Effective localization of limit cycles (from numerical to analytical results)

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Outline of the talk

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Context

We consider real planar polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \qquad (1)$$

Our aim is to prove that limit cycles that we have found numerically, actually exist.

To prove their existence we find analytic transversal curves which define Poincaré–Bendixson annular regions. Of course, this method also allows to locate them.

This is a work in progress.

Definition of transversal curve

Consider a smooth curve C in $\mathbb{R}^2$.

A contact point of C is a point of the curve at which the flow of the system and the curve are tangential.

We say that C is transversal if the flow of system (1) crosses it in the same sense on all its points, except maybe a finite set of contact points.

Definition of Poincaré–Bendixson annular region

Poincaré Annular Criterion

Suppose R is a finite region of the plane $\mathbb{R}^2$ lying between two simple closed curves C_1 and C_2. If

(i) the curves C_1 and C_2 are transversal and the flow crosses both of them towards the interior (or the exterior) of R, and

(ii) R contains no critical points.

Then, the system (1) has an odd number of limit cycles lying inside R (counted with multiplicity).

In such a case, we say that R is a Poincaré–Bendixson annular region for system (1).

A previous work

In the paper

H. Giacomini and M. Grau, *Transversal conics and the existence of limit cycles*, J. Math. Anal. Appl. (2015).

the authors dealt with the problem of finding Poincaré–Bendixson annular regions with boundaries given by transversal conics.

The authors treated several examples for which they "numerically knew" the existence of a limit cycle, but they did not use this information at all. Sometimes they could not ensure the existence of transversal conics.

Outline of the talk

Main result

From the "numerically evidence" of the existence of a stable or unstable hyperbolic limit cycle, we prove its actual existence.

Main theorem

Let $\{(\gamma_1(t), \gamma_2(t)) : t \in [0, T]\}$ be a hyperbolic limit cycle of the differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y). \qquad (1)$$

- $T > 0$ is the period of the limit cycle.
- Denote by

$$\mathrm{div}(x, y) := \frac{\partial P}{\partial x}(x, y) + \frac{\partial Q}{\partial y}(x, y)$$

the divergence of system (1).

Main theorem

- Define

$$\tilde{u}(t) = \frac{1}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \exp\left\{ \int_0^t \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds - \kappa t \right\},$$

with

$$\kappa = \frac{1}{T} \int_0^T \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds.$$

- Given $|\varepsilon| > 0$ small enough, define

$$\begin{aligned} \tilde{x}(t) &= \gamma_1(t) + \varepsilon\, \tilde{u}(t)\, \gamma_2'(t), \\ \tilde{y}(t) &= \gamma_2(t) - \varepsilon\, \tilde{u}(t)\, \gamma_1'(t). \end{aligned}$$

Main theorem

Main Theorem

Let $\{(\gamma_1(t), \gamma_2(t)) \,:\, t \in [0, T]\}$ be a hyperbolic limit cycle. Define

$$\begin{array}{rcl} \tilde{x}(t) & = & \gamma_1(t) + \varepsilon\, \tilde{u}(t)\, \gamma_2'(t), \\ \tilde{y}(t) & = & \gamma_2(t) - \varepsilon\, \tilde{u}(t)\, \gamma_1'(t). \end{array}$$

Then, the curve $\{(\tilde{x}(t), \tilde{y}(t)) \,:\, t \in [0, T]\}$

- is periodic of period T,
- if $|\varepsilon| > 0$ is small enough, then it is transversal to system (1).

Proof on the Main Theorem

The curve $(\tilde{x}(t), \tilde{y}(t))$ is periodic of period T.

Since $(\gamma_1(t), \gamma_2(t))$ is a periodic curve of period T, and

$$\begin{aligned} \tilde{x}(t) &= \gamma_1(t) + \varepsilon\, \tilde{u}(t)\, \gamma_2'(t), \\ \tilde{y}(t) &= \gamma_2(t) - \varepsilon\, \tilde{u}(t)\, \gamma_1'(t), \end{aligned}$$

we only need to check that $\tilde{u}(t)$ is periodic of period T. And it is so by definition

$$\tilde{u}(t) = \frac{1}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \exp\left\{ \int_0^t \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds - \kappa t \right\},$$

where

$$\kappa = \frac{1}{T} \int_0^T \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds.$$

Proof on the Main Theorem

The curve $(\tilde{x}(t), \tilde{y}(t))$ *is transversal to system* (1).

In order to show the transversality of the curve $(\tilde{x}(t), \tilde{y}(t))$ for $|\varepsilon| > 0$ small enough, we show that

$$P(\tilde{x}(t), \tilde{y}(t))\, \tilde{y}'(t) \; - \; Q(\tilde{x}(t), \tilde{y}(t))\, \tilde{x}'(t) \; = \; \kappa\, \mathcal{E}(t)\, \varepsilon \; + \; \mathcal{O}(\varepsilon^2),$$

where

$$\mathcal{E}(t) \; := \; \exp\left\{ \int_0^t \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds \; - \; \kappa t \right\} > 0.$$

A corollary

In fact, in the proof of the transversality it is not used the specific value of κ. We only have used that it is a nonzero constant.

