Poincaré half-maps in planar linear systems via inverse integrating factors

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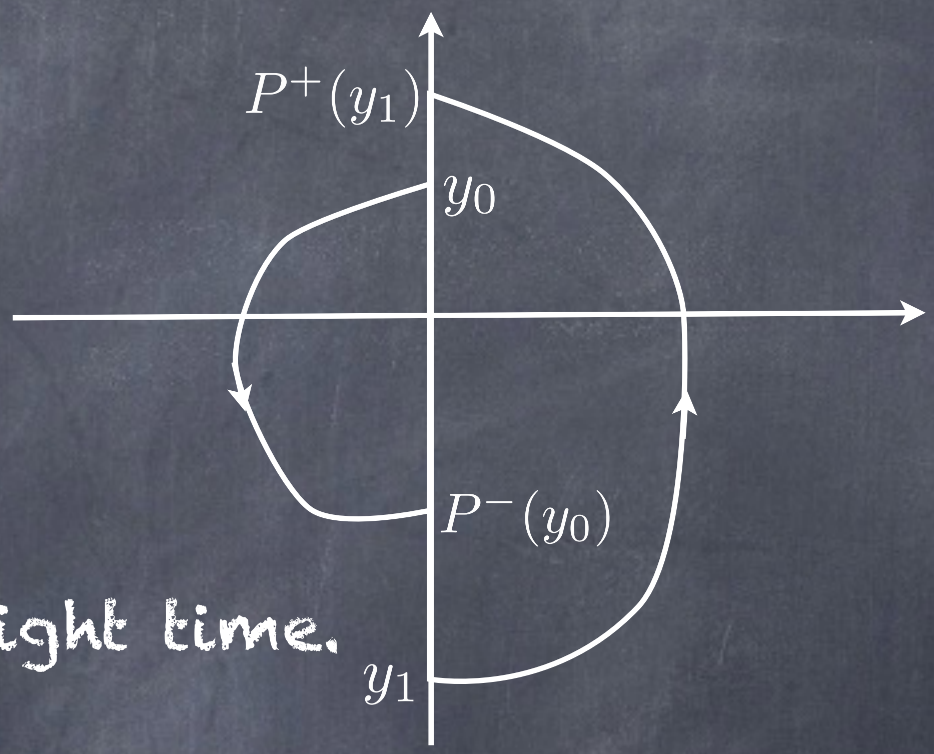
Poincaré half-maps and closing equations in Planar Piecewise Linear Dynamical Systems

$$y_1 = P^-(y_0), y_0 = P^+(y_1)$$



Two "problems":

- Implicit (nonlinear) dependance on the flight time.

- The final expression is strongly conditioned by the spectrum of the matrices of the system and many different cases appear.

Main goal of this work:

To presente an alternative way to override both previous "problems".

A brief review on Inverse Integrating Factors

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \Longrightarrow \quad g(x,y)dx - f(x,y)dy = 0$$

Integrating factor in the form $\dfrac{1}{V(x,y)}$ with $V(x,y) \neq 0$

If the ordinary differential equation

$$\frac{g(x,y)}{V(x,y)}dx - \frac{f(x,y)}{V(x,y)}dy = 0$$

is exact, then inverse integrating factor $V(x,y)$ satisfies the partial differential equation

$$f(x,y)\frac{\partial V}{\partial x}(x,y) + g(x,y)\frac{\partial V}{\partial y}(x,y) = \left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y)\right)V(x,y)$$

García, Isaac A.; Grau, Maite A survey on the inverse integrating factor. Qual. Theory Dyn. Syst. 9 (2010), no. 1-2, 115-166.

A brief review on Inverse Integrating Factors

$$(1) \quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

$V(x,y)$ is an inverse integrating factor of (1) if

$$f(x,y)\frac{\partial V}{\partial x}(x,y) + g(x,y)\frac{\partial V}{\partial y}(x,y) = \left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial g}{\partial y}(x,y)\right)V(x,y)$$

that is

$$\nabla V(x,y)\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = V(x,y)\,\mathrm{div}\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

A brief review on Inverse Integrating Factors

$$(1)\quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

If $V(x,y)$ is an inverse integrating factor of (1), then

a change of variable with

$$ds = V(x,y)dt$$

transforms system (1) into the Halmiltonian system

$$\begin{cases} \frac{dx}{ds} = \frac{f(x,y)}{V(x,y)} \\ \frac{dy}{ds} = \frac{g(x,y)}{V(x,y)} \end{cases}$$

A brief review on Inverse Integrating Factors

$$(1)\ \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \quad \overset{ds = V(x,y)dt}{\Longrightarrow} \begin{cases} \dfrac{dx}{ds} = \dfrac{f(x,y)}{V(x,y)} \\ \dfrac{dy}{ds} = \dfrac{g(x,y)}{V(x,y)} \end{cases}$$

is Hamiltonian

$V(x,y)$ is an inverse integrating factor of (1)

- The Halmintonian systems has no limit cycles.

The limit cycle of system (1), if any, must be contained in zero set of V, $V^{-1}(\{0\})$

A brief survey on Inverse Integrating Factors

$$(1)\ \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

$V(x,y)$ is an inverse integrating factor de (1)

The limit cycle of system (1), if any, must be contained in zero set of V, $V^{-1}(\{0\})$

Roughly speaking

If p_0 is a hyperbolic saddle point of (1), then V vanishes on the four separatrices.

If (1) has a monodromic graphic (homoclinic, heteroclinic cycle, ...), then V vanishes on this graphic.

