Chaos and non-trivial minimal sets for planar piecewise smooth systems

Claudio A. Buzzi
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Co-authors: Rodrigo D. Euzébio and Tiago de Carvalho

Non-trivial minimal sets in planar non-smooth vector fields

Definition of minimal set for smooth flows

Let $K \subset E$ be a nonempty set and ϕ_t the flow of system $\dot{x} = f(x)$, f of class C^1. We say that K is a **minimal set** for such system if K is compact, invariant for ϕ_t and there exists no proper subset of K satisfying these properties. The minimal set K is **trivial** if it is a equilibrium point or a periodic orbit. Otherwise, K is called a **non-trivial** minimal set.

Denjoy-Schwartz Theorem

A flow ϕ_t of system $\dot{x} = f(x)$, of class C^2 defined in a bi-dimensional compact connected boundaryless manifold M can not have a minimal set K different from a equilibrium point or a periodic orbit, unless $M = K$ is the torus.

Poincaré-Bendixson Theorem

Consider system $\dot{x} = f(x)$ with $f \in C^1$ in some open set of $\mathbb{R}^2$ and suppose that it has a trajectory Γ contained in a compact subset F on which f has only a finite number of equilibrium points. Then it follows that $\omega(\Gamma)$ is either a equilibrium point, a periodic orbit or a graphic of such system.

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In summary, it holds that:

- The minimal sets for planar systems defined in open sets of $\mathbb{R}^2$ are equilibrium points and periodic orbits
- The minimal sets are trivial
- Minimal sets are part of limit sets
- They are measure-zero set in $\mathbb{R}^2$

Questions

- How about the minimal sets of planar non-smooth vector fields?
- Are they all trivial?
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- For $p \in \Sigma^+ \backslash \Sigma$ and $p \in \Sigma^- \backslash \Sigma$ the trajectory is given by $\phi_Z(t,p) = \phi_X(t,p)$ and $\phi_Z(t,p) = \phi_Y(t,p)$ respectively, where $t \in I$.
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Definition of invariance for non-smooth systems

Consider $\mathbf{Z} \in \mathbf{\Omega}$. A set $A \subset \mathbb{R}^2$ is **invariant** for Z if, for each $p \in A$ and all global trajectory $\Gamma_Z(t,p)$ of Z passing through p, it holds $\Gamma_Z(t,p) \subset A$.

Definition of minimal sets for non-smooth systems

Consider $Z \in \Omega$ a non-smooth vector field. A set $M \subset \mathbb{R}^2$ is **minimal** for Z if

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is invariant for Z;

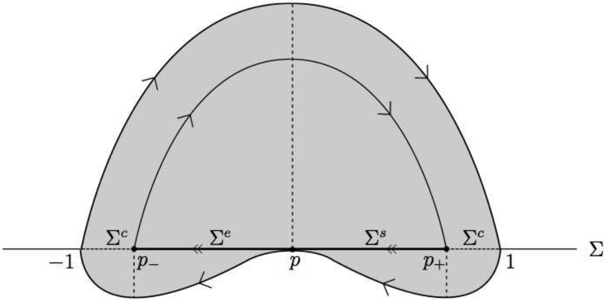
(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

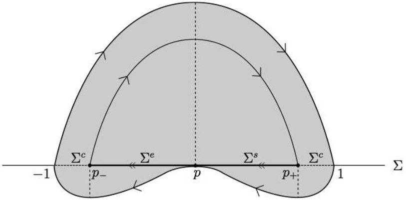
A non trivial minimal set

An example of non-trivial minimal set

Consider $Z = (X, Y) \in \Omega$, where $X(x, y) = (1, -2x)$, $Y(x, y) = (-2, 4x^3 - 2x)$, $f(x, y) = y$ and $\Sigma = f^{-1}(0) = \{(x, y) \in \mathbb{R}^2; y = 0\}$. Consider also

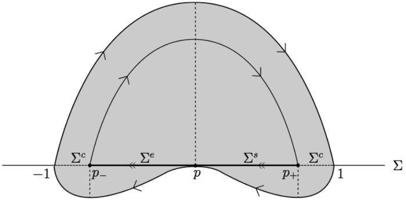
$$\Lambda = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1 \text{ and } x^4/2 - x^2/2 \leq y \leq 1 - x^2\}.$$