Corollary

Given an orbit $\{\gamma(t) \, : \, t \in [0, t_1]\}$ of system (1) and a nonzero constant K, then for $|\varepsilon| > 0$ small enough the curve

$$\begin{aligned} \tilde{x}(t) &= \gamma_1(t) + \varepsilon\, \hat{u}(t)\, \gamma_2'(t), \\ \tilde{y}(t) &= \gamma_2(t) - \varepsilon\, \hat{u}(t)\, \gamma_1'(t), \end{aligned}$$

for $t \in [0, t_1]$ where

$$\hat{u}(t) = \frac{1}{\gamma_1'(t)^2 + \gamma_2'(t)^2} \exp\left\{ \int_0^t \mathrm{div}\,(\gamma_1(s), \gamma_2(s))\, ds - Kt \right\}$$

is transversal to system (1).

This will be the useful result for non-smooth vector fields.

A corollary

As in the proof of the theorem we show that

$$P(\tilde{x}(t), \tilde{y}(t))\, \tilde{y}'(t) \; - \; Q(\tilde{x}(t), \tilde{y}(t))\, \tilde{x}'(t) \; = \; K\, \hat{\mathcal{E}}(t)\, \varepsilon \, + \, \mathcal{O}(\varepsilon^2).$$

The sign of $K\,\varepsilon$ gives whether the flow of system (1) crosses $(\tilde{x}(t), \tilde{y}(t))$ in one sense or another.

Outline of the talk

Step 1: the *numerical* limit cycle

We numerically compute the limit cycle $\{\varphi(t;(x_0,y_0)) \,:\, t \in [0, T(x_0,y_0)]\}$ and its period $T(x_0,y_0)$.

From now on, even though we do not have analytic but numerical expressions, we denote the limit cycle by $\gamma(t)$ and its period by T.

Step 2: the *numerical* transversal curve

We can numerically compute

$$\kappa = \frac{1}{T}\int_0^T \mathrm{div}\,(\gamma_1(s),\gamma_2(s))\,ds,$$

and a tabulation of the function

$$\tilde{u}(t) = \frac{1}{\gamma_1'(t)^2+\gamma_2'(t)^2}\exp\left\{\int_0^t \mathrm{div}\,(\gamma_1(s),\gamma_2(s))\,ds - \kappa t\right\},$$

which, by interpolation, provides the function $\tilde{u}(t)$.

We fix a rational ε and we construct

$$\begin{aligned}\tilde{x}(t) &= \gamma_1(t) + \varepsilon\,\tilde{u}(t)\,\gamma_2'(t),\\ \tilde{y}(t) &= \gamma_2(t) - \varepsilon\,\tilde{u}(t)\,\gamma_1'(t).\end{aligned}$$

Step 2: the *numerical* transversal curve

Given

$$\begin{aligned}\tilde{x}(t) &= \gamma_1(t) + \varepsilon\,\tilde{u}(t)\,\gamma_2'(t),\\ \tilde{y}(t) &= \gamma_2(t) - \varepsilon\,\tilde{u}(t)\,\gamma_1'(t),\end{aligned}$$

we numerically check whether the curve

$$\{(\tilde{x}(t), \tilde{y}(t)) \,:\, t \in [0, T]\}$$

is transversal to system (1).
If not, we take a smaller value of $|\varepsilon|$.

We take a natural n and, from these computations, we get a list of n points of the curve $(\tilde{x}(t), \tilde{y}(t))$.

Step 3: an approximated curve

We have a list of n points of the curve $(\tilde{x}(t), \tilde{y}(t))$.

We have chosen n odd and we define $m = (n-1)/2$.
We define the expressions

$$\begin{aligned} \tilde{x}_a(\theta) &= \tilde{c}_{00} + \sum_{i=1}^{m} \tilde{c}_{i0}\cos(i\theta) + \tilde{c}_{0i}\sin(i\theta) \\ \tilde{y}_a(\theta) &= \tilde{d}_{00} + \sum_{i=1}^{m} \tilde{d}_{i0}\cos(i\theta) + \tilde{d}_{0i}\sin(i\theta) \end{aligned}$$

with undefined coefficients $\tilde{c}_{ij}, \tilde{d}_{ij}$.
We have $2(2m+1)$ coefficients.

Step 3: an approximated curve

We have a list of n points of the curve $(\tilde{x}(t), \tilde{y}(t))$.

We have $2(2m+1)$ coefficients.