L.R. Berrone and H. Giacomini, On the vanishing set of inverse integrating factors, Qual. Th. Dyn. Systems 1 (2000), 211-230.
I.A. García and D.S. Shafer, Integral invariants and limit sets of planar vector fields, J. Differential Equations 217 (2005), 363-376.
H. Giacomini, J. Llibre and M. Viano, On the nonexistence, existence, and uniqueness of limit cycles, Nonlinearity 9 (1996), 501-516.
García, Isaac A.; Grau, Maite A survey on the inverse integrating factor. Qual. Theory Dyn. Syst. 9 (2010), no. 1-2, 115-166.

Inverse Integrating Factors for Linear Systems

$$(SL) \begin{cases} \dot{x} = a_{11}x + a_{12}y \\ \dot{y} = a_{21}x + a_{22}y \end{cases} \qquad (SL) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

$V(x,y)$ an inverse integrating factor of (SL)

V MUST VANISH ON THE INVARIANT MANIFOLDS OF (SL)

$$V(x,y) = \det \left(A \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right) = a_{21}x^2 + (a_{22} - a_{11})xy - a_{11}y^2$$

satisfies the equation

$$\nabla V(x,y) \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = V(x,y) \operatorname{div} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

that is, V is an inverse integrating factor of (SL).

Inverse Integrating Factors for Linear Systems

Nonhomegeneous system in Lienard canonical form

$$(FCL)\ \begin{cases} \dot{x} = Tx - y \\ \dot{y} = Dx - a \end{cases} \qquad D \neq 0, a \neq 0$$

(FCL) has a unique equilibrium point

$$(\bar{x}, \bar{y}) = \left(\frac{a}{D}, \frac{Ta}{D}\right)$$

The function

$$V(x,y) = D\left(x - \frac{a}{D}\right)^2 - T\left(x - \frac{a}{D}\right)\left(y - \frac{Ta}{D}\right) + \left(y - \frac{Ta}{D}\right)^2$$

is an inverse integrating factor of (FCL)

Inverse Integrating Factors for Linear Systems

Nonhomegeneous system in Lienard canonical form

$$(FCL)\ \left\{\begin{array}{l}\dot{x} = Tx - y\\ \dot{y} = Dx - a\end{array}\right. \quad D \neq 0 \qquad \overset{ds = V(x,y)dt}{\Longrightarrow} \qquad \left\{\begin{array}{l}\frac{dx}{ds} = \frac{Tx-y}{V(x,y)}\\ \frac{dy}{ds} = \frac{Dx-a}{V(x,y)}\end{array}\right.$$

es Hamiltoniano

$$V(x,y) = D\left(x - \frac{a}{D}\right)^2 - T\left(x - \frac{a}{D}\right)\left(y - \frac{Ta}{D}\right) + \left(y - \frac{Ta}{D}\right)^2$$

There exists a First Integral H(x,y)

$$\left\{\begin{array}{l}\frac{\partial H}{\partial y}(x,y) = \frac{Tx-y}{V(x,y)}\\ \frac{\partial H}{\partial x}(x,y) = -\frac{Dx-a}{V(x,y)}\end{array}\right.$$

$$H(x,y) = \int \frac{Tx-y}{V(x,y)} dy$$

H is constant on any orbit of system (FCL)

J. Llibre, E. Nuñez, A. E. Teruel. Limit cycles for planar piecewise linear differential systems via first integrals. Qual. Theory Dyn. Syst., 3(1), 29-50, 2002.

Poincaré half-maps in Planar Linear Dynamical Systems

$$(FCL)\begin{cases}\dot{x} = Tx - y\\ \dot{y} = Dx - a\end{cases} \qquad D \neq 0$$

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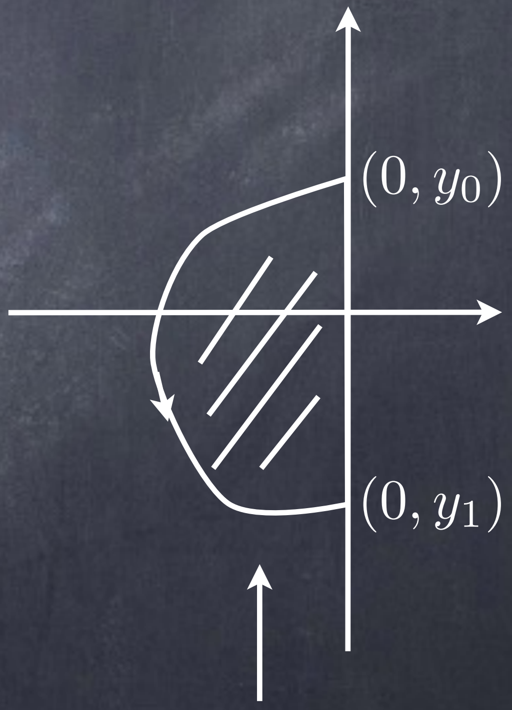
$$H(x,y) = \int \frac{Tx-y}{V(x,y)}dy$$

H is constant on any orbit of system (FCL)



$$H(0,y_0) = H(0,y_1)$$

No equilibrium points

Poincaré half-maps in Planar Linear Dynamical Systems

$$(FCL)\begin{cases}\dot{x} = Tx - y\\ \dot{y} = Dx - a\end{cases} \qquad D \neq 0$$

