligned} \tag{20}$$

as functions of the variable s_0 we will find conditions under which $f_+(s_0)$ has a zero. For this we start finding conditions in (ς, ω) under which this function has a double zero in a point $s_0 \in [0, 2\pi]$.
It is equivalent to find s_0 such that $f_+(s_0) = f'_+(s_0) = 0$, or to solve

$$\begin{aligned} c_+ \sin(s_0) + b_+ \cos(s_0) &= -a_+, \\ -b_+ \sin(s_0) + c_+ \cos(s_0) &= 0. \end{aligned}$$

This system has solution $\sin(s_0) = \dfrac{-a_+ c_+}{b_+^2 + c_+^2}$ and $\cos(s_0) = \dfrac{-a_+ b_+}{b_+^2 + c_+^2}$. So we must have

$$\frac{a_+^2 c_+^2}{(b_+^2 + c_+^2)^2} + \frac{a_+^2 b_+^2}{(b_+^2 + c_+^2)^2} = 1.$$

Or in an equivalent way

$$a_+^2 - (b_+^2 + c_+^2) = 0. \tag{21}$$

Solving $f_+(s_0) = 0$ in terms of s_0 we can see that this equation has:

a) a unique solution for $a_+^2 - (b_+^2 + c_+^2) = 0$;

b) two different solutions for $a_+^2 - (b_+^2 + c_+^2) < 0$;

c) no solution for $a_+^2 - (b_+^2 + c_+^2) > 0$.

Rewriting equation (21) in terms of ς and ω we have

$$a_2(\varsigma, \omega) \cot^2\left(\frac{\pi}{2\omega}\right) + a_1(\varsigma, \omega) \cot\left(\frac{\pi}{2\omega}\right) + a_0(\varsigma, \omega) = 0. \qquad (22)$$

where $a_2(\varsigma,\omega) = \frac{\tanh^2\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma^2} - \frac{1}{\varsigma^2+\omega^2}$, $a_1(\varsigma,\omega) = -2\left(\frac{\tanh^2\left(\frac{\pi\varsigma}{2\omega}\right)+\tanh\left(\frac{\pi\varsigma}{2\omega}\right)}{\varsigma}\right)$ and $a_0(\varsigma,\omega) = \tanh^2\left(\frac{\pi\varsigma}{2\omega}\right) + 2\tanh\left(\frac{\pi\varsigma}{2\omega}\right) + \frac{\varsigma^2}{\varsigma^2+\omega^2}$.

Note that equation (22) is quadratic in $\cot\left(\frac{\pi}{2\omega}\right)$. Solving it with respect to $\cot\left(\frac{\pi}{2\omega}\right)$ we obtain

$$\cot\left(\frac{\pi}{2\omega}\right) = \varsigma \quad \text{or} \quad \cot\left(\frac{\pi}{2\omega}\right) = \frac{\varsigma\left(-2\left(\varsigma^2+\omega^2\right)\sinh\left(\frac{\pi\varsigma}{\omega}\right) - \left(2\varsigma^2+\omega^2\right)\cosh\left(\frac{\pi\varsigma}{\omega}\right)+\omega^2\right)}{2\varsigma^2 - \omega^2\cosh\left(\frac{\pi\varsigma}{\omega}\right)+\omega^2}. \tag{23}$$

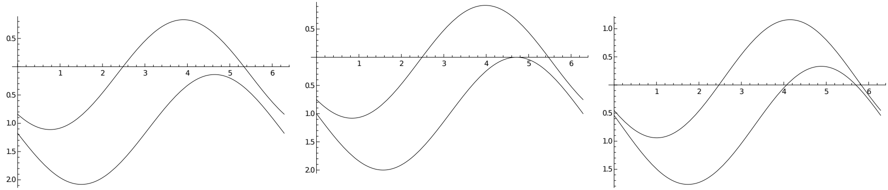


Figure: Functions $f_+(s_0)$ and $f_-(s_0)$ (see (20)) for $s_0 \in [0, 2\pi]$, $\omega = \frac{3}{2}$ and $\varsigma = 0,7 > \cot\left(\frac{\pi}{3}\right)$, $\varsigma = \cot\left(\frac{\pi}{3}\right) \simeq 0,577$ and $\varsigma = 0,3 < \cot\left(\frac{\pi}{3}\right)$ respectively. In all pictures the above curves represent function $f_-(s_0)$ and the bellow curves represent function $f_+(s_0)$.

Case 1: $\cot\left(\frac{\pi}{2\omega}\right) = \varsigma$.