We impose that the curve $(\tilde{x}_a(\theta), \tilde{y}_a(\theta))$ passes through the list of n points when $\theta = 2\pi\, i/n$ for $i = 0, 1, 2, \ldots, n$. We have $2n$ conditions. Since $n = 2m+1$, we have $2(2m+1)$ conditions.

We obtain a curve

$$\{(\tilde{x}_a(\theta), \tilde{y}_a(\theta)) \,:\, \theta \in [0, 2\pi]\}$$

which approximates the numerical transversal curve

$$\{(\tilde{x}(t), \tilde{y}(t)) \,:\, t \in [0, T]\}\,.$$

Step 4: a curve with rational coefficients

We take rational approximations of the coefficients in the expressions of $(\tilde{x}_a(\theta), \tilde{y}_a(\theta))$.
We obtain a closed curve

$$\{(x_a(\theta), y_a(\theta)) \,:\, \theta \in [0, 2\pi]\}$$

whose coefficients are rational.
That is,

$$\begin{aligned} x_a(\theta) &= c_{00} + \sum_{i=1}^{m} c_{i0}\cos(i\theta) + c_{0i}\sin(i\theta) \\ y_a(\theta) &= d_{00} + \sum_{i=1}^{m} d_{i0}\cos(i\theta) + d_{0i}\sin(i\theta) \end{aligned}$$

have c_{ij} and d_{ij} rational numbers which are approximations of the corresponding $\tilde{c}_{ij}$ and $\tilde{d}_{ij}$.

Step 5: a transversal curve

We have a closed curve

$$\{(x_a(\theta), y_a(\theta)) \,:\, \theta \in [0, 2\pi]\}$$

whose coefficients are rational.

We have that this curve is transversal to system (1) if

$$f(\theta) := P(x_a(\theta), y_a(\theta))\, y_a'(\theta) - Q(x_a(\theta), y_a(\theta))\, x_a'(\theta)$$

does not change sign for all $\theta \in [0, 2\pi]$.

Step 5: a transversal curve

We have that the curve is transversal to system (1) if

$$f(\theta) := P(x_a(\theta), y_a(\theta))\, y_a'(\theta) - Q(x_a(\theta), y_a(\theta))\, x_a'(\theta)$$

does not change sign for all $\theta \in [0, 2\pi]$.

To prove this,

- we expand $f(\theta)$ in powers of $\cos\theta, \sin\theta$ and we change $\cos\theta$ by u and $\sin\theta$ by v.
- Then we take the resultant of this expression with $u^2 + v^2 - 1$ with respect to v
- and finally we prove that the latter expression has no real roots for $u \in [-1, 1]$ by using Sturm's sequences.

Step 5: a transversal curve

If the obtained curve is not transversal, we repeat from steps 3 or 4 in order to obtain either

a list of n points of the curve $(\tilde{x}(t), \tilde{y}(t))$ with a bigger n,

or

sharper rational approximations of the coefficients of the curve.

Step 6: a Poincaré–Bendixson annular region

We repeat the process with an ε of different sign in order to obtain an inner transversal curve and an outer transversal curve to the limit cycle.

We have a Poincaré–Bendixson annular region which analytically shows the existence of at least one limit cycle in its interior.

We can take small values of ε which make this region narrower. Thus, we locate the limit cycle.

Outline of the talk

Examples

1. van der Pol system
2. Brusselator system
3. Rychkov system

Example 1: van der Pol system

We consider the van der Pol system with $\epsilon = 1$:

$$\dot{x} = y - \left(\frac{x^3}{3} - x\right), \quad \dot{y} = -x.$$

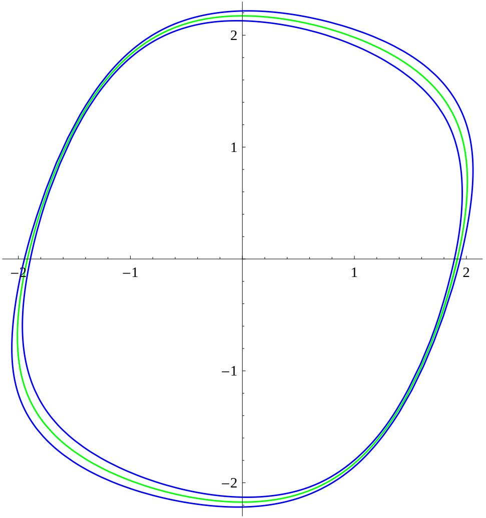
The limit cycle crosses the transversal section Σ at $x_0^* \sim 1.91928$ and it has period $T \sim 6.6632866$.

By our method we obtain an inner transversal curve and an outer transversal curve

$$(x_{in}(\theta), y_{in}(\theta)) \text{ and } (x_{ex}(\theta), y_{ex}(\theta)) \quad \text{with } \theta \in [0, 2\pi],$$

which provide a Poincaré–Bendixson annular region.
The inner transversal curve cuts Σ at ~ 1.89331 and the outer transversal curve at ~ 1.94543.