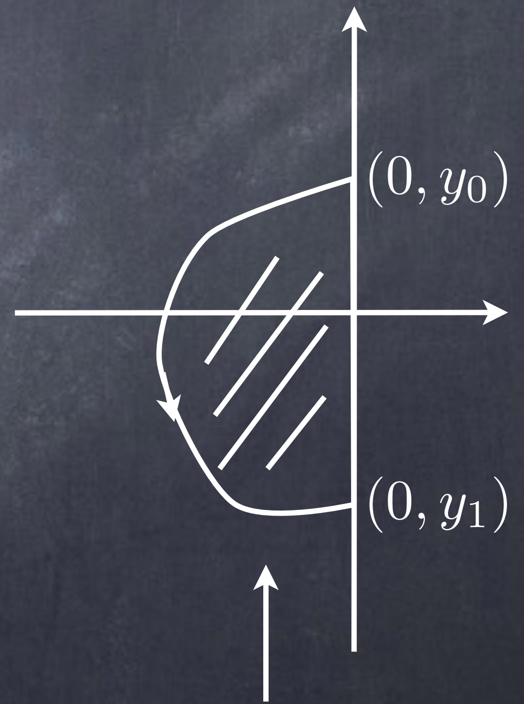
$$V(x,y) = D\left(x - \frac{a}{D}\right)^2 - T\left(x - \frac{a}{D}\right)\left(y - \frac{Ta}{D}\right) + \left(y - \frac{Ta}{D}\right)^2$$

$$H(x,y) = \int \frac{Tx - y}{V(x,y)}dy$$

$$H(0,y_0) = H(0,y_1)$$

$$\int_0^{y_0} \frac{-y}{V(0,y)}dy = \int_0^{y_1} \frac{-y}{V(0,y)}dy$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)}dy = 0$$



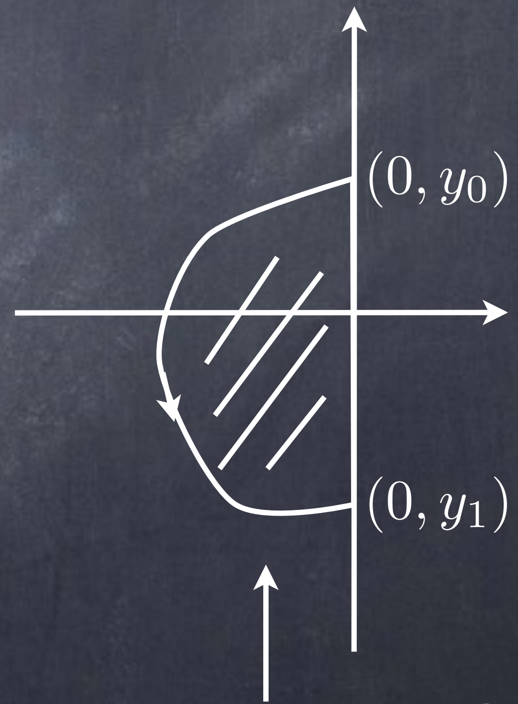
No equilibrium points

Poincaré half-maps in Planar Linear Dynamical Systems

$$(FCL)\begin{cases}\dot{x} = Tx - y\\ \dot{y} = Dx - a\end{cases} \qquad D \neq 0$$

$$V(x,y) = D\left(x - \frac{a}{D}\right)^2 - T\left(x - \frac{a}{D}\right)\left(y - \frac{Ta}{D}\right) + \left(y - \frac{Ta}{D}\right)^2$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)}dy = 0$$



$$\int_{y_1}^{y_0} \frac{-Dy}{a^2 - aTy + Dy^2}dy = 0$$

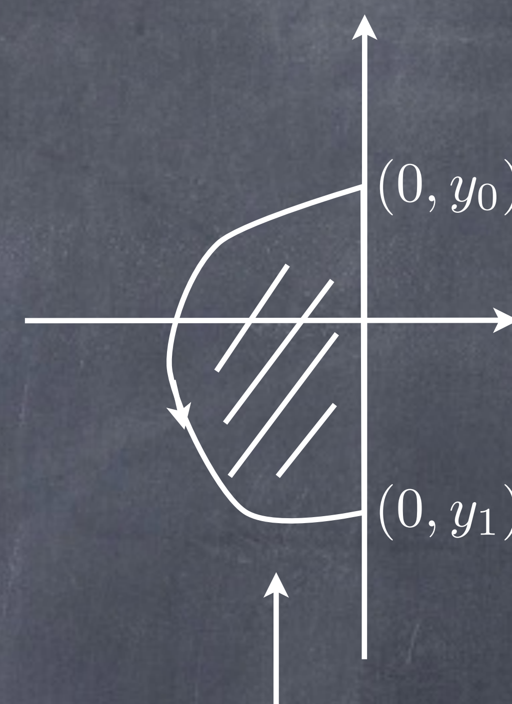
NO MORE FLIGHT TIME

No equilibrium points

Poincaré half-maps in Planar Linear Dynamical Systems

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$\int_{y_1}^{y_0} \frac{-Dy}{a^2 - aTy + Dy^2} dy = 0$$