Under this condition we have $f_+(s_0) = -1 - \sin(s_0)$. This function has double zero for $s_0 = 3\pi/2$. And, for this choice of s_0, considering (15) and (16) we have, respectively

$$\eta(\omega) = \frac{\omega^2}{\omega^2 + \csc^2\left(\frac{\pi}{2\omega}\right) - 1} - \tanh\left(\frac{\pi\cot\left(\frac{\pi}{2\omega}\right)}{2\omega}\right) \tag{24}$$

and

$$z_0 = -\frac{\cot\left(\frac{\pi}{2\omega}\right)}{\omega^2 + \cot^2\left(\frac{\pi}{2\omega}\right)} - \tan\left(\frac{\pi}{2\omega}\right)\tanh\left(\frac{\pi\cot\left(\frac{\pi}{2\omega}\right)}{2\omega}\right).$$

Now as $\varsigma = \cot\left(\frac{\pi}{2\omega}\right) > 0$ and $\omega > 0$ it follows that $z_0 < 0$. And from $X_+^2(0,0,z_0,3\pi/2) = -z_0\csc^2\left(\frac{\pi}{2\omega}\right) > 0$ we get that this point is a visible fold in $\partial\Sigma_c^+$.

Moreover the graph of $\eta(\omega)$ has the form

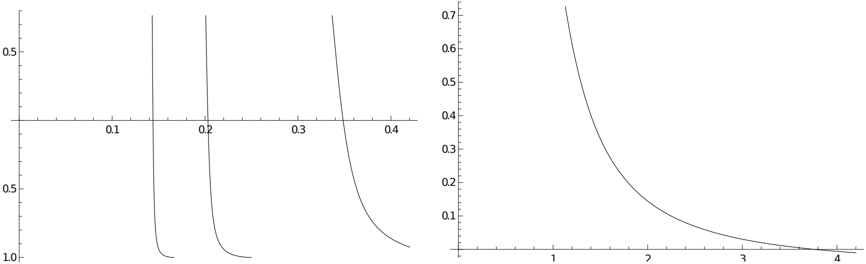


Figure: Function $\eta(\omega)$ for $\omega \in (1/8, 1/2)$ and $\omega \in (1, 4)$.

where $\eta(\omega) > 0$ if $\omega \in \left(\bigcup_{n\in\mathbb{N}} (1/(2n+1), \omega_n)\right) \cup (1, \omega^*)$.
So for $\varsigma = \cot\left(\frac{\pi}{2\omega}\right)$ with $\omega \in \left(\bigcup_{n\in\mathbb{N}} (1/(2n+1), \omega_n)\right) \cup (1, \omega^*)$ and η given by (24) system (2) undergoes a sliding-bifurcation at the point $(0, 0, z_0, 3\pi/2)$ where z_0 is given by (3) (see the second picture of Figure 1).

We also observe that if $\varsigma > \cot\left(\frac{\pi}{2\omega}\right)$ (that is equivalent to $a_+^2 - (b_+^2 + c_+^2) > 0$) system (2) has no one-turn crossing periodic orbit (see the first picture of Figure 1). It happens because under small perturbation of the parameter ς in such a way that ς becomes bigger then $\cot\left(\frac{\pi}{2\omega}\right)$ condition (17) is no longer satisfied and the associated system has a crossing-sliding solution (see [di Bernardo, et al 2007], [Guardia, et.al. (2010)]).

Now if $\varsigma < \cot\left(\frac{\pi}{2\omega}\right)$ (that is equivalent to $a_+^2 - (b_+^2 + c_+^2) < 0$) function $f_+(s_0)$ has two zeroes $\bar{s}_0$ and $\hat{s}_0$ given by (6) and, for $s_0 \in (\bar{s}_0, \hat{s}_0)$, $f_+(s_0)$ is positive (see the third picture of Figure 1). This implies that there exists an one-turn crossing periodic orbits for system (2) with η given by (4) since that condition (18) is satisfied.

Observe that for fixed (ς, ω) satisfying $\varsigma < \cot\left(\frac{\pi}{2\omega}\right)$ we have $(0, 0, z_0, s_0) \in \Sigma_c^+$ where s_0 is taking in the interval $(\bar{s}_0, \hat{s}_0)$ in such a way that $\eta(s_0) > 0$ where η is given by (4).

Moreover, as in a tangency point we have from (17) that $\sin(s_0) = \varsigma z_0 - \eta < 0$. It follows that $\pi < \bar{s}_0 < \frac{3\pi}{2}$ and so

$$X_+^2 h\,(0, 0, z_0, s_0) = -\left(1 + \varsigma^2\right) z_0 + \varsigma(1 + \sin s_0) - \omega \cos s_0 > 0.$$

This implies that $(0, 0, \bar{z}_0, \bar{s}_0)$ is a fold point. Also it is clear that, for $(0, 0, \hat{z}_0, \hat{s}_0)$ with $\hat{s}_0 \simeq \frac{3\pi}{2}$ we have a fold point.