Example 1: van der Pol system



The transversal curves are represented in blue and the (numerical) limit cycle in green.

Example 1: van der Pol system

The inner transversal curve is obtained with $\varepsilon = 0.01$ and $m = 12$. By the computations, we obtain the following table of $n = 2m + 1 = 25$ points:

$$\{(1.89451, 0.0056435), (1.76278, -0.488101), (1.59066, -0.939363),$$
$$(1.38198, -1.33813), (1.12999, -1.67325),$$
$$(0.819552, -1.92987), (0.424859, -2.08912),$$
$$(-0.093381, -2.12507), (-0.747354, -2.00013),$$
$$(-1.39679, -1.69586), (-1.80051, -1.27605),$$
$$(-1.9537, -0.788939), (-1.93903, -0.264387),$$
$$(-1.83453, 0.245845), (-1.68122, 0.719683),$$
$$(-1.49111, 1.14594), (-1.26215, 1.51447),$$
$$(-0.98337, 1.81246), (-0.634949, 2.02302),$$
$$(-0.183705, 2.12466), (0.40691, 2.08508),$$
$$(1.08983, 1.86873), (1.63579, 1.49426),$$
$$(1.90318, 1.04111), (1.96187, 0.527263)\}$$

Example 1: van der Pol system

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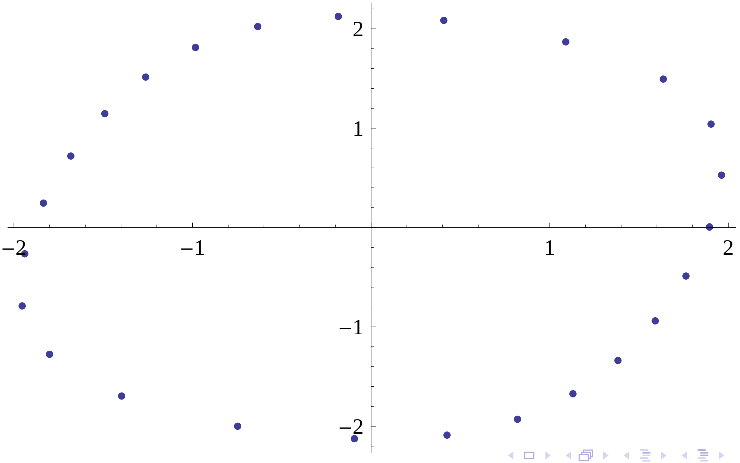


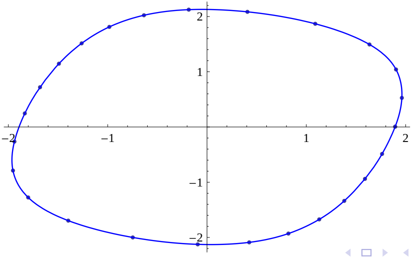
Example 1: van der Pol system

We find a curve of the form

$$\tilde{x}_a(\theta) = \tilde{c}_{00} + \sum_{i=1}^{12} \tilde{c}_{i0}\cos(i\theta) + \tilde{c}_{0i}\sin(i\theta)$$

$$\tilde{y}_a(\theta) = \tilde{d}_{00} + \sum_{i=1}^{12} \tilde{d}_{i0}\cos(i\theta) + \tilde{d}_{0i}\sin(i\theta)$$

which passes through these points.



Example 1: van der Pol system

We rationalize the coefficients of this curve and we have:

$$\begin{aligned}
x_{in}(\theta) = {} & \frac{1}{213892} + \frac{18566}{9395}\cos(\theta) + \frac{\cos(2\theta)}{117817} - \frac{1973}{35647}\cos(3\theta) \\
& - \frac{\cos(4\theta)}{84836} - \frac{337}{9801}\cos(5\theta) - \frac{\cos(6\theta)}{19746} + \frac{53}{5756}\cos(7\theta) \\
& - \frac{\cos(8\theta)}{420042} - \frac{\cos(9\theta)}{4738} + \frac{3\cos(10\theta)}{11954} - \frac{\cos(11\theta)}{776} + \frac{\cos(12\theta)}{5488} \\
& + \frac{1097}{13625}\sin(\theta) - \frac{\sin(2\theta)}{103485} - \frac{2003}{9487}\sin(3\theta) - \frac{\sin(4\theta)}{46332} \\
& + \frac{1317}{54185}\sin(5\theta) + \frac{\sin(6\theta)}{85313} + \frac{103}{24125}\sin(7\theta) + \frac{\sin(8\theta)}{8809} \\
& - \frac{29}{8781}\sin(9\theta) + \frac{\sin(10\theta)}{18036} + \frac{3\sin(11\theta)}{7760} - \frac{7\sin(12\theta)}{12512}.
\end{aligned}$$