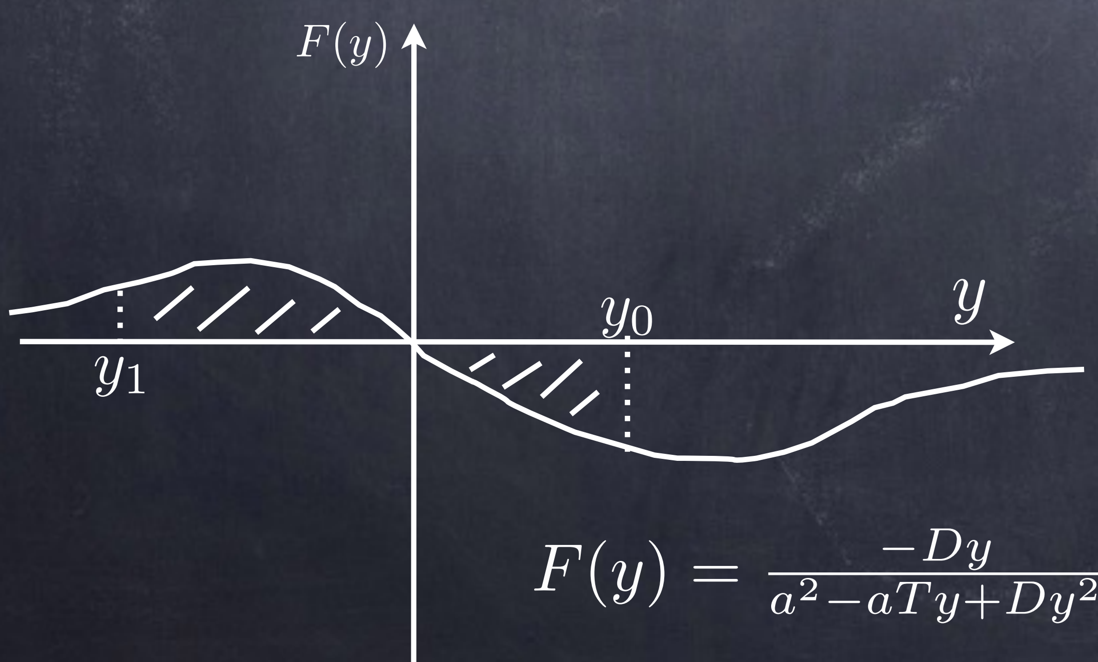


$$D \cdot V(0,y) = a^2 - aTy + Dy^2$$

A concrete case: Focus o center

$$a^2 - aTy + Dy^2 > 0 \quad \forall y \in \mathbb{R}$$

No equilibrium points



Fixed $y_0 \geq 0$, exists a unique $y_1 \leq 0$ such that

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

Poincaré half-maps in Planar Linear Dynamical Systems

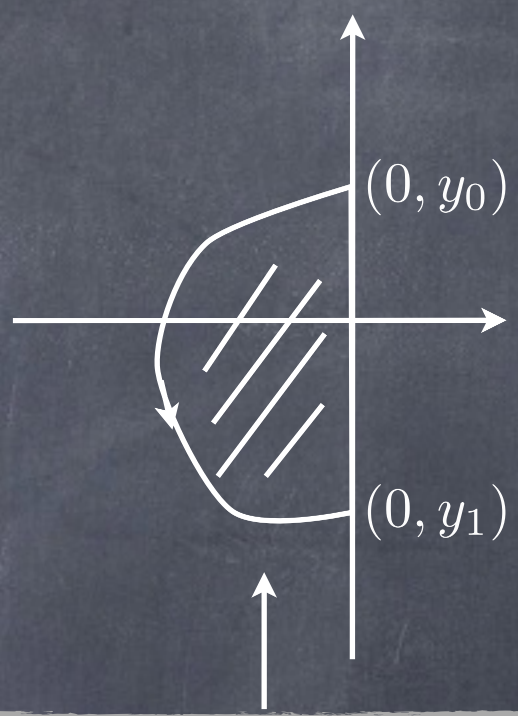
$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$D \cdot V(0,y) = a^2 - aTy + Dy^2$$

Let I be the conex component of set

$$\{y \in \mathbb{R} : D \cdot V(0,y) > 0\}$$

containing the origin.



No equilibrium points

Theorem

Given $y_0 \in I \cap [0, +\infty)$, there exists a unique $y_1 \in I \cap (-\infty, 0]$ such that

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

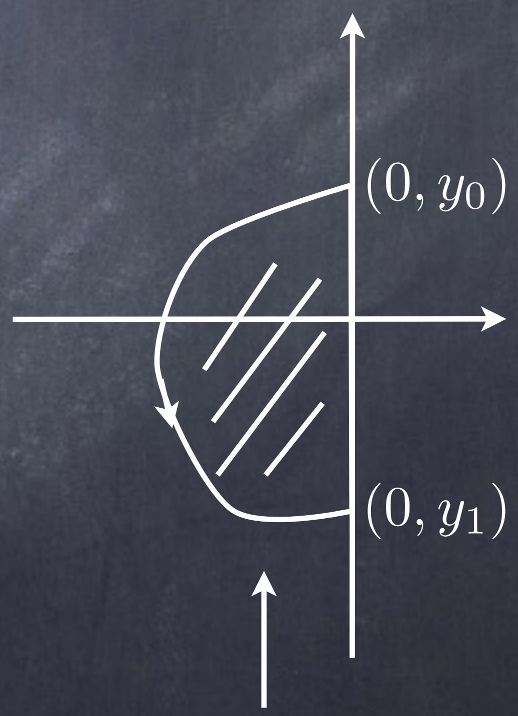
Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

Theorem (Definition of the Poincaré half maps)

Given $y_0 \in I \cap [0,+\infty)$, there exists a unique $y_1 \in I \cap (-\infty, 0]$ such that

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$\begin{array}{rccc} S: y_0 \in I \cap [0,+\infty) & \longrightarrow & (-\infty,0] \cap I \\ y_0 & \longmapsto & y_1 = S(y_0) \end{array}$$



S is analytic in $I \cap (0,+\infty)$.