Observe that the set Λ above satisfies:

- Z is a planar system
- Λ is minimal
- $med(\Lambda) > 0$
- There exists a coincidence of two tangencies, one visible and the other invisible
- The trajectory for any point in Λ meets p for future and past times
- Any two points can be connected by a positive or negative trajectory
- For each point in Λ, there exists infinitely many trajectories such that filled up Λ for future or past
- There exist infinitely many trajectories which are not dense in Λ



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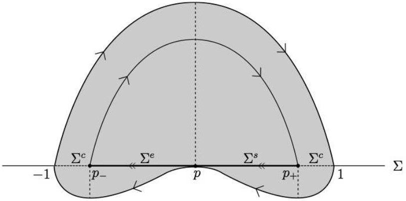

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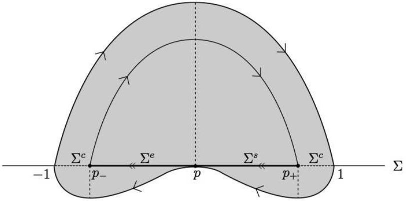

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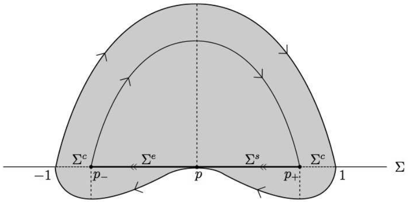

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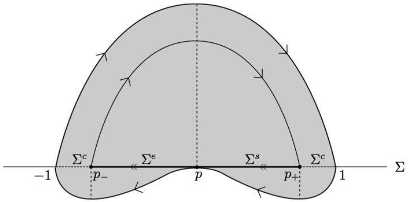

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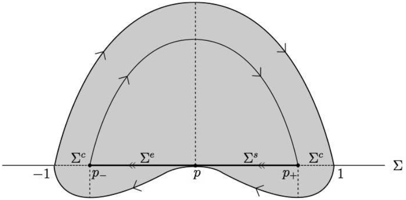

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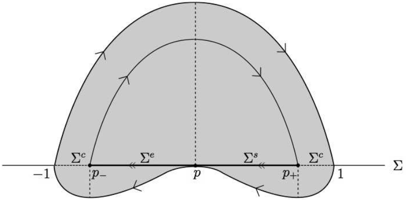

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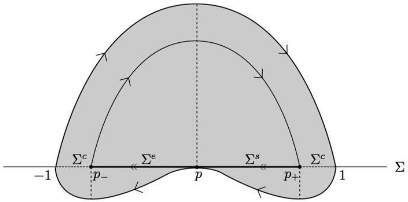

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Definition of positive/negative-invariance for non-smooth systems

A set $A \subset \mathbb{R}^2$ is **positive-invariant** (respectively, **negative-invariant**) if for each $p \in A$ and all positive global trajectory $\Gamma_Z^+(t,p)$ (respectively, negative global trajectory $\Gamma_Z^-(t,p)$) passing through p it holds $\Gamma_Z^+(t,p) \subset A$ (respectively, $\Gamma_Z^-(t,p) \subset A$).

Definition of positive/negative-minimal sets for non-smooth systems

Consider $\mathbf{Z} \in \mathbf{\Omega}$. A set $M \subset \mathbb{R}^2$ is **positive-minimal** (respectively, **negative-minimal**) if

(i) $M \neq \emptyset$;

(ii) M is compact;

(iii) M is positive-invariant (respectively, negative-invariant) for Z;

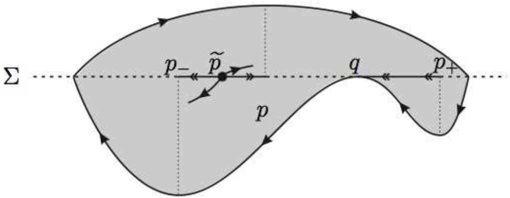
(iv) M does not contain proper subset satisfying (i), (ii) and (iii).