Example 1: van der Pol system

$$
\begin{aligned}
y_{in}(\theta) &= -\frac{1}{287689}+\frac{1207\cos(\theta)}{18761}+\frac{\cos(2\theta)}{180371}-\frac{721\cos(3\theta)}{11644}\\
&+\frac{\cos(4\theta)}{46468}+\frac{116\cos(5\theta)}{18697}-\frac{\cos(6\theta)}{85239}-\frac{27\cos(7\theta)}{11035}\\
&-\frac{\cos(8\theta)}{9627}-\frac{13\cos(9\theta)}{9450}-\frac{\cos(10\theta)}{19827}+\frac{7\cos(11\theta)}{12142}\\
&+\frac{\cos(12\theta)}{2425}-\frac{22778\sin(\theta)}{10867}+\frac{\sin(2\theta)}{106711}+\frac{295\sin(3\theta)}{14827}\\
&-\frac{\sin(4\theta)}{98567}+\frac{35\sin(5\theta)}{25042}-\frac{\sin(6\theta)}{20630}-\frac{21\sin(7\theta)}{7234}+\frac{\sin(8\theta)}{3180308}\\
&+\frac{21\sin(9\theta)}{14432}+\frac{2\sin(10\theta)}{9397}+\frac{7\sin(11\theta)}{9435}+\frac{\sin(12\theta)}{5087}
\end{aligned}
$$

Example 1: van der Pol system

We have that the curve is transversal to the system if the trigonometric polynomial

$$f(\theta) := P(x_{in}(\theta), y_{in}(\theta))\, y_{in}'(\theta) - Q(x_{in}(\theta), y_{in}(\theta))\, x_{in}'(\theta)$$

does not change sign for all $\theta \in [0, 2\pi]$.
Recall that we consider the van der Pol system with $\epsilon = 1$:

$$P(x,y) = y - \left(\frac{x^3}{3} - x\right), \quad Q(x,y) = -x.$$

$f(\theta)$ is a trigonometric polynomial of degree 48.

Example 1: van der Pol system

$f(\theta)$ is a trigonometric polynomial of degree 48.

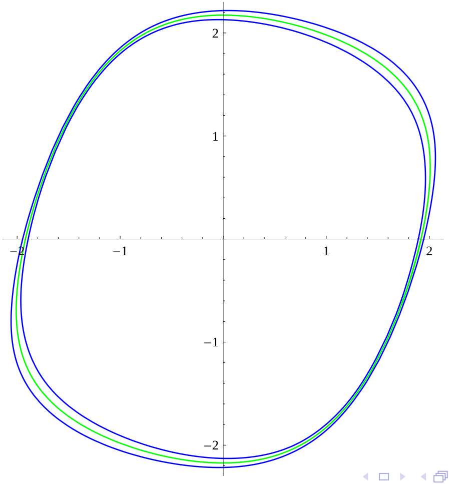
As we stated in the description of the method, we expand $f(\theta)$ in powers of $\cos\theta, \sin\theta$ and we change $\cos\theta$ by u and $\sin\theta$ by v, in order to get a polynomial $\tilde{f}(u,v)$ which is of degree 48.

Then we take the resultant of $\tilde{f}(u,v)$ with u^2+v^2-1 with respect to v. This resultant is a polynomial in u of degree 96.

Finally we prove that the latter expression has no real roots for $u \in [-1,1]$ by using Sturm's sequences.

Example 1: van der Pol system

To obtain the outer transversal curve, we choose $\varepsilon = -0.05$ and $m = 12$ and we repeat the process.



Examples

1. van der Pol system
2. Brusselator system
3. Rychkov system

Example 2: Brusselator system

We consider the system

$$\dot{x} = a - (b+1)x + x^2 y, \quad \dot{y} = bx - x^2 y,$$

with $a, b > 0$.

This system has a unique singular point at $(a, b/a)$.
The semi-axis $\Sigma := \{(x_0, b/a) : x_0 > a\}$ is transversal to the flow.
We take $a = 1$ and $b = 3$ and the system exhibits a hyperbolic stable limit cycle which cuts Σ at $x_0^* \sim 2.30354344$ and has period $T \sim 7.15691986$.

Example 2: Brusselator system

We consider the system

$$\dot{x} = a - (b+1)x + x^2 y, \quad \dot{y} = bx - x^2 y,$$

with $a = 1$ and $b = 3$.