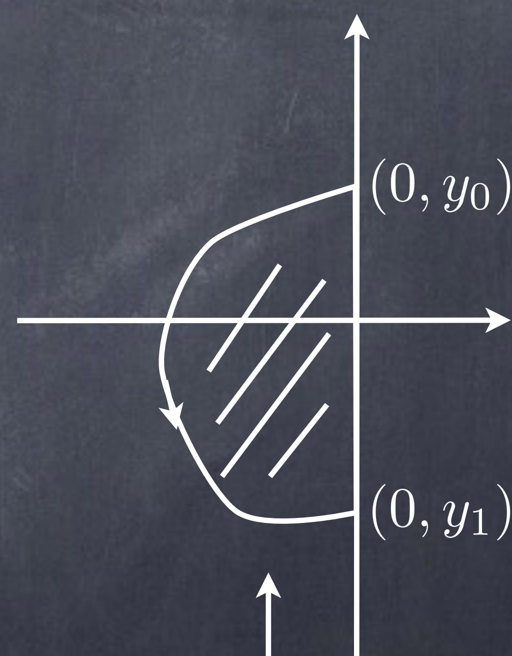
No equilibrium points

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$
\begin{array}{rcl} S : y_0 \in I \cap [0, +\infty) & \longrightarrow & (-\infty, 0] \cap I \\ y_0 & \longmapsto & y_1 = S(y_0) \end{array}
$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$y_1' = S'(y_0) = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}$$



$$
\left\{ \begin{array}{l} y_1' = \dfrac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(0) = 0 \end{array} \right.
$$

← Autonomous IVP

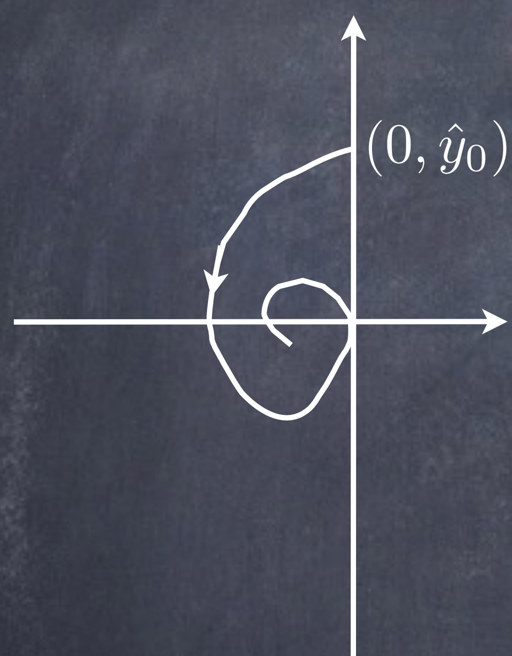
No equilibrium points

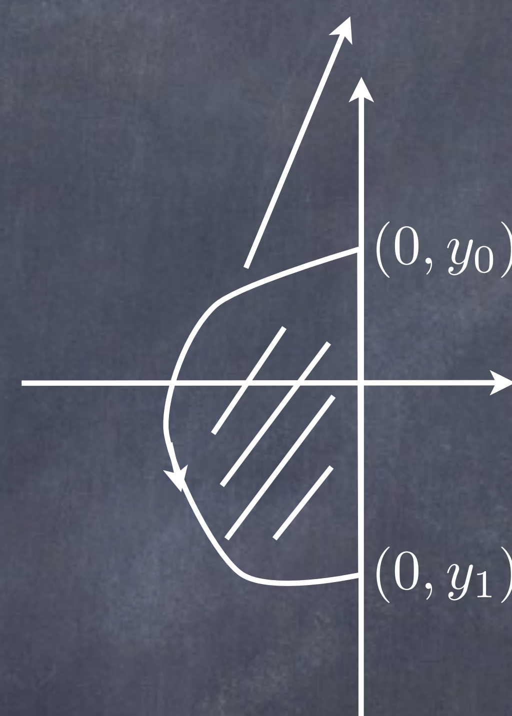
I. García, H. Giacomini, M. Grau. The Inverse Integrating Factor and the Poincaré Map. Trans. Amer. Math. Soc.c, 362(7), 3591-3612, 2010.

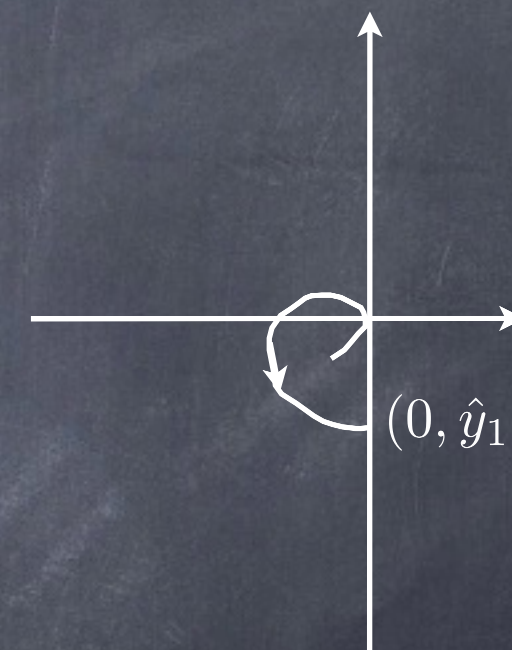
Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$(FCL)\begin{cases}\dot{x} = Tx - y\\ \dot{y} = Dx - a\end{cases}$$