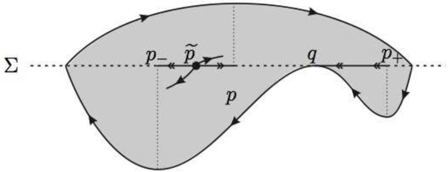
A non trivial minimal set

An example of non-trivial minimal set

Consider $Z_1 = (X, Y) \in \Omega$, where $X(x,y) = (1, -2x+1)$, $Y(x,y) = (-1, (-2+x)(-22+x(-7+4x)))$, $f(x,y) = y$ and $\Sigma = f^{-1}(0) = \{(x,y) \in \mathbb{R}^2; y = 0\}$. Consider also

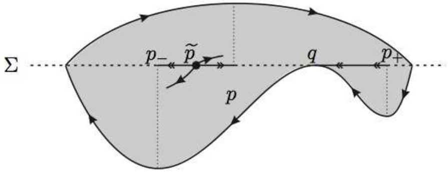
$$\begin{aligned}\Lambda_1 = \; & \{(x,y) \in \mathbb{R}^2; -3 \leq x \leq 4 \text{ and} \\ & (-4+x)(-2+x)^2(3+x) \leq y \leq -(-4+x)(3+x)\}.\end{aligned}$$





Observe that the set Λ_1 above satisfies:

- Z is a planar system having Λ_1 as minimal and $med(\Lambda_1) > 0$.
- There exists NO coincidence of tangencies
- The trajectory for any point in Λ meets q for FUTURE times. The trajectory for any point in Λ^o meets $\tilde{p}$ for PAST times
- For each point in Λ, there exists infinitely many trajectories which filled up Λ for simultaneous future AND past times
- Λ_1 is neither positive-minimal nor negative-minimal



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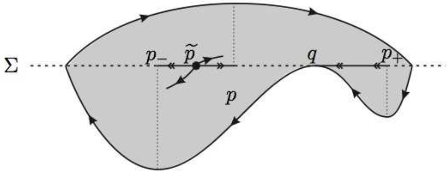

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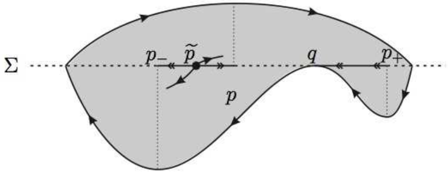

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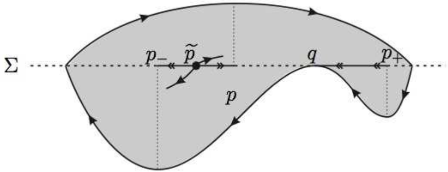

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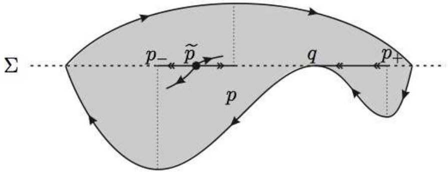

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Chaotic planar non-smooth systems having non-trivial minimal sets

Definition of topological transitivity

A discontinuous system Z is topologically transitive on an invariant set W if for every pair of nonempty, open sets U and V in W, there exist $q \in U$, $\Gamma_Z^+(t,q)$ a positive global trajectory and $t_0 > 0$ such that $\Gamma_Z^+(t_0,q) \in V$.

Definition of sentitive dependence

A discontinuous system Z exhibits sensitive dependence on a compact invariant set W if there is a fixed $r > 0$ satisfying $r < diam(W)$ such that for each $x \in W$ and $\varepsilon > 0$ there exist a $y \in B_\varepsilon(x) \cap W$ and positive global trajectories Γ_x^+ and Γ_y^+ passing through x and y, respectively, satisfying

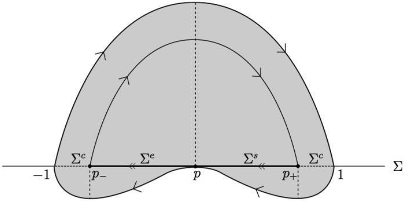
$$d_H(\Gamma_x^+, \Gamma_y^+) = \sup_{a \in \Gamma_x^+, b \in \Gamma_y^+} d(a,b) > r,$$

where $diam(W)$ is the diameter of W and d is the Euclidean distance.

Definition of chaos for non-smooth systems

A discontinuous system Z is chaotic on a compact invariant set W if it is topologically transitive and exhibits sensitive dependence on W.