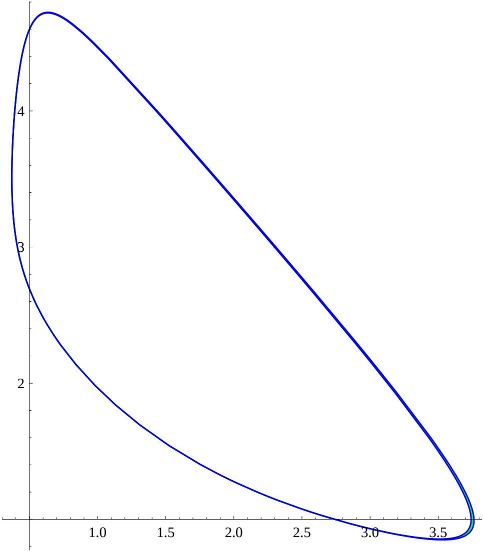
The system exhibits a hyperbolic stable limit cycle which cuts Σ at $x_0^* \sim 2.30354344$ and has period $T \sim 7.15691986$.

By our method we obtain an inner transversal curve and an outer transversal curve

$$(x_{in}(\theta), y_{in}(\theta)) \text{ and } (x_{ex}(\theta), y_{ex}(\theta)) \quad \text{with } \theta \in [0, 2\pi],$$

which provide a Poincaré–Bendixson annular region.
The inner transversal curve cuts Σ at ~ 2.2981 and the outer transversal curve at ~ 2.3091.

Example 2: Brusselator system



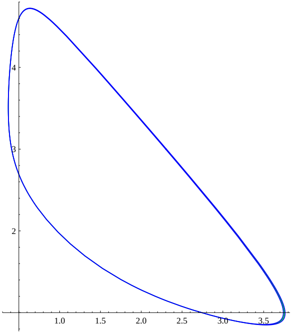
The transversal curves are represented in blue and the (numerical) limit cycle in green.

The three curves are indistinguishable.

Example 2: Brusselator system

The inner curve is obtained with $\varepsilon = 0.05$ and outer curve is obtained with $\varepsilon = -0.05$, and both of them with $m = 140$.

We have **not** been able to find a transversal curve with a lower value of m.



Examples

1. van der Pol system
2. Brusselator system
3. Rychkov system

Example 3: Rychkov system

We consider the system studied by Rychkov in 1975

$$\dot{x} = y - \left(x^5 - \mu x^3 + \delta x\right), \quad \dot{y} = -x,$$

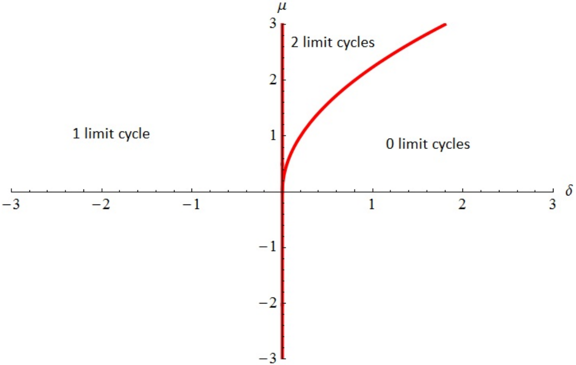
with $\delta, \mu \in \mathbb{R}$.

This system is also studied by Alsholm [1992] and Odani [1996]. It is known that it has 2 limit cycles if and only if $\delta > 0$ and $0 < \delta < \Delta^*(\mu)$, for some unknown function Δ^*.
For the value $\delta = \Delta^*(\mu)$ the system has a double limit cycle and, varying δ, it presents a *blue-sky bifurcation*.

Challenge

Find sharp estimations of $\Delta^*(\mu)$

Example 3: Rychkov system



The function $\delta = \Delta^*(\mu)$.

Odani proved that $\Delta^*(\mu) > \frac{\mu^2}{5}$.

Example 3: Rychkov system

We fix $\mu = 1$ in Rychkov system

$$\dot{x} = y - \left(x^5 - x^3 + \delta x\right), \quad \dot{y} = -x.$$

we want to find sharp bounds for $\delta^* := \Delta^*(1)$.
Notice that Odani's result says that $\delta^* > \frac{1}{5} = 0.2$. We prove:

Theorem

$$0.224 < \delta^* < 0.2249654$$

Example 3: Rychkov system

Since Rychkov system

$$\dot{x} = y - \left(x^5 - x^3 + \delta x\right), \quad \dot{y} = -x.$$

with $\delta \in \mathbb{R}$ is a semi-complete family of rotated vector fields with respect to δ, it holds that:

It for $\delta = \bar{\delta}$ the system has two limit cycles then $\bar{\delta} < \delta^*$.

Therefore, to prove the inequality $0.224 < \delta^*$, it suffices to prove that Rychkov system has two limit cycles for $\delta = 0.224$.

Similarly, if for $\delta = \hat{\delta}$ the system has no limit cycle then $\delta^* < \hat{\delta}$.

Then, to prove the inequality $\delta^* < 0.2249654$, it suffices to prove that Rychkov system has no limit cycle for $\delta = 0.2249654$.