No equilibrium points







$$\begin{cases}y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}\\ y_1(\hat{y}_0) = 0\end{cases}$$

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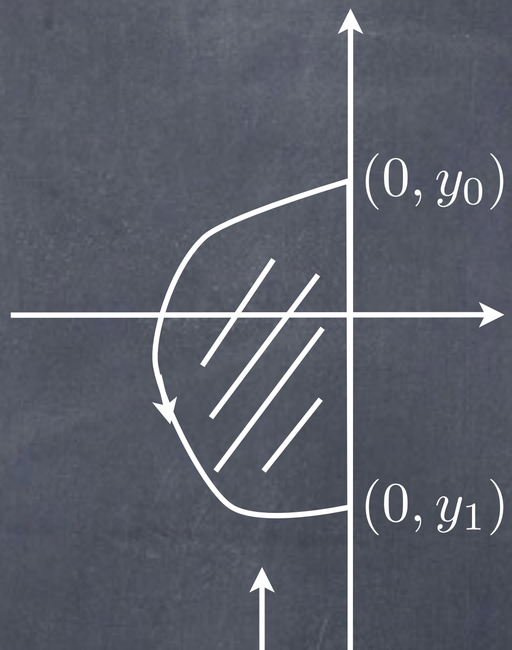
$$\begin{cases}y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}\\ y_1(0) = \hat{y}_1\end{cases}$$

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(0) = 0 \end{cases}$$

← Autonomous IVP



↑ No equilibrium points

$$D \cdot V(0,y) = a^2 - aTy + Dy^2 > 0, \quad y \in I$$

$$y_1'(y_0) = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} < 0, \quad y_0 \cdot y_1 < 0$$

The Poincaré half-map is decreasing

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$\begin{array}{rcl} S : y_0 \in I \cap [0,+\infty) & \longrightarrow & (-\infty, 0] \cap I \\ y_0 & \longmapsto & y_1 = S(y_0) \end{array}$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$y_1' = S'(y_0) = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}$$

$$\text{sign}\,(y_1''(y_0)) = \text{sign}(y_0 + y_1) = \text{sign}(-aDT)$$

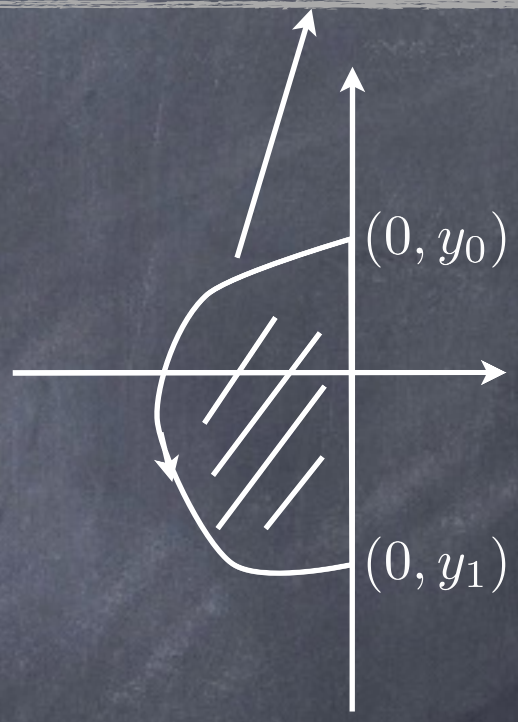
The sign of the second derivative of function f is constant

NO MORE "MANY CASES"

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$(FCL)\begin{cases}\dot{x}=Tx-y\\ \dot{y}=Dx-a\end{cases}\qquad D\neq 0, a\neq 0$$

No equilibrium points



$$\begin{cases}y_1'=\frac{y_0V(0,y_1)}{y_1V(0,y_0)}=\frac{y_0}{y_1}\frac{a^2-aTy_1+Dy_1^2}{a^2-aTy_0+Dy_0^2}\\ y_1(0)=0\end{cases}$$

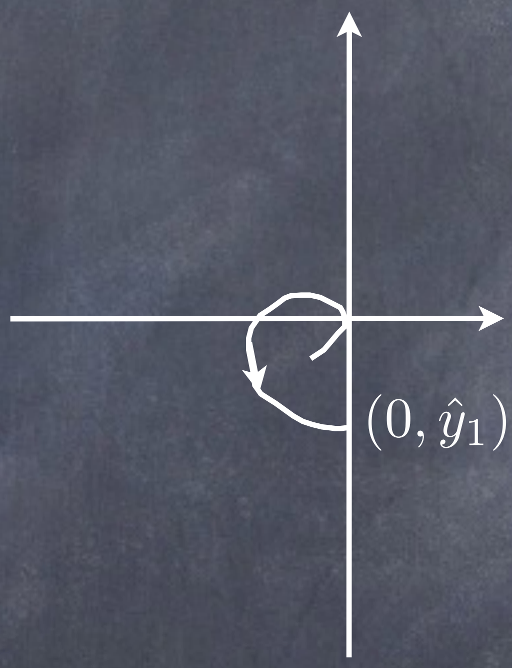
$$y_1(y_0)=-y_0-\frac{2T}{3a}y_0^2-\frac{4T^2}{9a^2}y_0^3+\frac{2\left(9DT-22T^3\right)}{135a^3}y_0^4+\frac{4\left(27DT^2-26T^4\right)}{665a^3}y_0^5+\cdots$$