In planar smooth systems there is no chaotic behavior, due to the Jordan curve Theorem. However, in non-smooth ones it may occur. See next Example.



Observe that, once any two points in Λ can be connected, it holds that

- Any two open sets in Λ can be connected. Then there exist topological transitivity in Λ
- Points arbitrarily closed x and y can be connected to others x' and y', respectively, in such way that $d(x', y')$ is sufficient large. Then there exist sensitive dependence in Λ
- Each point in Λ can be connected to itself. Then the set of periodic points in Λ coincides to it

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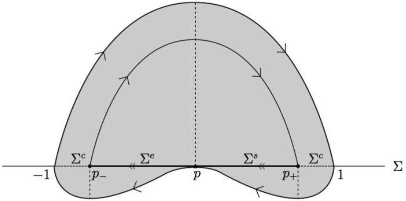

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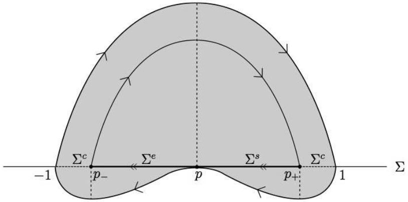

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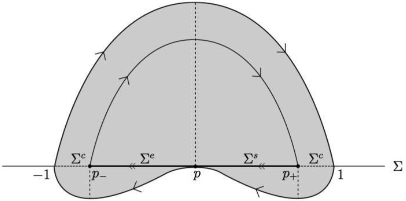

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Theorem A: Connection between minimal sets and chaotic systems

Let Z be a planar non-smooth vector field and $\Lambda \subset \mathbb{R}^2$ a compact invariant set. If Λ is simultaneously positive-minimal and negative-minimal satisfying $med(\Lambda) > 0$, then Z is chaotic on Λ.

In order to prove such results, we use the following lemmas:

Lemma 1

Under the same hypotheses of Theorem A, it holds that for any $x, y \in \Lambda$, there exist a global trajectory $\Gamma(t, y)$ passing through y and $t^* > 0$ such that $\Gamma^+(t^*, y) = x$.

Lemma 2

Under the same hypotheses of Theorem A, if any two points of Λ can be connected by a global trajectory of Z, then Z is chaotic on Λ.

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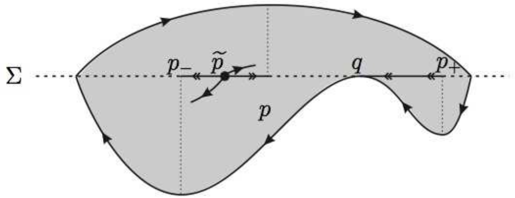
Under the same hypotheses of Theorem A, if any two points of Λ can be connected by a global trajectory of Z, then Z is chaotic on Λ.

A minimal set without chaotic behavior

Next example shows that Theorem A can not be extended for ordinary minimal sets. Indeed, the set Λ_1 presented here is minimal for Z_1 but this vector field is not chaotic on Λ_1.



Poincaré-Bendixson and Denjoy-Theorem for planar non-smooth vector fields

Poincaré-Bendixson for non-smooth systems

Let $Z = (X, Y) \in \Omega$. Assume that Z does not have sliding motion and it has a global trajectory $\Gamma_Z(t, p)$ whose positive trajectory $\Gamma_Z^+(t, p)$ is contained in a compact subset $K \subset V$. Suppose also that X and Y have a finite number of critical points in K, no one of them in Σ, and a finite number of tangency points with Σ. Then, the ω-limit set $\omega(\Gamma_Z(t, p))$ of $\Gamma_Z(t, p)$ is one of the following objects:

(i) an equilibrium of X or Y;

(ii) a periodic orbit of X or Y;

(iii) a graph of X or Y;

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Lemma

Under the same hypothesis of the last theorem, the ω-limit set $\omega(p)$ of a point $p \in V$ is one of the objects described in items (i), (ii), (iii), (iv), (v) and (vi) or a union of them.

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Denjoy-Schwartz Theorem for non-smooth systems

Under the same hypothesis of the last theorem, the minimal sets of a given non-smooth systems are trivial and given by one of the following objects:

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Thank you for your attention!