Example 3: Rychkov system

Proof of $\quad 0.224 < \delta^* < 0.2249654$

- Inequality $0.224 < \delta^*$ is proved by using the tools introduced in this paper.
- Inequality $\delta^* < 0.2249654$ is proved by constructing a polynomial function $R(x,y) = \sum_{j=0}^{300} R_j(x)\, y^j$, such that

$$\dot{R}(x,y) = \left(y - \left(x^5 - x^3 + \frac{2\,249\,654}{10\,000\,000}x\right)\right)\frac{\partial R(x,y)}{\partial x} - x\frac{\partial R(x,y)}{\partial y} = M(x) \geq 0,$$

for some polynomial $M(x)$ of degree 1496. This method is proposed and already used for general classical Liénard systems by Cherkas and also by Giacomini-Neukirch.

Example 3: Rychkov system

We illustrate our results taking $\delta = 0.2$, $\delta = 0.22$ and $\delta = 0.224$.

In the three cases the smaller limit cycle is hyperbolic and unstable and the bigger one is hyperbolic and stable.

Example 3: Rychkov system

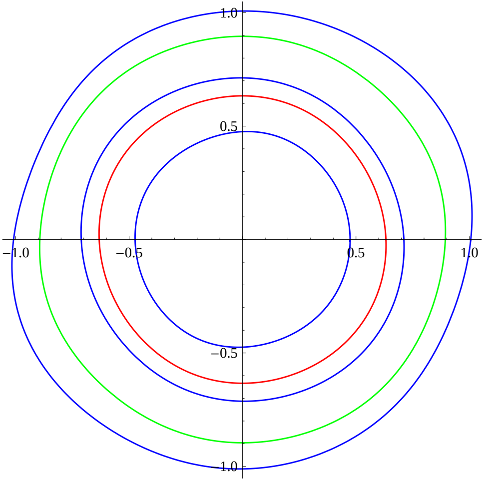
Rychkov system with $\delta = 0.2$.

The limit cycles cut Σ at $x_0 \sim 0.632018$ and $x_0 \sim 0.893787$.

- The interior transversal curve cuts Σ at $x_0 = 0.474059$, it has been obtained from the unstable limit cycle taking $\varepsilon = 0.1$ and $m = 5$.
- The transversal curve in the middle cuts Σ at $x_0 = 0.711158$, it has been obtained from the unstable limit cycle taking $\varepsilon = -0.05$ and $m = 7$.
- The exterior transversal curve cuts Σ at $x_0 = 1.00597$, it has been obtained from the stable limit cycle taking $\varepsilon = -0.1$ and $m = 5$.

Example 3: Rychkov system

Rychkov system with $\delta = 0.2$.



Transversal curves are represented in blue.

Example 3: Rychkov system

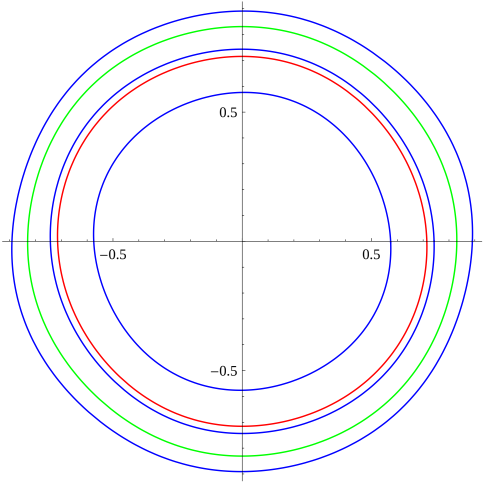
Rychkov system with $\delta = 0.22$.

The limit cycles cut Σ at $x_0 \sim 0.714276$ and $x_0 \sim 0.830266$.

- The interior transversal curve cuts Σ at $x_0 = 0.57421$, it has been obtained from the unstable limit cycle taking $\varepsilon = 0.1$ and $m = 7$.
- The transversal curve in the middle cuts Σ at $x_0 = 0.74227$, it has been obtained from the unstable limit cycle taking $\varepsilon = -0.02$ and $m = 7$.
- The exterior transversal curve cuts Σ at $x_0 = 0.8905$, it has been obtained from the stable limit cycle taking $\varepsilon = -0.05$ and $m = 7$.

Example 3: Rychkov system

Rychkov system with $\delta = 0.22$.



Transversal curves are represented in blue.

Example 3: Rychkov system

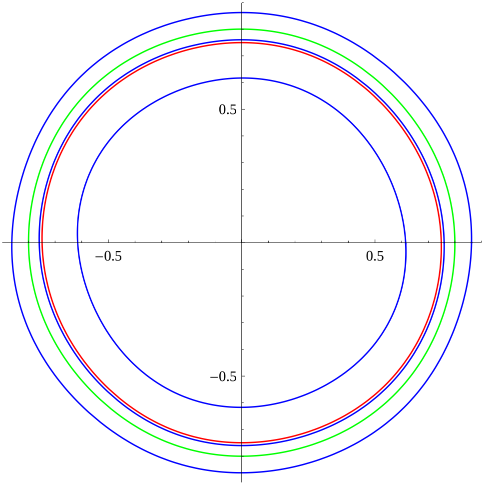
Rychkov system with $\delta = 0.224$.