E. Freire, E. Ponce & F. Torres, Canonical Discontinuous Planar Piecewise Linear Systems, SIAM J. App. Dyn. Syst. 2012, 11, 181-211

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$$(FCL)\begin{cases}\dot{x}=Tx-y\\ \dot{y}=Dx-a\end{cases}\qquad D\neq 0, a\neq 0$$

$$\begin{cases}y_1'=\dfrac{y_0V(0,y_1)}{y_1V(0,y_0)}\\ y_1(0)=\hat{y}_1\end{cases}$$



$$y_1(y_0)=\hat{y}_1+\frac{DV(0,\hat{y}_1)}{a^2\hat{y}_1}y_0^2+\frac{2TDV(0,\hat{y}_1)}{a^3\hat{y}_1}y_0^3-\frac{3DV(0,\hat{y}_1)\left(a^2+\left(D-2T^2\right)\hat{y}_1^2\right)}{a^4\hat{y}_1^3}y_0^4+\cdots$$

E. Freire, E. Ponce & F. Torres, Canonical Discontinuous Planar Piecewise Linear Systems, SIAM J. App. Dyn. Syst. 2012, 11, 181-211

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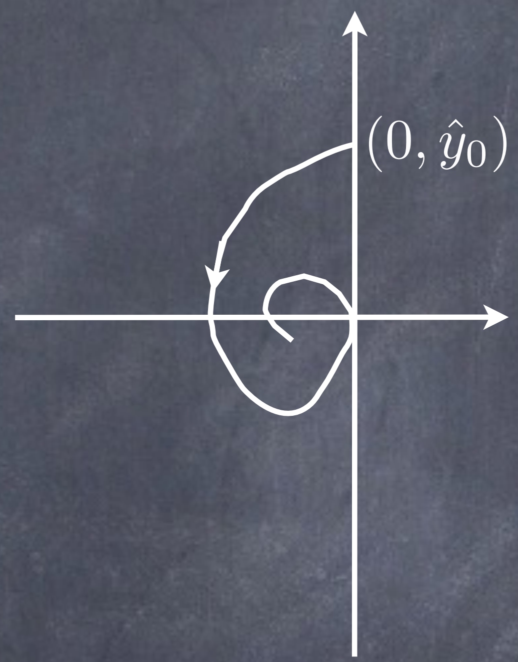


$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}\\ y_1(\hat{y}_0) = 0\end{cases}$$

$$\begin{aligned} y_1(y_0) &= \sqrt{\frac{2a^2\hat{y}_0}{V(0,\hat{y}_0)}}|y_0-\hat{y}_0|^{1/2} - \frac{2aT\hat{y}_0}{3V(0,\hat{y}_0)}|y_0-\hat{y}_0| \\ &+ \frac{\left(9a^2+\left(9D+2T^2\right)\hat{y}_0^2)\right)}{18\sqrt{2}a^2\hat{y}_0^2}\left(\frac{a^2\hat{y}_0}{V(0,\hat{y}_0)}\right)^{3/2}|y_0-\hat{y}_0|^{3/2}+\cdots\end{aligned}$$

E. Freire, E. Ponce & F. Torres, Canonical Discontinuous Planar Piecewise Linear Systems, SIAM J. App. Dyn. Syst. 2012, 11, 181-211

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

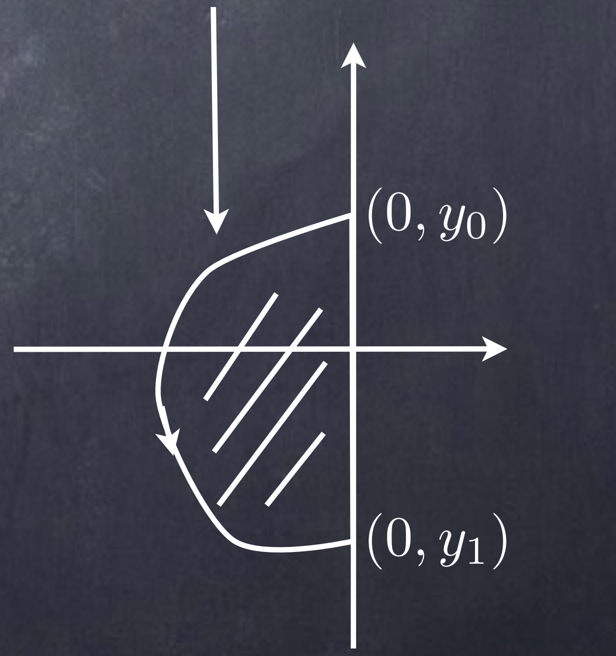
$$\frac{V(0,y_1)}{V(0,y_0)} = \frac{a^2-aTy_1+Dy_1^2}{a^2-aTy_0+Dy_0^2} = \exp\left(\int_{y_1}^{y_0} \frac{aT}{a^2-aTy+Dy^2}dy\right)$$