So by the same reason that in the case $\varsigma = \cot\left(\frac{\pi}{2\omega}\right)$, we have a crossing-sliding bifurcation to system (2).

Proof of Theorem 2

First considering system (13) as a two dimensional system defined in all $\mathbb{R}^2$ it is straightforward to show that this system with $\eta \neq 0$ has a periodic orbit given by (8). Moreover this is the unique periodic orbit of system (13) when $\eta \neq 0$.

Proof of Theorem 2

First considering system (13) as a two dimensional system defined in all $\mathbb{R}^2$ it is straightforward to show that this system with $\eta \neq 0$ has a periodic orbit given by (8). Moreover this is the unique periodic orbit of system (13) when $\eta \neq 0$.

It is important to observe that the sliding system (13) is defined only in Σ_s. So for seeing if this orbit is an orbit of system (2) in Σ_s we have to study the Lie derivatives X_+h and X_-h on the points of φ_s.
Doing this it follows that

$$\begin{aligned} X_+h(\varphi_s(t)) &= -\eta\left(1+\frac{a}{b}\cos(\omega t+s_0)+\frac{c}{b}\sin(\omega t+s_0)\right), \\ X_-h(\varphi_s(t)) &= \eta\left(1-\frac{a}{b}\cos(\omega t+s_0)-\frac{c}{b}\sin(\omega t+s_0)\right). \end{aligned} \tag{25}$$

Taking $f(t,\varsigma,\eta,\omega) = X_+h(\varphi_s(t))X_-h(\varphi_s(t))$ we have that in order to $\varphi_s(t) \subset \Sigma_s$ we must have $f(t,\varsigma,\eta,\omega) \leq 0$ for all t.

From (25) it is not difficult to see that this inequality follows if and only if $\dfrac{a^2+c^2}{b^2} \leq 1$.

Taking $f(t,\varsigma,\eta,\omega) = X_+h(\varphi_s(t))X_-h(\varphi_s(t))$ we have that in order to $\varphi_s(t) \subset \Sigma_s$ we must have $f(t,\varsigma,\eta,\omega) \leq 0$ for all t.

From (25) it is not difficult to see that this inequality follows if and only if $\dfrac{a^2+c^2}{b^2} \leq 1$.

Moreover, $\varphi_s(t)$ intersects $\partial\Sigma_c^{\pm}$ for $f(t,\varsigma,\eta,\omega) = 0$. From this we have that when $\dfrac{a^2+c^2}{b^2} = 1$ (that is equivalent to $(\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 = 0$) φ_s is tangent to $\partial\Sigma_c^{\pm}$ at a cusp point.

Solving equations $X_+h(\varphi_s(t)) = 0$ and $X_-h(\varphi_s(t)) = 0$ with respect to t under the condition $(\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 = 0$ we obtain t_+ and t_-.

This implies that the tangency points are given by $\varphi_s(t_+) \in \partial\Sigma_+$ and $\varphi_s(t_-) \in \partial\Sigma_-$.

Solving equations $X_+h(\varphi_s(t)) = 0$ and $X_-h(\varphi_s(t)) = 0$ with respect to t under the condition $(\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 = 0$ we obtain t_+ and t_-.

This implies that the tangency points are given by $\varphi_s(t_+) \in \partial\Sigma_+$ and $\varphi_s(t_-) \in \partial\Sigma_-$.

Also, for $\eta \in (0,1)$ the periodic orbit escapes from Σ_s when $(\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 \geq 0$ and for $\eta > 1$ the periodic orbit escapes from Σ_s when $(\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2 \leq 0$.

This implies that system (2) performs an adding-sliding bifurcation under small perturbation of the parameters in such a way that $(\eta-1)\left((\eta+1)\omega^2 + (\eta-1)\varsigma^2 - 2\right) \leq 0$. This concludes the proof of the theorem.

References

- M. L. di Bernardo, C. J. Budd, A. R. Champneys, P. Kowalczyk. Piecewise-Smooth Dynamical Systems: Theory and Applications. Springer, 2007.
- A.F. Filippov. Differential Equations with Discontinuous Right-hand Sides. Mat. Sb. 51 (93), 99178, 1960.
- M. Guardia, S. J. Hogan, and T. M. Seara, *An Analytical Approach to Codimension-2 Sliding Bifurcations in the Dry-Friction Oscillator.* SIAM J. Appl. Dyn. Syst. **9** -3, (2010), 769-798.

References

A.F. Filippov. Differential Equations with Discontinuous Right-hand Sides. Mat. Sb. 51 (93), 99178, 1960.

M. Guardia, S. J. Hogan, and T. M. Seara, *An Analytical Approach to Codimension-2 Sliding Bifurcations in the Dry-Friction Oscillator.* SIAM J. Appl. Dyn. Syst. **9** -3, (2010), 769-798.

THANK YOU!