The limit cycles cut Σ at $x_0 \sim 0.748705$ and $x_0 \sim 0.799588$.

- The interior transversal curve cuts Σ at $x_0 = 0.615043$, it has been obtained from the unstable limit cycle taking $\varepsilon = 0.1$ and $m = 7$.
- The transversal curve in the middle cuts Σ at $x_0 = 75939$, it has been obtained from the unstable limit cycle taking $\varepsilon = -0.008$ and $m = 10$.
- The exterior transversal curve cuts Σ at $x_0 = 0.862111$, it has been obtained from the stable limit cycle taking $\varepsilon = -0.05$ and $m = 7$.

Example 3: Rychkov system

Rychkov system with $\mu = 1$ and $\delta = 0.224$.

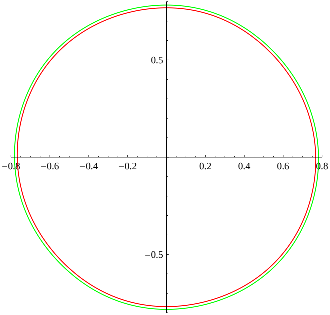


Transversal curves are represented in blue.

Example 3: Rychkov system

Rychkov system with $\delta = 0.2249$.

The limit cycles cut Σ at $x_0 \sim 0.767488$ and $x_0 \sim 0.781773$.



The figure represents the limit cycles in the case $\delta = 0.2249$, numerically found.

We can find a *numerical* transversal curve in the intermediate region between the two limit cycles but we have **not** been able yet to prove that the associated curve with rational coefficients is transversal.

Outline of the talk

The value of m

In the presented method, we cannot a priori predict the value of m necessary for the approximated curve to be transversal.

Question

Which is the difference between the limit cycle in the van der Pol system and the limit cycle in the Brusselator system that force that the value of m is so different?
Recall that in the first one we have got good approaches taking $m = 12$ while in the second one $m = 140$.

We have taken random samples of n points in the case of the Brusselator system and we always need $m = 140$ or higher.

The value of m

We prove that the value of m depends on the moduli of coefficients of the Fourier series expansion of the limit cycles.

By computing (numerically) the Fourier expansions of both limit cycles we have found an explanation of the different computational difficulties for finding approximations of the limit cycles of the van der Pol and Brusselator systems.

Outline of the talk

Piecewise linear system with 3 limit cycles

In the work

J. LLIBRE AND E. PONCE, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*. Dynam. Contin. Discrete Impuls. Systems. Ser. B **19** (2012) 325–335.

the authors provide a rigorous computer assisted proof that a certain example of a piecewise linear differential system with two zones can have 3 nested limit cycles of crossing type surrounding a unique equilibrium.

Essentially the example was introduced and numerically studied in

S.-M. HUAN AND X.-S. YANG, *On the number of limit cycles in general planar piecewise linear systems*, Discrete Contin. Dyn. S-A **32** (2012) 2147–2164.

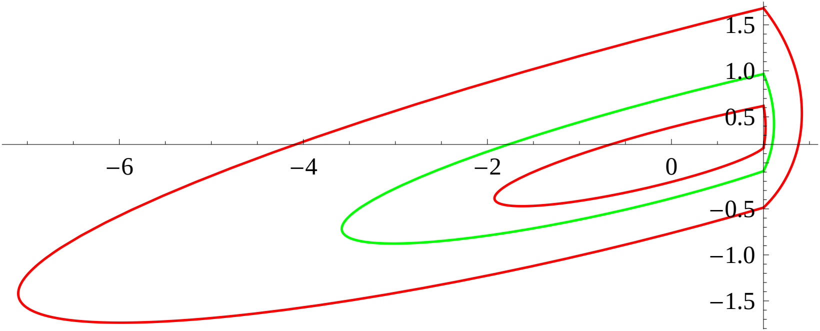
Piecewise linear system with 3 limit cycles

If $x \geq 1$

$$\dot{x} = \frac{19}{50}x - y$$
$$\dot{y} = x + \frac{19}{50}y$$

if $x < 1$

$$\dot{x} = \frac{4}{3}x - \frac{20}{3}y$$
$$\dot{y} = \frac{377}{750}x + \frac{26}{15}y$$



Piecewise linear system with 3 limit cycles

Work in progress

We want to prove the existence of the three limit cycles with our method.

Here, we can glue portions of transversal curves in order to form a Poincaré–Bendixson annular region. The value of K needs to be taken so that the cuts of these transversal curves with the straight line $x = 1$ gives the desired stabilities. We will need the Corollary of our main result.

Thank you very much for your attention.

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