$$y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)}$$

$$y_1' = \frac{y_0}{y_1}\exp(T\tau)$$

τ is the half-flight time

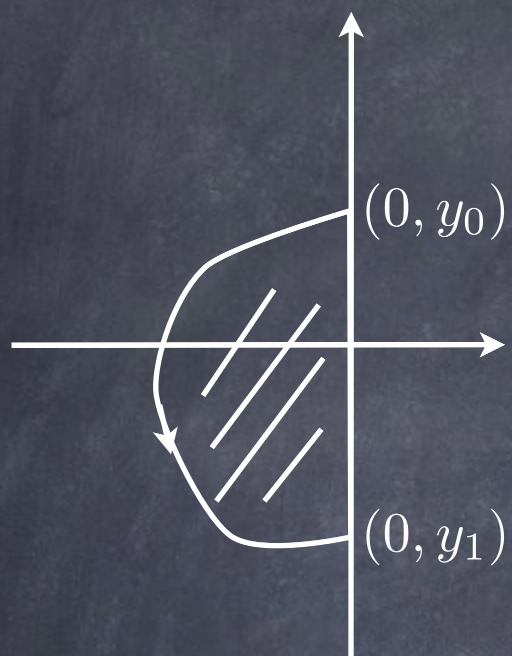
No equilibrium points

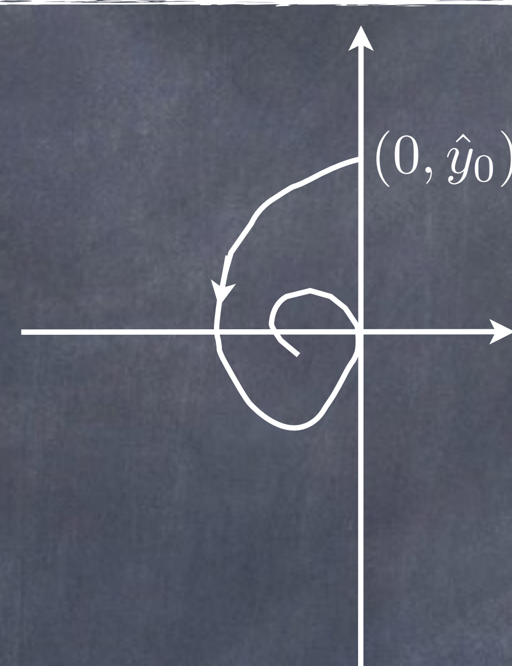


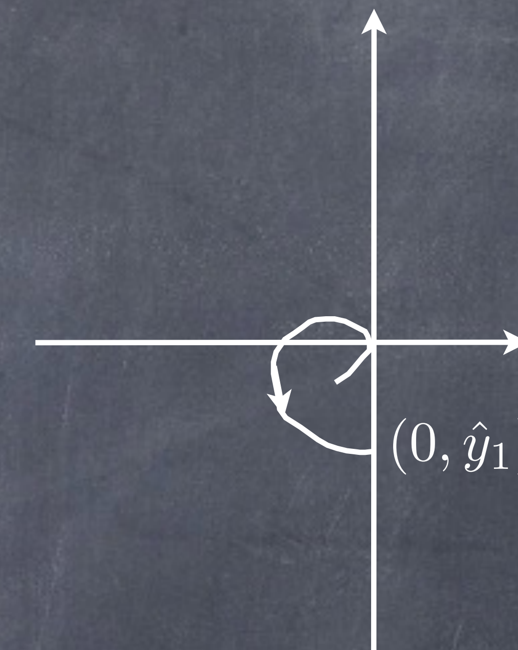
$$\tau = \int_{y_1}^{y_0} \frac{a}{a^2-aTy+Dy^2}dy = \int_{y_1}^{y_0} \frac{a}{DV(0,y)}dy$$

$$T \neq 0$$

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors







$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(0) = 0 \end{cases}$$

$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(\hat{y}_0) = 0 \end{cases}$$

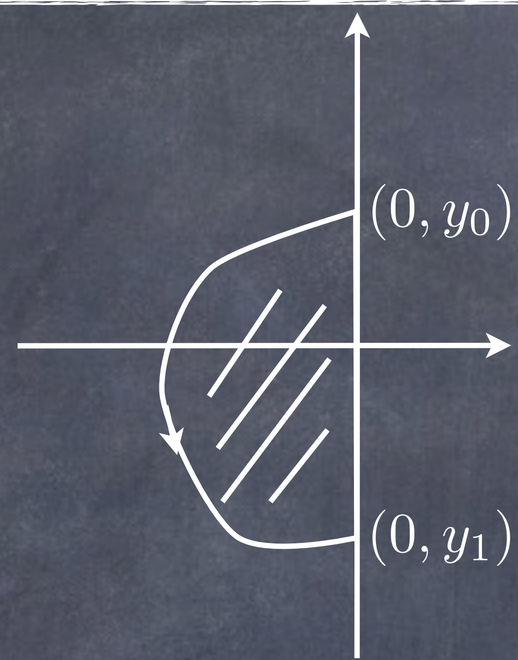
$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(0) = \hat{y}_1 \end{cases}$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = 0$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = -\frac{T}{\sqrt{T^2-4D}} \pi \operatorname{sign}(a)$$

Poincaré half-maps in Planar Linear Dynamical Systems via Inverse Integrating Factors

$a = 0$



$$\begin{cases} y_1' = \frac{y_0 V(0,y_1)}{y_1 V(0,y_0)} \\ y_1(0) = 0 \end{cases}$$

$$\begin{cases} y_1' = \frac{y_1}{y_0} \\ y_1(0) = 0 \end{cases}$$

$$\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = \frac{T}{2\sqrt{T^2-4D}}\pi$$

Poincaré half-maps in planar linear systems via inverse integrating factors

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Thank you very much!

UNIVERSIDAD DE SEVILLA

1ST JOINT MEETING BRAZIL-SPAIN IN MATHEMATICS

Fortaleza, December 7-10, 2015